Spectrification of Khovanov homology II

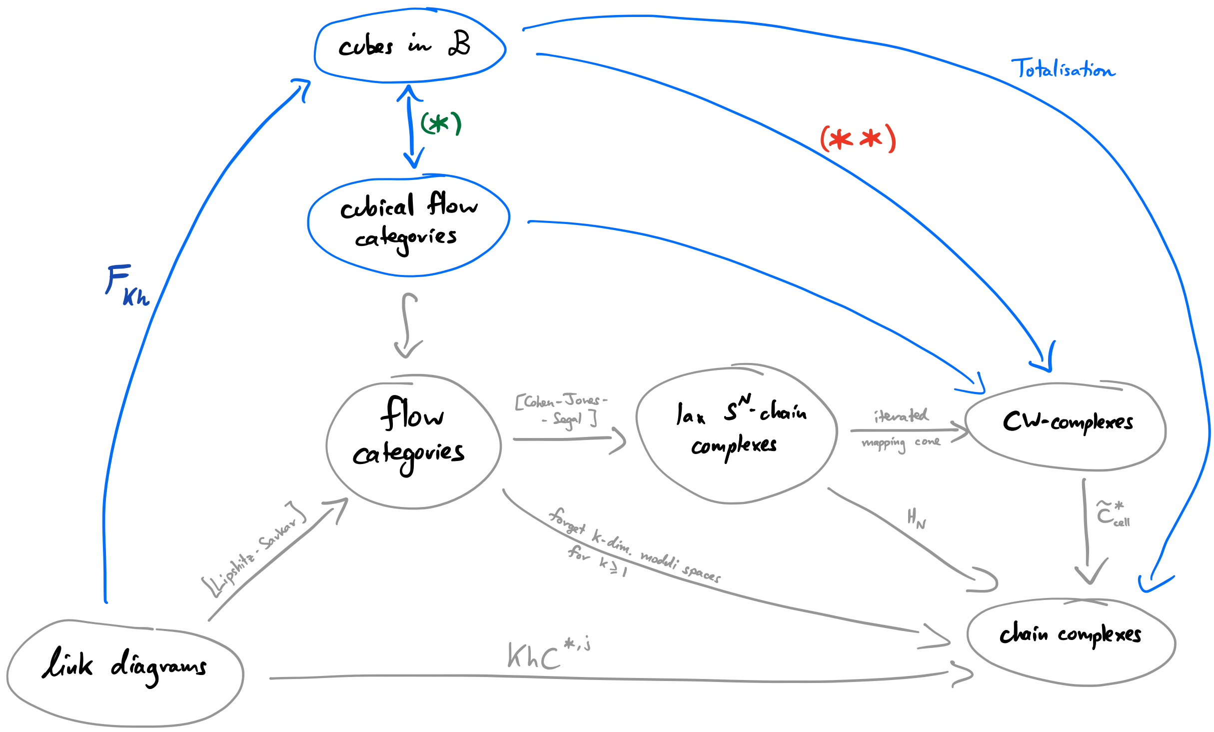
GeMAT seminar, IMAR
24 March 2022

Plan

(last time) I The original construction of Lipshitz – Sarkar

(today) II A second, simpler construction (Lawson - Lipshitz – Sarkar, Hu - Kriz - Kriz)

(8 April) III An extension to $\underbrace{\text{tangles}}_{(0,1)-\mathrm{TQFT}}$ and $\underbrace{\text{tangle cobordisms}}_{(0,1,2)-\mathrm{TQFT}}$



Plan

Setup

1. Flow categories & cubical flow categories
2. The Burnside category $\mathcal{B}$, cubical diagrams
3. The correspondence (∗)

Construction

4. The Khovanov functor $F_{Kh}(D)$ of a link diagram D
5. Invariance of F_{Kh}
6. The realisation (∗∗)

Applications

7. Corollary : $\sqcup$, $\#$, mirrors of links
8. Corollary : non-trivial Sq^n
9. Universal Khovanov homology (deformations of Kh) & other questions

1. Flow categories

A <u>flow category</u> $\mathcal{C}$ is a category where

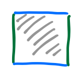
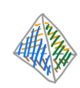
- its object set $ob(\mathcal{C})$ is graded by
  $gr : ob(\mathcal{C}) \longrightarrow \mathbb{Z}$
- $Hom(x,x) = \{id_x\}$
- if $x \neq y$ and $gr(x) - gr(y) = k+1$
  then $Hom(x,y)$ is a compact k-dimensional <u>$\langle k \rangle$-manifold</u>
- the disjoint union of composition maps

$$\coprod_{\substack{z \neq x \\ z \neq y \\ gr(z) - gr(y) = i}} Hom(z,y) \times Hom(x,z) \longrightarrow Hom(x,y)$$

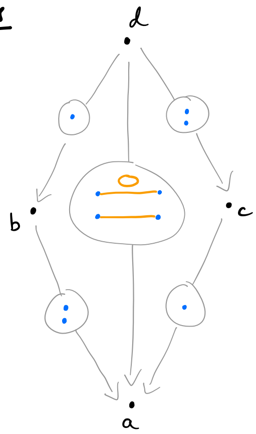
is a diffeomorphism onto $\partial_i Hom(x,y)$

