Mapping class group representations via Heisenberg, Schrödinger and Stone-von Neumann

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30 March 2022

Based on joint work with

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arXiv: 2109.00515

Background / motivation / goal

Σ = surface

$\mathcal{M}(\Sigma)$ = its mapping class group

$= \pi_0(\mathrm{Diff}_\partial(\Sigma))$

$= \{$ isotopy classes of self-diffeomorphisms of Σ that fix $\partial\Sigma$ pointwise $\}$

Goal: Construct "interesting" $\underbrace{\text{homological}}_{\text{defined via an action on a space, followed by taking (some variety of) homology}}$ representations of $\mathcal{M}(\Sigma)$.

Why representations?

Understand $\mathcal{M}(\Sigma)$ via the "symmetries" of simpler objects,

e.g. · vector spaces

· R-modules

Why homological representations?

- The representation theory of $\mathcal{M}(\Sigma)$ is "wild"
  (no classification of irred. rep.'s with finitely many parameters)

- Constructing representations topologically / homologically allows us to use topological tools to study them...

When $\Sigma = \mathbb{D}_k = \mathbb{D}^2 \setminus \{p_1, \dots, p_k\}$, $\qquad \mathcal{M}(\mathbb{D}_k) \cong B_k$

In 2000, Bigelow & Krammer (independently) constructed finite-dimensional, faithful representations of B_k (over a field)

$$B_k \hookrightarrow GL_N(\mathbb{R})$$

$\Rightarrow$ "braid groups are linear"

For almost all other surfaces Σ, linearity of $\mathcal{M}(\Sigma)$ is an open question.

$$\Downarrow$$

More concrete & ambitious goal :

Construct faithful, fin.-dim. homological representations of $\mathcal{M}(\Sigma)$.

We will focus on $\Sigma = \Sigma_{g,1} =$  for $g \geq 1$.

For convenience suppose that $g \geq 4$.

Theorem (Blanchet - P. - Shaukat)

1. For each $n \geq 2$, there is a $\underbrace{\text{complex unitary representation}}$:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[r] \arrow[d] & U(\mathcal{V}_n(W)) \arrow[d] \\
\mathcal{M}(\Sigma) \arrow[r] & PU(\mathcal{V}_n(W))
\end{tikzcd}$$

universal central extension

projective representation

$$\mathcal{V}_n(W) = \underbrace{H_n^{BM}}_{\text{Borel-Moore homology}}\big(\underbrace{C_n(\Sigma)}_{\text{configuration space}}, C_n^{I}(\Sigma); \underbrace{W}_{\text{Schrödinger representation of the discrete Heisenberg group } \mathcal{H}(\Sigma)}\big)$$

Rmk $\mathcal{V}_n(W)$ is an infinite-dim. Hilbert space.

We also have a fin.-dim variant:

$$\widetilde{\mathcal{M}}(\Sigma) \longrightarrow U(\mathcal{V}_n(W_N))$$

$$\dim(\mathcal{V}_n(W_N)) = \binom{2g+n-1}{n} . N^g .$$

2. For each $n \geq 2$ and each representation V (over $\mathbb{R}$) of the discrete Heisenberg group $\mathcal{H}(\Sigma)$, there is a $\underbrace{\text{representation}}$

$$\mathrm{Torelli}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{V}_n(V))$$

where $\mathcal{V}_n(V) = H_n^{BM}(C_n(\Sigma), C_n^{I}(\Sigma); V)$.

Kernels

$\bigcap_n J(n) = \{1\}$

The Johnson filtration is an exhaustive filtration :

$$\mathcal{M}(\Sigma) \supseteq J(1) \supseteq J(2) \supseteq J(3) \supseteq \cdots$$

$$J(1) = \mathrm{Torelli}(\Sigma)$$

Def • $J(n) = \ker\left(\mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\left(\pi_1(\Sigma) / \Gamma_{n+1}(\pi_1(\Sigma))\right)\right)$

• $\mathrm{Magnus}(\Sigma) = \ker\left(\mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\left(\frac{\pi_1(\Sigma)}{\pi_1(\Sigma)^{(2)}}\right)\right)$

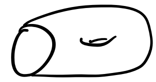
"Magnus kernel"

Proposition

Each of our representations (for given $n \geq 2$) has kernel contained in $J(n) \cap \mathrm{Magnus}(\Sigma)$.

