Mapping class group representations via Heisenberg, Schrödinger and Stone-von Neumann

Based on joint work with
- Christian Blanchet (Paris)
- Awais Shaukat (Lahore)

Fudan topology seminar
10 March 2022

arXiv: 2109.00515

Background / motivation / goal

Σ = surface

$\mathcal{M}(\Sigma)$ = its mapping class group

$= \pi_0(\mathrm{Diff}_\partial(\Sigma))$

$= \{$isotopy classes of self-diffeomorphisms of Σ that fix $\partial\Sigma$ pointwise$\}$

Goal: Construct "interesting" homological representations of $\mathcal{M}(\Sigma)$.

defined via an action on a space, followed by taking (some variety of) homology

Why representations?

Understand $\mathcal{M}(\Sigma)$ via the "symmetries" of simpler objects,

e.g.
- vector spaces
- R-modules

"non-complicated" ring R

e.g. $R = \mathbb{Z}[\mathbb{Z}^k]$ (Laurent polynomials)

$R = \mathbb{Z}[G]$ (G nilpotent group)

(NOT $R = \mathbb{Z}[\mathcal{M}(\Sigma)]$)

Why homological representations?

- The representation theory of $\mathcal{M}(\Sigma)$ is "wild"
  (no classification of irred. rep.'s with finitely many parameters)
- Constructing representations topologically / homologically allows us to use topological tools to study them...

When $\Sigma = \mathbb{D}_k = \mathbb{D}^2 \setminus \{p_1, \dots, p_k\}$, $\quad \mathcal{M}(\mathbb{D}_k) \cong \underset{\text{Artin's braid group}}{B_k}$

In 2000, Bigelow & Krammer (independently) constructed finite-dimensional, faithful representations of B_k (over a field)

$$B_k \hookrightarrow GL_N(\mathbb{R}) \qquad [N \text{ depending on } k]$$

$\Rightarrow$ "braid groups are linear"

For almost all other surfaces Σ, $\underbrace{\text{linearity of } \mathcal{M}(\Sigma)}_{\text{i.e. whether } \exists \text{ fin. dim. faithful repr. of } \mathcal{M}(\Sigma) \text{ over a field}}$ is an open question.

More concrete & ambitious goal:

Construct faithful, fin.-dim. homological representations of $\mathcal{M}(\Sigma)$.

We will focus on $\Sigma = \Sigma_{g,1} =$  for $g \geq 1$.

For convenience suppose that $g \geq 4$. (But everything works for $g \in \{1,2,3\}$ too.)

Theorem (Blanchet - P. - Shaukat)

① For each $n \geq 2$, there is a complex unitary representation:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[r] \arrow[d, "\text{universal central extension}"'] & U(\mathcal{V}_n(W)) \arrow[d] \\
\mathcal{M}(\Sigma) \arrow[r, "\text{projective representation}"'] & PU(\mathcal{V}_n(W))
\end{tikzcd}$$

$$\mathcal{V}_n(W) = \overbrace{H}^{\text{Borel-Moore homology}}{}^{BM}_n\big(\underbrace{C_n(\Sigma)}_{\text{configuration space}}, C_n^{I}(\Sigma); \underbrace{W}_{\text{Schrödinger representation of the discrete Heisenberg group } \mathcal{H}(\Sigma)}\big)$$

Rmk $\mathcal{V}_n(W)$ is an infinite-dim. Hilbert space, essentially because W is infinite-dim. But there is also a variant of our construction using a finite-dim analogue W_N of W. ($N \geq 2$ even integer)
In this case we obtain a unitary representation

$$\widetilde{\mathcal{M}}(\Sigma) \longrightarrow U(\mathcal{V}_n(W_N))$$

on a complex Hilbert space $\mathcal{V}_n(W_N)$ of dimension $\binom{2g+n-1}{n} . N^g$.