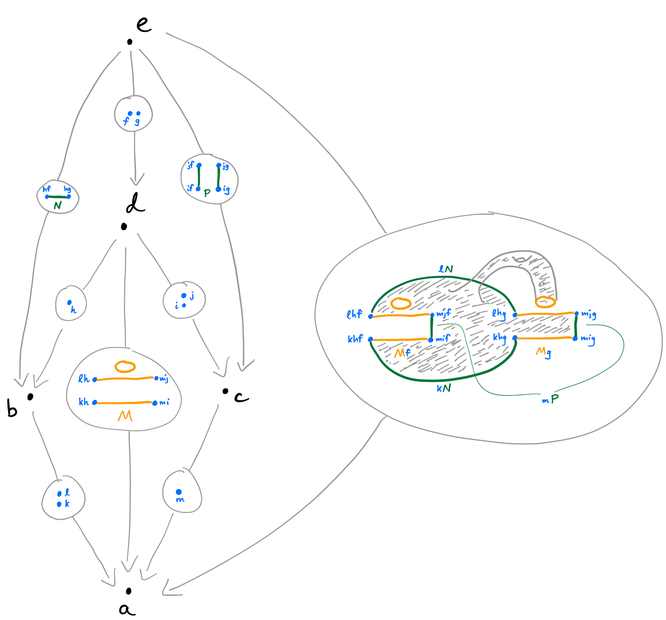
a d-dimensional $\langle c \rangle$-manifold is a manifold with corners M (modelled on $\mathbb{R}^d_+ = [0,\infty)^d$), equipped with a decomposition of ∂M into $\partial_1 M, \ldots, \partial_c M$, where each $\partial_i M$ is a disjoint union of codimension-1 faces.

<u>Eg</u>.

-  2-dim $\langle 2 \rangle$-mfld
-  2-dim $\langle 3 \rangle$-mfld
- $[0,1]^n$ n-dim $\langle n \rangle$-mfld
-  3-dim $\langle 4 \rangle$-mfld
- Δ^n n-dim $\langle n+1 \rangle$-mfld

<u>Examples</u>





A framed flow category is a flow category $\mathcal{C}$ together with framed embeddings of each $\langle k \rangle$-manifold $\mathrm{Hom}(x,y)$ into some Euclidean space with corners, and these should be compatible with composition in $\mathcal{C}$.

Note:

When $gr(x) - gr(y) = 1$, $\mathrm{Hom}(x,y)$ is a compact 0-dim. $\langle 0 \rangle$-manifold, i.e. a finite set.

A framed embedding of $\mathrm{Hom}(x,y)$ into Euclidean space determines a sign for each point of $\mathrm{Hom}(x,y)$.

framed flow categories $\longrightarrow$ chain complexes

objects $\longrightarrow$ generators

$\left.\begin{array}{l}\mathrm{Hom}(x,y)\\ gr(x)-gr(y)=1\end{array}\right\} \longrightarrow$ differential $\qquad \delta(x) = \sum_y \underbrace{\#\mathrm{Hom}(x,y)}_{\text{counted with sign}} . \, y$

Def

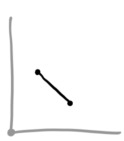
Consider the $n!$ points $(\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n$ for $\sigma \in \Sigma_n$.

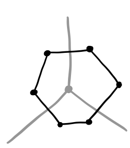
Note: they all lie on the affine subspace $\{ \underline{x} \in \mathbb{R}^n \mid \sum x_i = \frac{1}{2} n(n+1) \}$.

The $(n-1)$-dimensional permutohedron Π_{n-1} is the convex hull of these points.

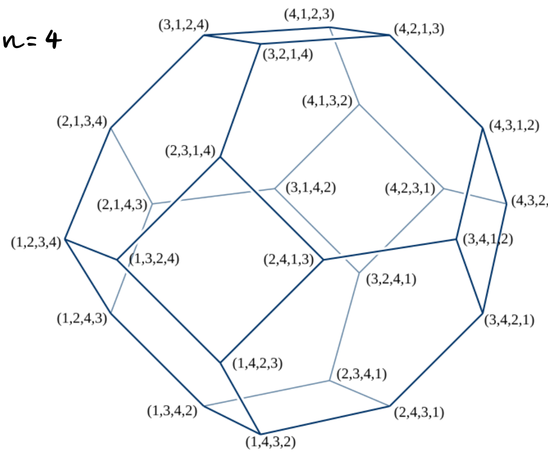
E.g.

$n=1$

$n=2$ 

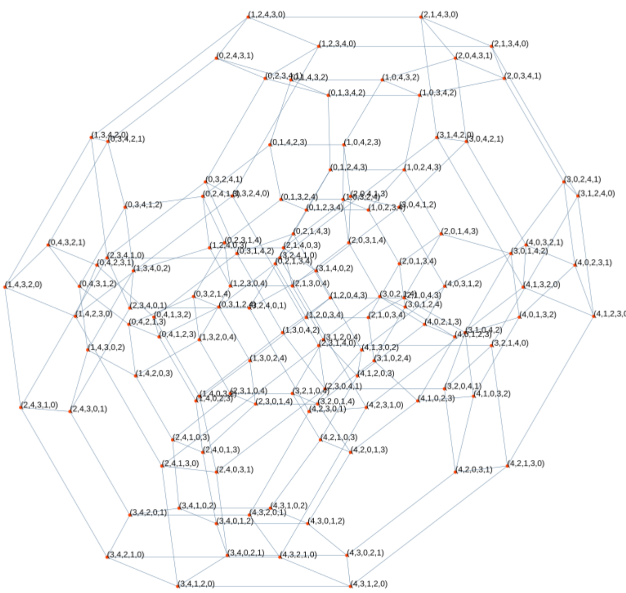
$n=3$ 

$n = 4$



(source : Wikipedia)