(This uses [Moriyama '07], who identifies $J(n)$ with the kernel of a certain (simpler) homological representation of $\mathcal{M}(\Sigma)$, and [Suzuki '05], who does the same for $\mathrm{Magnus}(\Sigma)$.)

Calculations

$g = 1$ $\quad$ ($\Sigma =$ )

$n = 2$

$$\mathcal{H}(\Sigma) = \langle a, b, u \mid u \text{ central}, \ [a,b] = u^2 \rangle \cong \begin{pmatrix} 1 & \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} V &= \text{regular representation} \\ &= \mathbb{Z}[\mathcal{H}(\Sigma)] = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle \big/ (aba^{-1}b^{-1} - u^2) =: R \end{aligned}$$

In this case $\mathcal{V}_2(V) \cong R^3$ as an R-module, so

$$\mathrm{Torelli}(\Sigma_{1,1}) \longrightarrow GL_3(R)$$

and the Dehn twist around the boundary $T_{\partial\Sigma}$ acts by

$$\begin{pmatrix}
\begin{array}{c} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{array} &
\begin{array}{c} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{array} &
\begin{array}{c} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+ \\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+ \\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+ \\ (-u^{-3}+u^{-4})a^{-2} \end{array} \\
\begin{array}{c} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{array} &
\begin{array}{c} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{array} &
\begin{array}{c} (-u^{-6}+u^{-7})a^{-2}b+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+ \\ (-u^{-5}+u^{-6})a^{-2} \end{array} \\
\begin{array}{c} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{array} &
\begin{array}{c} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{array} &
\begin{array}{c} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+ \\ (-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{array}
\end{pmatrix}$$

Note (exercise!): This simplifies to the identity if we set $a = b = u^2 = 1$.

Plan

- General recipe for construction (1st version)

  ↳ Burau, Lawrence - Krammer - Bigelow rep.'s of B_K.

- General recipe (2nd version; more general) $\rightsquigarrow$ twisted representations.

  Idea: · apply this recipe
  · use $\underbrace{\text{further structure/properties}}_{\text{Stone - von Neumann theorem}}$ to "untwist" it.

General recipe

$$\begin{array}{c} \pi_1 C_n(\Sigma) \\ \Big\downarrow\!\!\downarrow \phi \\ Q \end{array}$$

$\mathcal{M}(\Sigma)$ – invariant quotient

$$\begin{array}{c} \widetilde{C_n(\Sigma)}^{\phi} \\ \big\downarrow \\ C_n(\Sigma) \end{array}$$

regular covering with Deck $= Q$ (principal Q - bundle)

$\mathcal{M}(\Sigma)$ - action lifts and commutes with the Q - action

Q - representation V (over R)

$$\begin{array}{c} \mathbb{Z}\left[\widetilde{C_n(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[Q]} V \\ \big\downarrow \\ C_n(\Sigma) \end{array}$$

bundle of R-modules (local system over R)

$\mathcal{M}(\Sigma)$ acts on $C_n(\Sigma)$ equipped with this local system

twisted homology

$H_i(C_n(\Sigma); V)$

$\mathcal{M}(\Sigma)$ - representation over R

twisted relative Borel-Moore homology

$S \subseteq \partial\Sigma$

$C_n^S(\Sigma) := \{ c \in C_n(\Sigma) \mid c \cap S \neq \emptyset \}$

<u>NB:</u> This is preserved (setwise) by $\mathcal{M}(\Sigma)$.