② For each $n \geq 2$ and each representation V (over $\mathbb{R}$) of the discrete Heisenberg group $\mathcal{H}(\Sigma)$, there is a representation

$$\mathrm{Torelli}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{V}_n(V))$$

Torelli group = kernel of $\mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H_1(\Sigma;\mathbb{Z}))$

where $\mathcal{V}_n(V) = H_n^{BM}(C_n(\Sigma), C_n^{I}(\Sigma); V)$.

Universal central extension

- An extension $1 \to K \to H \to G \to 1$ of G is central if $K \subseteq \mathcal{Z}(H)$.
- The universal central extension $\widetilde{G} \to G$ is the initial object in the category of central extensions of G, i.e. for any central extension $H \to G$, $\exists$ unique
$$\begin{tikzcd} \widetilde{G} \arrow[rr, dashed] \arrow[dr] & & H \arrow[dl] \\ & G & \end{tikzcd}$$
.
- It exists $\iff$ $H_1(G;\mathbb{Z}) = 0$ and the kernel of $\widetilde{G} \to G$ is $H_2(G;\mathbb{Z})$.
- $$\left.\begin{aligned} H_1(\mathcal{M}(\Sigma_{g,1}); \mathbb{Z}) &= 0 \\ H_2(\mathcal{M}(\Sigma_{g,1}); \mathbb{Z}) &\cong \mathbb{Z} \end{aligned}\right\} \text{ for } g \geqslant 4$$

  So the universal central extension is of the form
$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathcal{M}}(\Sigma) \longrightarrow \mathcal{M}(\Sigma) \longrightarrow 1$$

Discrete Heisenberg group

- Central extension $1 \to \mathbb{Z} \longrightarrow \mathcal{H}(\Sigma) \longrightarrow H_1(\Sigma;\mathbb{Z}) \to 1$
- Concretely $\mathcal{H}(\Sigma) = \mathbb{Z} \times H_1(\Sigma;\mathbb{Z})$ as a set, with operation
$$(a,x)(b,y) = (a+b+x\cdot y,\ x+y)$$
  where $\cdot : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{Z}$ is the intersection cocycle.

Schrödinger representation of $\mathcal{H}(\Sigma)$

- $W = L^2(\mathbb{R}^g)$
- $W_N = L^2((\mathbb{Z}/N)^g) \cong \mathbb{R}^{N^g}$

(more details later...)

Kernels

$\bigcap_n J(n) = \{1\}$

The Johnson filtration is an exhaustive filtration :

$$\mathcal{M}(\Sigma) \supseteq \underset{\overset{\|}{\mathrm{Torelli}(\Sigma)}}{J(1)} \supseteq J(2) \supseteq J(3) \supseteq \cdots\cdots$$

Def • $J(n) = \ker\left(\mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\Big(\underbrace{\pi_1(\Sigma)/\Gamma_{n+1}(\pi_1(\Sigma))}_{}\Big)\right)$

the largest n-nilpotent quotient of $\pi_1(\Sigma) \cong F_{2g}$

• $\mathrm{Magnus}(\Sigma) = \ker\left(\mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\Big(\underbrace{\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}}_{}\Big)\right)$

"Magnus kernel"

the largest 2-solvable ("metabelian") quotient of $\pi_1(\Sigma)$

Proposition

Each of our representations (for given $n \geq 2$) has kernel contained in $J(n) \cap \mathrm{Magnus}(\Sigma)$.

(This uses [Moriyama '07], who identifies $J(n)$ with the kernel of a certain (simpler) homological representation of $\mathcal{M}(\Sigma)$, and [Suzuki '05], who does the same for $\mathrm{Magnus}(\Sigma)$.)