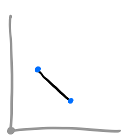
$n = 5$



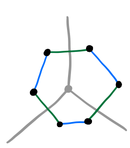
(source : Wikipedia)

Lemma: Π_k is a k-dimensional $\langle k \rangle$-manifold

Eg: Π_1



Π_2

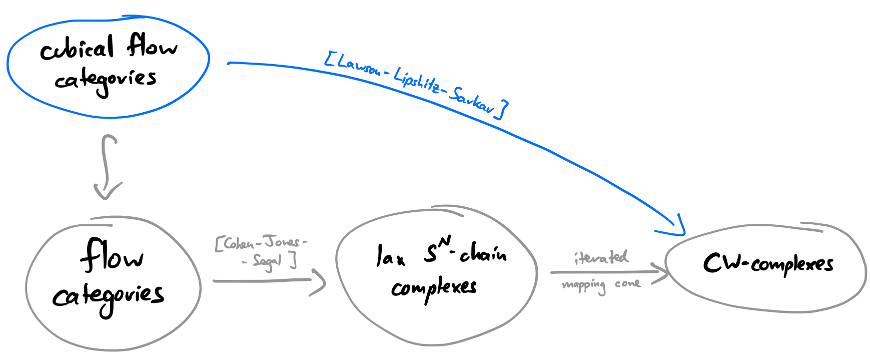


Proposition: There is a flow category $\mathcal{C}(n)$ with
- objects $= \{0,1\}^n$
- grading $gr(v_1, \dots, v_n) = \sum_i v_i$
- when $gr(u) - gr(v) = k > 0$, $\operatorname{Hom}(u,v) = \begin{cases} \Pi_{k-1} & \text{if } u_i \geq v_i \\ \varnothing & \text{otherwise} \end{cases}$

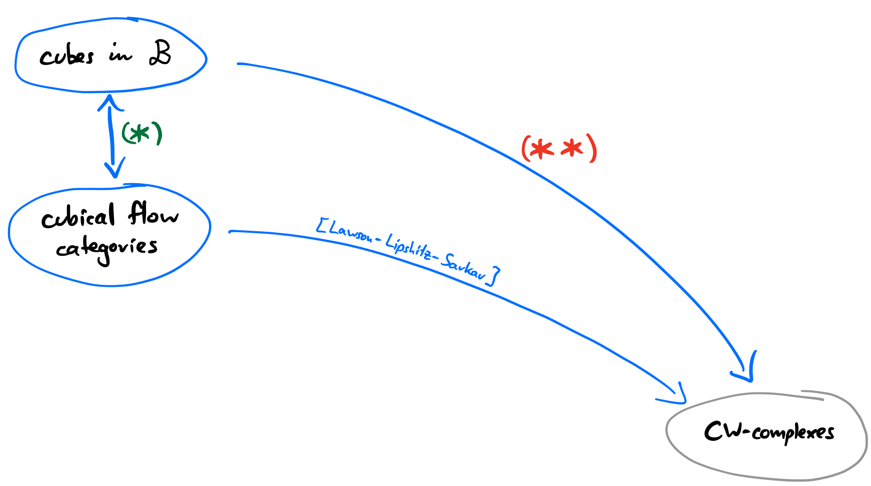
Rmk: The morphism spaces $\mathrm{Hom}(u,v)$ in $\mathscr{C}(n)$ may be thought of as moduli spaces of "broken Morse flows" on the n-cube $[0,1]^n$.

Def: A <u>cubical flow category</u> is a flow category $\mathscr{C}$ and a functor $f: \mathscr{C} \longrightarrow \mathscr{C}(n)$ that preserves the grading up to a global shift, such that each $f: \mathrm{Hom}_{\mathscr{C}}(x,y) \longrightarrow \underbrace{\mathrm{Hom}_{\mathscr{C}(n)}(f(x), f(y))}_{\text{permutohedron or empty}}$ is a trivial covering.

Rmk
- Cubical flow categories are very rigid compared with general flow categories.
- This allows [LLS] to give a construction of the <u>realisation</u> of a cubical flow category that is much simpler than the one of [CJS], which works for all flow categories:



- We'll see this construction later, via <u>cubes in $\mathcal{B}$</u>:



2. The Burnside category $\mathcal{B}$ and cubical diagrams

Def $\mathcal{B}$ has objects $=$ finite sets

$\mathrm{Hom}(A,B)$ $=$ diagrams of functions of the form
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$
$=$ correspondences

2-morphisms from
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$
to
$$\begin{tikzcd} A & & B \\ & X' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$
are bijections $X \cong X'$ such that
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, phantom, "\cong"] & \\ A & & B \\ & X' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$
commutes.