$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); V)$

$\mathcal{M}(\Sigma)$ - representation over R

When $\Sigma = \mathbb{D}_k$

n=1

$$\phi : \pi_1(\mathbb{D}_k) = F_k \xrightarrow{ab} \mathbb{Z}^k \xrightarrow{+} \mathbb{Z} = Q$$

$$\underbrace{F_k \twoheadrightarrow \mathbb{Z}}_{\text{total winding number}}$$

$$V = \mathbb{Z}[Q] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$$

(regular representation of Q)

$S = I$, interval $\subseteq \partial \mathbb{D}_k$

$H_1^{BM}(\mathbb{D}_k, I; \mathbb{Z}[t^{\pm 1}])$ $\quad$ B_k - representation over $\mathbb{Z}[t^{\pm 1}]$

the **Burau representation**

(isomorphic as a $\mathbb{Z}[t^{\pm 1}]$ - module to $\mathbb{Z}[t^{\pm 1}]^k$)

n ⩾ 2

$$\phi : \pi_1 C_n(\mathbb{D}_k) \twoheadrightarrow \mathbb{Z}^2 = Q$$

(total winding number, writhe)

$$V = \mathbb{Z}[Q] = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$$

$S = I$, interval $\subseteq \partial \mathbb{D}_k$

$H_n^{BM}(C_n(\mathbb{D}_k), C_n^I(\mathbb{D}_k); \mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$ $\quad$ B_k - representation over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$

the n^{th} **Lawrence - Bigelow representation**

(isomorphic as a $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ - module to $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]^{\binom{n+k-1}{n}}$)

Theorem (Bigelow, Krammer)

Faithful B_k - representation when $n = 2$.

In general,

$$\pi_1(C_n(\Sigma))^{ab} \cong \begin{cases} \underbrace{H_1(\Sigma)}_{\text{winding numbers of particles around loops on the surface}} \times \underbrace{\mathbb{Z}}_{\text{writhe}} & \Sigma \text{ planar} \\ H_1(\Sigma) \times \mathbb{Z}/(2n-2) & \Sigma = S^2 \\ H_1(\Sigma) \times \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

Thus any abelian quotient Q of $\pi_1(C_n(\Sigma_{g,1}))$ $(g \geq 1)$ can only remember the writhe mod 2.

$\Rightarrow$ Look for non-abelian quotients of $\pi_1(C_n(\Sigma_{g,1}))$.

Lemma: For $n \geq 2$ there is a natural quotient

$$\phi : \pi_1(C_n(\Sigma_{g,1})) \twoheadrightarrow \underbrace{\mathcal{H}(\Sigma_{g,1})}_{\text{discrete Heisenberg group}}$$

$\mathbb{Z} \times H_1(\Sigma)$

$(a,x)(b,y) = (a+b+x.y,\ x+y)$

Moreover, $\ker(\phi)$ is $\mathcal{M}(\Sigma)$-invariant.

But ϕ itself is not.

Idea of proof

General recipe II

differences from first version highlighted

$\pi_1 C_n(\Sigma)$

$\downarrow \phi$ (surjective)

Q

$\ker(\phi)$ fixed (setwise) by $\mathcal{M}(\Sigma)$

$\downarrow$

$\widetilde{C_n(\Sigma)}^{\phi}$

$\downarrow$

$C_n(\Sigma)$

regular covering with Deck $= Q$ (principal Q-bundle)

$\mathcal{M}(\Sigma)$-action lifts to the covering

(but does not commute with the Q-action)

Q-representation V (over R)

$\downarrow$

$\mathbb{Z}\left[\widetilde{C_n(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[Q]} V$

$\downarrow$

$C_n(\Sigma)$

bundle of R-modules

(local system over R)

$\mathcal{M}(\Sigma)$ acts on $C_n(\Sigma)$, but does not preserve the local system

twisted homology

$H_i(C_n(\Sigma); V)$

twisted $\mathcal{M}(\Sigma)$-representation over R

$S \subseteq \partial\Sigma$

$C_n^S(\Sigma) := \{ c \in C_n(\Sigma) \mid c \cap S \neq \emptyset \}$

NB: This is preserved (setwise) by $\mathcal{M}(\Sigma)$.

twisted relative Borel-Moore homology

$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); V)$

twisted $\mathcal{M}(\Sigma)$-representation over R

Q: What does "twisted $\mathcal{M}(\Sigma)$-representation over R" mean?