Calculations

$g = 1$ $\quad\Big(\Sigma =$  $\Big)$

$n = 2$

$$\mathcal{H}(\Sigma) = \langle a, b, u \mid u \text{ central}, [a,b] = u^2 \rangle \cong \begin{pmatrix} 1 & \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

V = regular representation

$$= \mathbb{Z}[\mathcal{H}(\Sigma)] = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle \big/ (aba^{-1}b^{-1} - u^2) =: R$$

In this case $\mathcal{V}_2(V) \cong R^3$ as an R-module, so

$$\mathrm{Torelli}(\Sigma_{1,1}) \longrightarrow GL_3(R)$$

and the Dehn twist around the boundary $T_{\partial\Sigma}$ acts by

$$\begin{pmatrix}
\begin{array}{c} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{array}
&
\begin{array}{c} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{array}
&
\begin{array}{c} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+ \\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+ \\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+ \\ (-u^{-3}+u^{-4})a^{-2} \end{array}
\\
\begin{array}{c} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{array}
&
\begin{array}{c} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{array}
&
\begin{array}{c} (-u^{-6}+u^{-7})a^{-2}b+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+ \\ (-u^{-5}+u^{-6})a^{-2} \end{array}
\\
\begin{array}{c} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{array}
&
\begin{array}{c} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{array}
&
\begin{array}{c} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+ \\ (-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{array}
\end{pmatrix}$$

Note (exercise!): This simplifies to the identity if we set $a = b = u^2 = 1$.

Plan

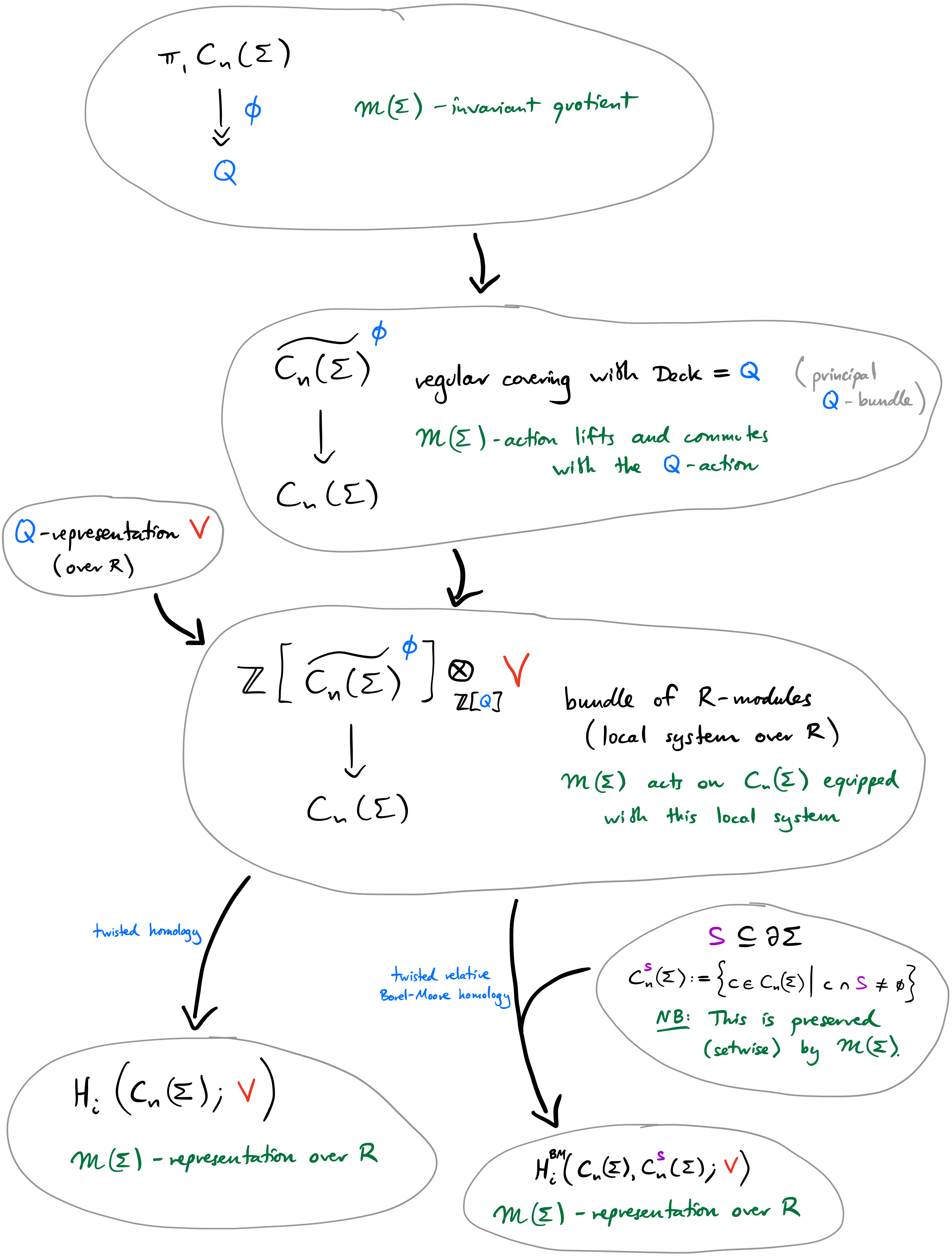
- General recipe for construction (1st version)