Composition is given by
$$X \times_B Y = \{(x,y) \in X \times Y \mid t(x) = s'(y)\}$$
$$\begin{tikzcd} & & X \times_B Y \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl, "s"'] \arrow[dr, "t"] & & Y \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\ A & & B & & C \end{tikzcd}$$

The cube category has objects $\{0,1\}^n$
$$\mathrm{Hom}(u,v) = \begin{cases} \{\varphi_{u,v}\} & \text{if } u_i \geq v_i \\ \emptyset & \text{otherwise} \end{cases}$$

Notation: $\underline{2}^n$

E.g.: $n=2$
$$\begin{tikzcd} 11 \arrow[r] \arrow[d] \arrow[dr] & 10 \arrow[d] \\ 01 \arrow[r] & 00 \end{tikzcd}$$

$n=3$
$$\begin{tikzcd} & 111 \arrow[rr] \arrow[dl] \arrow[dd] & & 110 \arrow[dl] \arrow[dd] \\ 011 \arrow[rr] \arrow[dd] & & 010 \arrow[dd] & \\ & 101 \arrow[rr] \arrow[dl] & & 100 \arrow[dl] \\ 001 \arrow[rr] & & 000 & \end{tikzcd}$$

A **cube in $\mathcal{B}$** consists of

- for each object $v \in \{0,1\}^n$, a finite set $F(v)$
- for each pair $u > v$, a correspondence
$$\begin{tikzcd} & F(u>v) \arrow[dl] \arrow[dr] & \\ F(u) & & F(v) \end{tikzcd}$$
- for each triple $u > v > w$, a bijection
$$\begin{tikzcd} & & F(u>w) \arrow[ddll, bend right] \arrow[ddrr, bend left] \arrow[d, phantom, "\cong" rotate=90] & & \\ & & F(u>v) \times_{F(v)} F(v>w) \arrow[dl] \arrow[dr] & & \\ F(u) & F(u>v) \arrow[l] \arrow[r] & F(v) & F(v>w) \arrow[l] \arrow[r] & F(w) \end{tikzcd}$$
- satisfying a certain condition for any quadruple $u > v > w > x$

(Formally, this is a strictly unitary lax 2-functor $\underline{2}^n \longrightarrow \mathcal{B}$.)

Lemma: It is enough to specify:

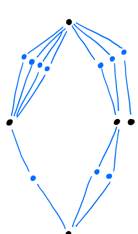
- $F(v)$ for each vertex u,
- $F(u>v)$ for each edge $u \to v$,
- the composite $F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \cong F(u>v') \times_{F(v')} F(v'>w)$ for each 2-face $u \begin{smallmatrix} \nearrow v \searrow \\ \searrow v' \nearrow \end{smallmatrix} w$.
$$\underbrace{F(u>v) \times_{F(v)} F(v>w) \longrightarrow F(u>v') \times_{F(v')} F(v'>w)}_{\cong}$$

The **underlying chain complex** of a cube in $\mathcal{B}$: (also called its **totalisation**)

- First construct a cube of free abelian groups by
$$\begin{tikzcd} & F(u) \\ F(u>v) \arrow[ur, "s"] \arrow[dr, "t"'] & \\ & F(v) \end{tikzcd} \longmapsto \begin{tikzcd} \mathbb{Z}\langle F(u) \rangle \arrow[d] \\ \mathbb{Z}\langle F(v) \rangle \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] \\ \displaystyle\sum_{y \in F(v)} \#(s^{-1}(x) \cap t^{-1}(y)) \cdot y \end{tikzcd}$$

- Then add signs to certain edges of the cube and sum.

E.g.



$$\longmapsto \begin{tikzcd} & \mathbb{Z} \arrow[dl, "4"'] \arrow[dr, "\binom{2}{1}"] & \\ \mathbb{Z} \arrow[dr, "-1"'] & \oplus & \mathbb{Z}^2 \arrow[dl, "(2\ 0)"] \\ & \mathbb{Z} & \end{tikzcd}$$

3. The correspondence cubes in $\mathcal{B}$ $\longleftrightarrow$ cubical flow categories

Recall

- In cubical flow categories, morphism spaces are disjoint unions of permutohedra.
- So most of the complication is taken care of by the combinatorics of permutohedra.
- The additional information is exactly encoded by a cube in $\mathcal{B}$.

cubes in $\mathcal{B}$ $\longleftarrow$ cubical flow categories

$$\mathcal{C} \xrightarrow{f} \underbrace{\mathcal{C}(n)}_{\text{objects} = \{0,1\}^n} \quad \text{cubical flow category}$$

$$\downarrow$$

$$F : \underline{2}^n \longrightarrow \mathcal{B} \qquad F(v) = f^{-1}(v)$$

$$F(u>v) = \coprod_{\substack{x \in f^{-1}(u) \\ y \in f^{-1}(v)}} \pi_0 \underbrace{\mathrm{Hom}_{\mathcal{C}}(x,y)}_{\text{disjoint union of permutohedra}}$$

$$f^{-1}(u) = F(u)$$

$$f^{-1}(v) = F(v)$$

$$F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \quad \text{induced from composition in } \mathcal{C}.$$

cubes in $\mathcal{B}$ $\longrightarrow$ cubical flow categories

$$F: \underline{2}^n \longrightarrow \mathcal{B} \qquad \longrightarrow \qquad \mathcal{C} \xrightarrow{f} \mathcal{C}(n)$$

- $ob(\mathcal{C}) = \coprod_{v \in \{0,1\}^n} F(v)$

- For $u > v$ and $x \in F(u)$, $y \in F(v)$, consider
$$\begin{tikzcd} & F(u>v) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ F(u) & & F(v) \end{tikzcd}$$