A: An action on the collection of R-modules $\left\{ H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \varphi^* V) \mid \varphi \in \mathcal{M}(\Sigma) \right\}$

For each $\varphi, \psi \in \mathcal{M}(\Sigma)$ we have an isomorphism (not automorphism)

$$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \psi^* \varphi^* V) \longrightarrow H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \varphi^* V)$$

Formally, this is a Mod_R-valued functor on the translation groupoid of $\mathcal{M}(\Sigma)$.

or action groupoid

Set ϕ to be the quotient $\pi_1(C_n(\Sigma_{g,1})) \twoheadrightarrow \mathcal{H}(\Sigma_{g,1}) = Q$

and $S = I$ interval $\subseteq \partial \Sigma_{g,1}$

Then we obtain a twisted $\mathcal{M}(\Sigma)$-representation $H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V)$

for each $n \geq 2$ and $\mathcal{H}(\Sigma)$-representation V.

Moreover, we understand $H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V)$ completely as a module:

Proposition:

$$\boxed{H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V) \cong \bigoplus_{\binom{n+2g-1}{n}} V \quad \text{as } R\text{-modules}}$$

explicit generating set

Final step:

How to "untwist" the $\underbrace{\mathcal{M}(\Sigma)\text{-action}}$ when $V = W =$ Schrödinger representation??

(after passing to $\widetilde{\mathcal{M}}(\Sigma)$)

The Stone-von Neumann theorem

$$\underline{\text{discrete}}\ \text{Heisenberg group} = \mathcal{H}(\Sigma) = \underset{b_1,\dots,b_g}{\mathbb{Z}^g} \ltimes \underset{a_1,\dots,a_g,\,u}{\mathbb{Z}^{g+1}} \qquad b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j \quad (j \neq i) \end{cases}$$

$$\big\downarrow$$

$$\underline{\text{continuous}}\ \text{Heisenberg group} = \mathcal{H}(\Sigma)_{\mathbb{R}} = \underset{b_1,\dots,b_g}{\mathbb{R}^g} \ltimes \underset{a_1,\dots,a_g,\,u}{\mathbb{R}^{g+1}} \qquad b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j \quad (j \neq i) \end{cases}$$

Definition

Fix a real number $\hbar > 0$ ("Planck's constant")

$$\begin{tikzcd}
\mathbb{R}^{g+1} \arrow[r, two heads, "pr_{g+1}"] \arrow[d, hook] & \mathbb{R} \arrow[r, "t \mapsto e^{\hbar i t/2}"'] & S^1 = U(1) \\
\mathbb{R}^g \ltimes \mathbb{R}^{g+1} \arrow[rr, "\rho_W"] & & U(\overbrace{L^2(\mathbb{R}^g)}^{W})
\end{tikzcd}$$

Schrödinger representation

Explicit formula:

$$\rho_W\Big(\sum_{i=1}^{g} q_i b_i + \sum_{i=1}^{g} p_i a_i + t u\Big)(\psi)(s) = e^{i\hbar\left(p\cdot s + \frac{1}{2}(t - p\cdot q)\right)}\psi(s-q)$$

$p, q, s \in \mathbb{R}^g$
$t \in \mathbb{R}$
$\psi \in L^2(\mathbb{R}^g) = W$

Theorem (Stone-von Neumann)

$\forall\, \varphi \in \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}})$

$\exists$ unique inner automorphism $T(\varphi)$ of $U(W)$ such that

$$\begin{tikzcd}
\mathcal{H}(\Sigma)_{\mathbb{R}} \arrow[r, "\rho_W"] \arrow[d, "\varphi"'] & U(W) \arrow[d, "T(\varphi)"] \\
\mathcal{H}(\Sigma)_{\mathbb{R}} \arrow[r, "\rho_W"] & U(W)
\end{tikzcd}$$

commutes.