  ↳ Burau, Lawrence-Krammer-Bigelow rep.'s of B_k.

- General recipe (2nd version; more general) ⇝ *twisted* representations.

Idea:
- apply this recipe
- use $\underbrace{\text{further structure/properties}}_{\text{Stone-von Neumann theorem}}$ to "untwist" it.

General recipe



$$\begin{tikzcd}\pi_1 C_n(\Sigma) \arrow[d, two heads, "\phi"] \\ Q\end{tikzcd}$$

$\mathcal{M}(\Sigma)$ - invariant quotient

$$\begin{tikzcd}\widetilde{C_n(\Sigma)}^{\phi} \arrow[d] \\ C_n(\Sigma)\end{tikzcd}$$

regular covering with Deck $= Q$ (principal Q - bundle)

$\mathcal{M}(\Sigma)$ - action lifts and commutes with the Q - action

Q - representation V (over R)

$$\begin{tikzcd}\mathbb{Z}\left[\widetilde{C_n(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[Q]} V \arrow[d] \\ C_n(\Sigma)\end{tikzcd}$$

bundle of R - modules (local system over R)

$\mathcal{M}(\Sigma)$ acts on $C_n(\Sigma)$ equipped with this local system

twisted homology

$H_i\left(C_n(\Sigma); V\right)$

$\mathcal{M}(\Sigma)$ - representation over R

twisted relative Borel-Moore homology

$S \subseteq \partial\Sigma$

$C_n^S(\Sigma) := \{c \in C_n(\Sigma) \mid c \cap S \neq \emptyset\}$

NB: This is preserved (setwise) by $\mathcal{M}(\Sigma)$.

$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); V)$

$\mathcal{M}(\Sigma)$ - representation over R

When $\Sigma = \mathbb{D}_k$

n = 1

$$\begin{tikzcd} \phi : \pi_1(\mathbb{D}_k) = F_k \arrow[r, two heads, "ab"] \arrow[rr, two heads, bend right, "\text{total winding number}"'] & \mathbb{Z}^k \arrow[r, two heads, "+"] & \mathbb{Z} = Q \end{tikzcd}$$

$V = \mathbb{Z}[Q] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$

(regular representation of Q)

$S = I$, interval $\subseteq \partial \mathbb{D}_k$

$H_1^{BM}(\mathbb{D}_k, I; \mathbb{Z}[t^{\pm1}])$ B_k - representation over $\mathbb{Z}[t^{\pm1}]$

$\uparrow$ the **Burau representation**

(isomorphic as a $\mathbb{Z}[t^{\pm1}]$ - module to $\mathbb{Z}[t^{\pm1}]^k$)

n ⩾ 2

$$\begin{tikzcd} \phi : \pi_1 C_n(\mathbb{D}_k) \arrow[r, two heads, "{(\text{total winding number, writhe})}"'] & \mathbb{Z}^2 = Q \end{tikzcd}$$