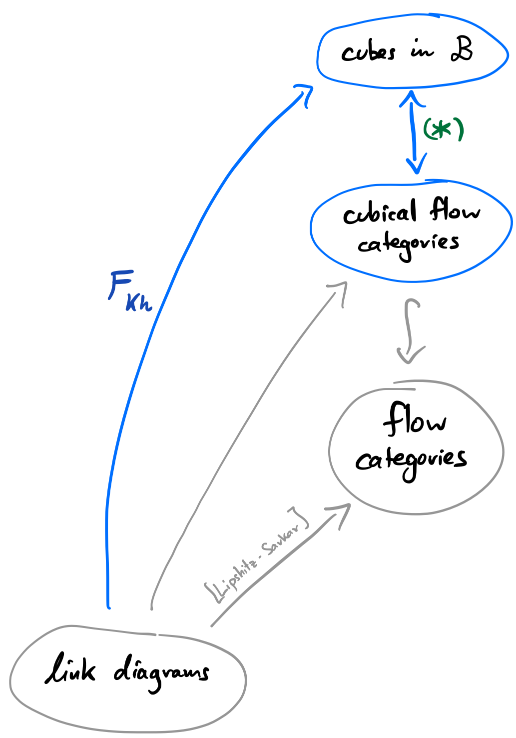
  and let $\mathrm{Hom}_{\mathcal{C}}(x,y) = \left(s^{-1}(x) \cap t^{-1}(y)\right) \times \underbrace{\mathrm{Hom}_{\mathcal{C}(n)}(u,v)}_{\text{permutohedron}}$

- Composition is defined using $\begin{cases} F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \text{ from } F \\ \text{composition in } \mathcal{C}(n) \end{cases}$

Lemma: These constructions are inverse to each other.
In other words:

$$\{\text{cubical flow categories}\} = \begin{matrix} (\text{the standard permutohedron flow category } \mathcal{C}(n)) \\ + \\ \{\text{cubes in the Burnside category } \mathcal{B}\} \end{matrix}$$

The flow categories constructed by [Lipshitz-Sarkar] are cubical, so we may describe them equivalently as cubes in $\mathscr{B}$:



Next : describe F_{Kh} ...

4. The Khovanov functor of a link diagram

Warm-up /recollection : link diagram $D \rightsquigarrow$ Khovanov cube $F^{Ab}_{Kh}(D)$ of abelian groups

(• add signs to certain edges
• take $\bigoplus$ within each homological grading
$\rightsquigarrow$ Khovanov chain complex)

- label crossings by $\{1, 2, \dots, n\}$
- $v \in \{0,1\}^n \longrightarrow D_v$ = perform 0-resolution  at the i^{th} crossing if $v_i = 0$

  perform 1-resolution  at the i^{th} crossing if $v_i = 1$

  This is an embedded disjoint union of circles in $\mathbb{R}^2$.

$$F^{Ab}_{Kh}(D)(v) = \bigotimes_{\pi_0(D_v)} \mathbb{Z}\langle x_+, x_- \rangle$$

- $u \longrightarrow v$ edge of the cube
  - if $D_u \rightsquigarrow D_v$ merges two circles, apply m : $\begin{array}{l} x_+ \otimes x_+ \longmapsto x_+ \\ x_+ \otimes x_- \longmapsto x_- \\ x_- \otimes x_+ \longmapsto x_- \\ x_- \otimes x_- \longmapsto 0 \end{array}$ (and identity on other components)
  - if $D_u \rightsquigarrow D_v$ splits a circle in two, apply Δ : $\begin{array}{l} x_+ \longmapsto x_+ \otimes x_- + x_- \otimes x_+ \\ x_- \longmapsto x_- \otimes x_- \end{array}$ (and identity on other components)

Aim : link diagram $D \rightsquigarrow$ Khovanov cube $F_{Kh}(D)$ in $\mathcal{B}$

- $F_{Kh}(D)(v) = \{x_+, x_-\}^{\pi_0(D_v)}$
- $u \longrightarrow v$ edge of the cube

$$F^{Ab}_{Kh}(D)(u \to v) : \mathbb{Z}\langle F_{Kh}(D)(u) \rangle \longrightarrow \mathbb{Z}\langle F_{Kh}(D)(v) \rangle$$

all coefficients are 0 or 1 by construction, so there is no choice :

$$F_{Kh}(D)(u \to v) = \{(x,y) \in F_{Kh}(D)(u) \times F_{Kh}(D)(v) \mid \text{coeff of } y \text{ in the image of } x \text{ is } 1\}$$

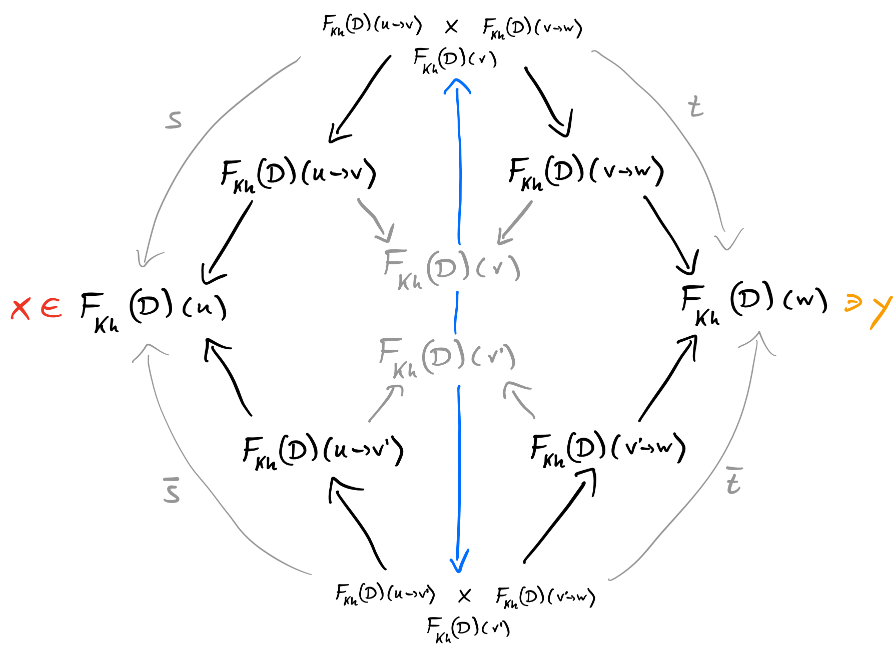
$$\begin{tikzcd} & F_{Kh}(D)(u \to v) \arrow[dl] \arrow[dr] & \\ F_{Kh}(D)(u) & & F_{Kh}(D)(v) \end{tikzcd}$$