$$\rightsquigarrow\ T : \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}}) \longrightarrow \mathrm{Inn}(U(W)) = PU(W)$$

This is the **Segal-Shale-Weil projective representation** of $\mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}})$

Lift to a unitary representation of $\widetilde{\mathcal{M}}(\Sigma)$:

$\mathbb{Z}$-central extension (universal)

S^1-central extension

S^1-central extension

$$
\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[r] \arrow[dr] & & \overline{\mathcal{M}}(\Sigma) \arrow[rrr] \arrow[dl] \arrow[drr, phantom, "\lrcorner", very near start] & & & U(W) \arrow[d] \\
& \mathcal{M}(\Sigma) \arrow[rr] & & \mathrm{Aut}(\mathcal{H}(\Sigma)) \subseteq \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}}) \arrow[rr, "T"] & & PU(W)
\end{tikzcd}
$$

induced action on $\mathcal{H}(\Sigma)$ via the quotient $\phi : \pi_1 C_n(\Sigma) \twoheadrightarrow \mathcal{H}(\Sigma)$

<u>Notation</u>: $\beta : \widetilde{\mathcal{M}}(\Sigma) \longrightarrow U(W)$

<u>Recall</u>:

In the general construction, we have a bundle of Hilbert spaces

$$
\begin{array}{c}
\mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}] \underset{\mathbb{Z}[\mathcal{H}(\Sigma)]}{\otimes} W \\
\downarrow \\
C_n(\Sigma)
\end{array}
\qquad (*)
$$

but the action of $\mathcal{M}(\Sigma)$ on $C_n(\Sigma)$ does <u>not</u> preserve this local system.

Write $\alpha : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\left(\mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}]\right)$ for the lifted action of $\mathcal{M}(\Sigma)$

(which does <u>not</u> commute with the $\mathbb{Z}[\mathcal{H}(\Sigma)]$-module structure)

Key theorem:

We may use β to untwist α.

Precisely: the $\mathcal{M}(\Sigma)$-action on $C_n(\Sigma)$ may be lifted to a well-defined $\widetilde{\mathcal{M}}(\Sigma)$-action on $(*)$ by bundle automorphisms via the formula:

$$\varphi(v \otimes w) = \alpha\big(\pi(\varphi)\big)(v) \otimes \beta(\varphi)(w)$$

$$\begin{array}{c} \widetilde{\mathcal{M}}(\Sigma) \\ \downarrow{\scriptstyle \pi} \\ \mathcal{M}(\Sigma) \end{array}$$

$\varphi \in \widetilde{\mathcal{M}}(\Sigma)$

$v \in \mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}]$

$w \in W$

Corollary ($=$ main theorem)

After taking twisted BM-homology, we have an untwisted (unitary) representation of $\widetilde{\mathcal{M}}(\Sigma)$.

Bonus:

- **Lemma:** The universal central extension $\begin{array}{c}\widetilde{\mathcal{M}}(\Sigma)\\ \downarrow \\ \mathcal{M}(\Sigma)\end{array}$ is trivial when restricted to the Torelli group.

  Idea of proof:

  - One may compute that the inclusion $\mathrm{Torelli}(\Sigma) \hookrightarrow \mathcal{M}(\Sigma)$ induces the 0 map on $H^2(-;\mathbb{Z})$.
  - Thus any $\mathbb{Z}$-central extension of $\mathcal{M}(\Sigma)$ becomes trivial when restricted to $\mathrm{Torelli}(\Sigma)$.

- Hence we obtain an untwisted representation of $\mathrm{Torelli}(\Sigma)$ on $H_n^{BM}(C_n(\Sigma), C_n^{\mathcal{I}}(\Sigma); W)$ without passing to a central extension.

- In fact, using different methods, we obtain an untwisted representation of $\mathrm{Torelli}(\Sigma)$ on $H_n^{BM}(C_n(\Sigma), C_n^{\mathcal{I}}(\Sigma); V)$ for any $\mathcal{H}(\Sigma)$-representation V, not just $V = W$.