$V = \mathbb{Z}[Q] = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[t^{\pm1}, q^{\pm1}]$

$S = I$, interval $\subseteq \partial \mathbb{D}_k$

$H_n^{BM}(C_n(\mathbb{D}_k), C_n^I(\mathbb{D}_k); \mathbb{Z}[t^{\pm1}, q^{\pm1}])$ B_k - representation over $\mathbb{Z}[t^{\pm1}, q^{\pm1}]$

$\uparrow$ the n^{th} **Lawrence - Bigelow representation**

(isomorphic as a $\mathbb{Z}[t^{\pm1}, q^{\pm1}]$ - module to $\mathbb{Z}[t^{\pm1}, q^{\pm1}]^{\binom{n+k-1}{n}}$)

Theorem (Bigelow, Krammer)

Faithful B_k - representation when $n = 2$.

In general,

$$\pi_1(C_n(\Sigma))^{ab} \cong \begin{cases} H_1(\Sigma) \times \mathbb{Z} & \Sigma \text{ planar} \\ H_1(\Sigma) \times \mathbb{Z}/(2n-2) & \Sigma = S^2 \\ H_1(\Sigma) \times \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

winding numbers of particles around loops on the surface

writhe

Thus any abelian quotient Q of $\pi_1(C_n(\Sigma_{g,1}))$ $(g \geq 1)$ can only remember the writhe mod 2.

$\Rightarrow$ Look for non-abelian quotients of $\pi_1(C_n(\Sigma_{g,1}))$.

Lemma: For $n \geq 2$ there is a natural quotient

$$\phi : \pi_1(C_n(\Sigma_{g,1})) \twoheadrightarrow \underbrace{\mathcal{H}(\Sigma_{g,1})}$$

discrete Heisenberg group

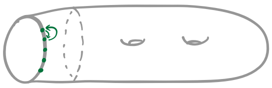
$\mathbb{Z} \times H_1(\Sigma)$

$(a,x)(b,y) = (a+b+x.y\,,\, x+y)$

Moreover, $\ker(\phi)$ is $\mathcal{M}(\Sigma)$-invariant.

But ϕ itself is not.