Rmk So far, this is the same information as $F_{Kh}^{Ab}(D)$, repackaged.

The difference lies in the last part of the construction of $F_{Kh}(D)$.

- $\begin{tikzcd}[row sep=small] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}$ 2-face of the cube

We need to choose a bijection:



Unpacking this, for each x (labelling of $\pi_0(D_u)$ by $\{x_+, x_-\}$)

y (labelling of $\pi_0(D_w)$ by $\{x_+, x_-\}$)

we need to choose a bijection:

$$s^{-1}(x) \cap t^{-1}(y) = \{\text{labellings of } \pi_0(D_v), \text{ compatible with } F_{Kh}(D)(\underset{x}{u} \to v \to \underset{y}{w})\}$$

$$\updownarrow$$

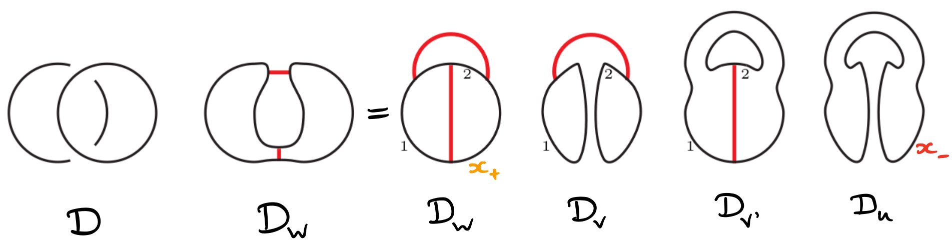
$$\bar{s}^{-1}(x) \cap \bar{t}^{-1}(y) = \{\text{labellings of } \pi_0(D_{v'}), \text{ compatible with } F_{Kh}(D)(\underset{x}{u} \to v' \to \underset{y}{w})\}$$

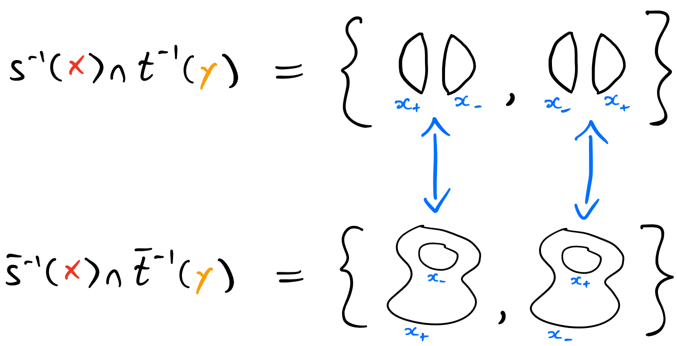
In all cases, these sets:

- are both empty, or
- both have size 1, or
- both have size 2.

no choice!

The last case occurs when:





- This is called the "ladybug matching"
- It is the only extra information about the link diagram D that is remembered by $\underbrace{F_{Kh}(D) : \underline{2}^n \to \mathcal{B}}_{\text{Khovanov spectrum}}$ compared with $\underbrace{F^{Ab}_{Kh}(D) : \underline{2}^n \longrightarrow Ab}_{\text{Khovanov homology}}$.

5. Invariance of F_{Kh}

Thm (Lawson-Lipshitz-Sarkar)

$$\begin{tikzcd}
\{\text{link diagrams}\}/_{\text{isotopy}} \arrow[r, "F_{Kh}"] \arrow[d, two heads] & \{\text{cubes in } \mathcal{B}\}/_{\text{isomorphism}} \arrow[d, two heads] \\
\{\text{links}\}/_{\text{isotopy}} \arrow[r] & \{\text{cubes in } \mathcal{B}\}/_{\text{stable equivalence}}
\end{tikzcd}$$

Def "stable equivalence" is the equivalence relation on cubes in $\mathcal{B}$ generated by:

- natural transformations $\begin{tikzcd} \underline{2}^n \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{B} \end{tikzcd}$ whose induced map on underlying chain complexes is a chain homotopy equivalence.

- if $\underline{2}^N \to \mathcal{B}$ is obtained from $\underline{2}^n \to \mathcal{B}$, $n < N$, by sending all vertices of $\underline{2}^N \setminus \underline{2}^n$ to $\emptyset$.

Idea of proof: Copy the proof of invariance of Khovanov homology under the Reidemeister moves and upgrade to a stable equivalence at each stage.

Rmk: We will next construct $\{\text{cubes in } \mathcal{B}\} \xrightarrow{(**)} \{\text{CW-complexes}\}$, which will send stable equivalences to stable homotopy equivalences.

$$\{\text{links}\}/_{\text{isotopy}} \longrightarrow \{\text{cubes in } \mathcal{B}\}/_{\text{stable equivalence}} \xrightarrow{\overline{(**)}} \{\text{spectra}\}$$

Question: How much information about a link does $\overline{(**)}$ forget ??

6. The realisation of a cube in the Burnside category

Box maps

Fix
$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$

"box" $= \prod_{i=1}^{k} [a_i, b_i] \subseteq \mathbb{R}^k$

and "boxes" $B_x \quad x \in X$, $B_y \quad y \in Y$] pairwise disjoint

$B_a \subseteq B_{s(a)} \quad a \in A$] pairwise disjoint

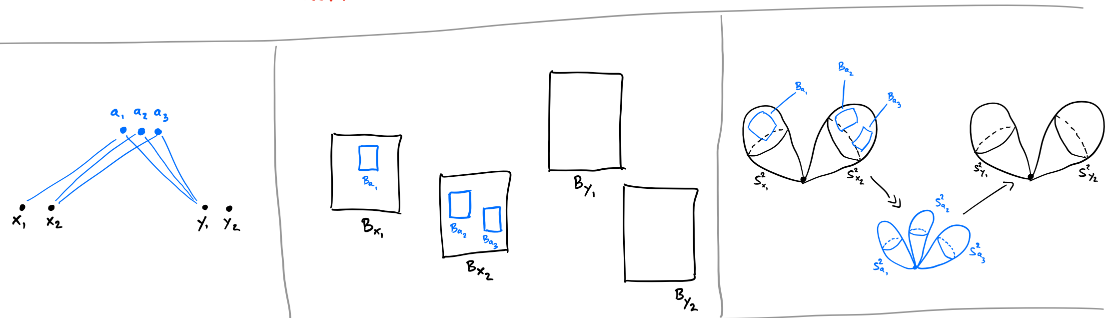
The corresponding box map is

$$\bigvee_{x \in X} S^k = \coprod_{x \in X} B_x \Big/ \partial \longrightarrow \bigvee_{a \in A} B_a / \partial B_a = \bigvee_{a \in A} S^k \longrightarrow \bigvee_{y \in Y} S^k$$

collapse the complement of $\coprod_{a \in A} \mathring{B}_a$ to a point

determined by t

Example



Lemma For fixed
$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$
, the space of box maps

$$\bigvee_{x \in X} S^k \longrightarrow \bigvee_{y \in Y} S^k$$ is <u>$(k-2)$-connected</u>.

Def For a cube $\underline{2}^n \xrightarrow{F} \mathcal{B}$, a <u>k-dim. spatial refinement</u> is a <u>homotopy coherent diagram</u> $\underline{2}^n \xrightarrow{\widetilde{F}} \mathrm{Top}_\bullet$

- for each object $v \in \underline{2}^n$, a space $\widetilde{F}(v)$
- for each sequence of morphisms

$$v_0 \xrightarrow{f_1} v_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} v_n$$

a continuous map

$$\widetilde{F}(f_n, \dots, f_1) : [0,1]^{n-1} \times \widetilde{F}(v_0) \longrightarrow \widetilde{F}(v_n)$$

- satisfying some conditions

where $\widetilde{F}(v) = \bigvee_{F(v)} S^k$

$$\widetilde{F}(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) : \bigvee_{F(v_0)} S^k \longrightarrow \bigvee_{F(v_n)} S^k$$

is a box map associated to the correspondence

$$F(v_0) \longleftarrow F(f_n \circ \cdots \circ f_1) \longrightarrow F(v_n)$$

Proposition If $k \geq n+1$, k-dim. spatial refinements exist and are unique up to hty equivalence of hty coherent diagrams.

<u>Idea</u>: construct it recursively, using the fact that the space of box maps is highly-connected.

Rmk Homotopy coherent diagrams $\underline{2}^n \xrightarrow{\tilde{F}} \mathrm{Top}_\bullet$ have well-defined iterated mapping cones $|\tilde{F}| \in \mathrm{Top}_\bullet$.

([Vogt, 1973])

Construction

$$\underline{2}^n \xrightarrow{F} \mathcal{B} \rightsquigarrow \underline{2}^n \xrightarrow{\tilde{F}} \mathrm{Top}_\bullet \rightsquigarrow |\tilde{F}| \in \mathrm{Top}_\bullet$$

Rmk
- Both steps depend on the higher faces of the cube.
- This is essential for preserving the extra info that $\underline{2}^n \longrightarrow \mathcal{B}$ captures!

Proposition If F, G are stably equivalent cubes in $\mathcal{B}$, then $|\tilde{F}|, |\tilde{G}|$ are stably homotopy equivalent (pointed) spaces, i.e. they determine homotopy equivalent (suspension) spectra.

Proposition The composition

$$\{\text{links}\}/_{\text{isotopy}} \xrightarrow{F_{Kh}} \{\text{cubes in } \mathcal{B}\}/_{\text{stable equivalence}} \xrightarrow{(**)} \{\text{spectra}\}$$

$$L \longmapsto X_{Kh}(L)$$

- comes equipped with a decomposition $X_{Kh}(L) = \bigvee_{j \in \mathbb{Z}} X^j_{Kh}(L)$
- recovers the Khovanov spectrum of [Lipshitz-Sarkar]
- in particular, $\tilde{H}^*(X^j_{Kh}(L)) \cong Kh^{*,j}(L)$.