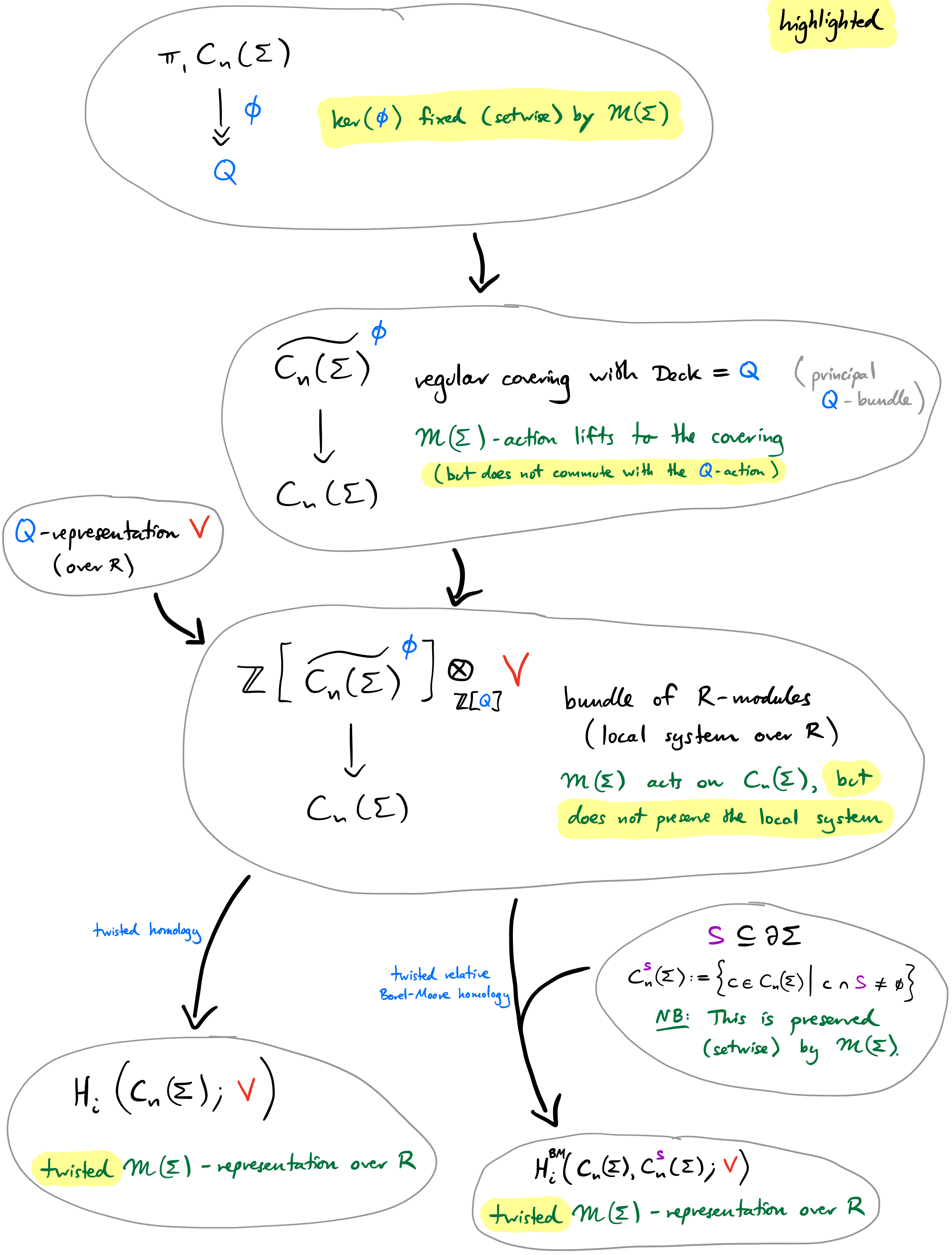
Idea of proof:

- $\sigma_1 =$  $N = \langle\langle [\sigma_1, x] \rangle\rangle$
- $\pi_1(C_n(\Sigma_{g,1}))/N \cong \mathcal{H}(\Sigma)$
- σ_1 fixed by $\mathcal{M}(\Sigma)$

$\Rightarrow N = \ker(\phi)$ is fixed (setwise) by $\mathcal{M}(\Sigma)$.

General recipe II

differences from
first version
highlighted



$$\begin{tikzcd}\pi_1 C_n(\Sigma) \arrow[d, two heads, "\phi"] \\ Q\end{tikzcd}$$

ker(ϕ) fixed (setwise) by $\mathcal{M}(\Sigma)$

$$\begin{tikzcd}\widetilde{C_n(\Sigma)}^{\phi} \arrow[d] \\ C_n(\Sigma)\end{tikzcd}$$

regular covering with Deck = Q (principal Q-bundle)

$\mathcal{M}(\Sigma)$-action lifts to the covering

(but does not commute with the Q-action)

Q-representation V
(over R)

$$\begin{tikzcd}\mathbb{Z}\left[\widetilde{C_n(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[Q]} V \arrow[d] \\ C_n(\Sigma)\end{tikzcd}$$

bundle of R-modules
(local system over R)

$\mathcal{M}(\Sigma)$ acts on $C_n(\Sigma)$, but
does not preserve the local system

twisted homology

twisted relative
Borel-Moore homology

$S \subseteq \partial\Sigma$

$C_n^S(\Sigma) := \{c \in C_n(\Sigma) \mid c \cap S \neq \emptyset\}$

NB: This is preserved
(setwise) by $\mathcal{M}(\Sigma)$.