7. Corollary : $\amalg$, $\#$, mirrors of links

This new, much simpler construction of $\mathcal{X}_{Kh}(L)$ allows [LLS] to prove:

Thm [LLS]

- $\mathcal{X}_{Kh}(L_1 \amalg L_2) \simeq \mathcal{X}_{Kh}(L_1) \wedge \mathcal{X}_{Kh}(L_2)$

  smash product

- $\mathcal{X}_{Kh}(L_1 \# L_2) \simeq \mathcal{X}_{Kh}(L_1) \underset{\mathbb{S} \vee \mathbb{S}}{\otimes} \mathcal{X}_{Kh}(L_2)$

  tensor product over $\mathbb{S} \vee \mathbb{S} = \mathcal{X}_{Kh}(\text{unknot})$

- $\mathcal{X}^{j}_{Kh}(\underbrace{m(L)}_{\text{mirror of } L}) \simeq \mathcal{X}^{-j}_{Kh}(L)^{\vee}$

  Spanier-Whitehead dual

Rmk In each case, the proof reduces to a statement about cubes in $\mathcal{B}$ instead of flow categories, which is what makes it tractable.

8. Corollary^2: non-trivial Sq^n

<u>Thm</u> [LLS] $\forall n$ $\exists$ link L_n such that its Khovanov homology has a non-trivial $Sq^n: Kh^{i,j}(L_n) \longrightarrow Kh^{i+n,j}(L_n)$ operation.

<u>Proof</u>
- We need to find L_n such that the spectrum $\mathcal{X}_{Kh}(L_n)$ has a non-trivial Sq^n in its cohomology.
- The space $\underbrace{\mathbb{RP}^2 \wedge \cdots \wedge \mathbb{RP}^2}_{n \text{ copies}}$ has a non-trivial Sq^n.
- $\Rightarrow$ Enough to find L_n with $\mathcal{X}_{Kh}(L_n) \simeq Z \vee \Sigma^k(\mathbb{RP}^2 \wedge \cdots \wedge \mathbb{RP}^2)$
- Calculation of Lipshitz-Sarkar:

  $\mathcal{X}_{Kh}(\text{left trefoil}) \simeq Z' \vee \Sigma^{-4}\mathbb{RP}^2$
- By the previous corollary we may take

  L_n = disjoint union of n left trefoils. $\square$

9. Universal Khovanov homology (deformations of Kh) & other questions

Khovanov homology may be upgraded to universal Khovanov homology :

- label crossings by $\{1, 2, \dots, n\}$
- $v \in \{0,1\}^n \longrightarrow D_v =$ perform 0-resolution $\times \rightsquigarrow)($ at the i^{th} crossing if $v_i = 0$

  perform 1-resolution $\times \rightsquigarrow \asymp$ at the i^{th} crossing if $v_i = 1$

  This is an embedded disjoint union of circles in $\mathbb{R}^2$.

$$F^{Ab}_{Kh}(D)(v) = \bigotimes_{\pi_0(D_v)} \mathbb{Z}[h,t]\langle x_+, x_- \rangle$$

- $u \longrightarrow v$ edge of the cube
  - if $D_u \rightsquigarrow D_v$ merges two circles, apply m :
    $$\begin{aligned} x_+ \otimes x_+ &\longmapsto x_+ \\ x_+ \otimes x_- &\longmapsto x_- \\ x_- \otimes x_+ &\longmapsto x_- \\ x_- \otimes x_- &\longmapsto h x_- + t x_+ \end{aligned}$$
    (and identity on other components)
  - if $D_u \rightsquigarrow D_v$ splits a circle in two, apply Δ :
    $$\begin{aligned} x_+ &\longmapsto x_+ \otimes x_- + x_- \otimes x_+ - h . x_+ \otimes x_+ \\ x_- &\longmapsto x_- \otimes x_- + t . x_+ \otimes x_+ \end{aligned}$$
    (and identity on other components)

- add signs to certain edges
- take $\oplus$ within each homological grading

  $\rightsquigarrow$ universal Khovanov chain complex

Quantum gradings :

x_+	$+1$
x_-	-1
h	-2
t	-4

set $(h,t) = (0,0)$ and take homology $\longrightarrow$ Khovanov homology (over $\mathbb{Z}$) (bigraded)

set $(h,t) = (0,1)$ and take homology $\longrightarrow$ Lee homology (over $\mathbb{Z}$) (graded + filtered)

set $t = 0$ and take homology $\longrightarrow$ Bar-Natan homology (over $\mathbb{Z}[h]$) (bigraded)

if we set: (homological, quantum) grading of $h = (0,-2)$

take homology $\longrightarrow$ universal Khovanov homology (over $\mathbb{Z}[h,t]$) (bigraded)

if we set: (homological, quantum) grading of $h = (0,-2)$
(homological, quantum) grading of $t = (0,-4)$

Question Can any of these deformations of Khovanov homology be spectrified ?

$\hookrightarrow$ Question If we specialize (h,t) to integers, can the Khovanov cube in Ab be lifted to a cube in the Burnside category ?

Rmk Clearly no unless $h = 0$ and $t \geq 0$ (because sets cannot have negative cardinality).

[LLS] Also impossible for $(h,t) = (0,1)$

Question

- Can Seidel-Smith's description of Khovanov homology via Floer theory be upgraded to produce a flow category refining the Khovanov chain complex and then a Khovanov spectrum via the construction of Cohen-Jones-Segal ?
- If yes, is it homotopy equivalent to $\underbrace{\mathcal{X}_{Kh}(-)}_{\text{(Lawson-) Lipshitz-Sarkar}}$?