$H_i(C_n(\Sigma); V)$

twisted $\mathcal{M}(\Sigma)$-representation over R

$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); V)$

twisted $\mathcal{M}(\Sigma)$-representation over R

Q: What does "twisted $\mathcal{M}(\Sigma)$-representation over R" mean?

A: An action on the collection of R-modules $\left\{ H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \varphi^* V) \mid \varphi \in \mathcal{M}(\Sigma) \right\}$

For each $\varphi, \psi \in \mathcal{M}(\Sigma)$ we have an isomorphism (not automorphism)

$$H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \psi^* \varphi^* V) \longrightarrow H_i^{BM}(C_n(\Sigma), C_n^S(\Sigma); \varphi^* V)$$

Formally, this is a Mod_R-valued functor on the translation groupoid of $\mathcal{M}(\Sigma)$.

Set ϕ to be the quotient $\pi_1(C_n(\Sigma_{g,1})) \twoheadrightarrow \mathcal{H}(\Sigma_{g,1}) = Q$

and $S = I$ interval $\subseteq \partial \Sigma_{g,1}$

Then we obtain a twisted $\mathcal{M}(\Sigma)$-representation $H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V)$

for each $n \geq 2$ and $\mathcal{H}(\Sigma)$-representation V.

Moreover, we understand $H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V)$ completely as a module:

Proposition:

$$\boxed{H_n^{BM}(C_n(\Sigma), C_n^I(\Sigma); V) \cong \bigoplus_{\binom{n+2g-1}{n}} V \qquad \text{as } R\text{-modules}}$$

Final step:

How to "untwist" the $\underbrace{\mathcal{M}(\Sigma)\text{-action}}_{\text{(after passing to } \widetilde{\mathcal{M}}(\Sigma)\text{)}}$ when $V = W =$ Schrödinger representation??

The Stone-von Neumann theorem

$$\begin{tikzcd}
\text{discrete Heisenberg group} = \mathcal{H}(\Sigma) = \underset{b_1,\dots,b_g}{\mathbb{Z}^g} \ltimes \underset{a_1,\dots,a_g,\,u}{\mathbb{Z}^{g+1}} \arrow[d, hook] \\
\text{continuous Heisenberg group} = \mathcal{H}(\Sigma)_{\mathbb{R}} = \underset{b_1,\dots,b_g}{\mathbb{R}^g} \ltimes \underset{a_1,\dots,a_g,\,u}{\mathbb{R}^{g+1}}
\end{tikzcd}$$

$$b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j & (j \neq i) \end{cases} \qquad\qquad b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j & (j \neq i) \end{cases}$$

Definition

Fix a real number $\hbar > 0$ ("Planck's constant")

$$\begin{tikzcd}
\mathbb{R}^{g+1} \arrow[r, two heads, "pr_{g+1}"] \arrow[d, hook] & \mathbb{R} \arrow[r, "t \mapsto e^{\hbar i t/2}"'] & S^1 = U(1) \\
\mathbb{R}^g \ltimes \mathbb{R}^{g+1} \arrow[rr, "\rho_W"] & & U(\overbrace{L^2(\mathbb{R}^g)}^{W}) \arrow[ll, bend left, leftrightarrow]
\end{tikzcd}$$

(1-dim. unitary representation)

(induced representation)

This is the <u>Schrödinger representation</u> of $\mathcal{H}(\Sigma)_{\mathbb{R}}$

Explicit formula :

$$\rho_W\left(\sum_{i=1}^g q_i b_i + \sum_{i=1}^g p_i a_i + tu\right)(\psi)(s) = e^{i\hbar(p\cdot s + \frac{1}{2}(t - p\cdot q))}\psi(s-q)$$

$p, q, s \in \mathbb{R}^g$

$t \in \mathbb{R}$

$\psi \in L^2(\mathbb{R}^g) = W$

Theorem (Stone-von Neumann)

$\forall\, \varphi \in \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}})$

$\exists$ unique inner automorphism $T(\varphi)$ of $U(W)$ such that

$$\begin{tikzcd}
\mathcal{H}(\Sigma)_{\mathbb{R}} \arrow[r, "\rho_W"] \arrow[d, "\varphi"'] & U(W) \arrow[d, "T(\varphi)"] \\
\mathcal{H}(\Sigma)_{\mathbb{R}} \arrow[r, "\rho_W"] & U(W)
\end{tikzcd}$$

commutes.

$$\rightsquigarrow\ T : \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}}) \longrightarrow \mathrm{Inn}(U(W))$$
$$= PU(W)$$

projective unitary group

This is the <u>Segal-Shale-Weil projective representation</u> of $\mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}})$

Lift to a unitary representation of $\widetilde{\mathcal{M}}(\Sigma)$:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[r] \arrow[dr] & \overline{\mathcal{M}}(\Sigma) \arrow[d] \arrow[rrr, "\text{pullback}"'] & & & U(W) \arrow[d] \\
& \mathcal{M}(\Sigma) \arrow[r] & \mathrm{Aut}(\mathcal{H}(\Sigma)) \arrow[r, phantom, "\subseteq"] & \mathrm{Aut}(\mathcal{H}(\Sigma)_{\mathbb{R}}) \arrow[r, "T"] & PU(W)
\end{tikzcd}$$

$\mathbb{Z}$-central extension (universal)

S^1-central extension

S^1-central extension

induced action on $\mathcal{H}(\Sigma)$ via the quotient $\phi : \pi_1 C_n(\Sigma) \twoheadrightarrow \mathcal{H}(\Sigma)$

Notation: $\beta : \widetilde{\mathcal{M}}(\Sigma) \longrightarrow U(W)$

Recall:

In the general construction, we have a bundle of Hilbert spaces

$$\begin{tikzcd}
\mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}] \underset{\mathbb{Z}[\mathcal{H}(\Sigma)]}{\otimes} W \arrow[d] \\
C_n(\Sigma)
\end{tikzcd} \qquad (*)$$

but the action of $\mathcal{M}(\Sigma)$ on $C_n(\Sigma)$ does not preserve this local system.

Write $\alpha : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}\left(\mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}]\right)$ for the lifted action of $\mathcal{M}(\Sigma)$

(which does not commute with the $\mathbb{Z}[\mathcal{H}(\Sigma)]$-module structure)

Key theorem:

We may use β to untwist α.

Precisely: the $\mathcal{M}(\Sigma)$-action on $C_n(\Sigma)$ may be lifted to a well-defined $\widetilde{\mathcal{M}}(\Sigma)$-action on $(*)$ by bundle automorphisms via the formula:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[d, "\pi"] \\
\mathcal{M}(\Sigma)
\end{tikzcd}$$

$\varphi \in \widetilde{\mathcal{M}}(\Sigma)$
$v \in \mathbb{Z}[\widetilde{C_n(\Sigma)}^{\phi}]$
$w \in W$

$$\varphi(v \otimes w) = \alpha(\pi(\varphi))(v) \otimes \beta(\varphi)(w)$$

Corollary (= main theorem)

After taking twisted BM-homology, we have an untwisted (unitary) representation of $\widetilde{\mathcal{M}}(\Sigma)$.

Bonus :

- **Lemma**: The universal central extension

$$\begin{tikzcd} \widetilde{\mathcal{M}}(\Sigma) \arrow[d] \\ \mathcal{M}(\Sigma) \end{tikzcd}$$

is trivial when restricted to the Torelli group.

Idea of proof:

- One may compute that the inclusion $\mathrm{Torelli}(\Sigma) \hookrightarrow \mathcal{M}(\Sigma)$ induces the 0 map on $H^2(-;\mathbb{Z})$.
- Thus any $\mathbb{Z}$-central extension of $\mathcal{M}(\Sigma)$ becomes trivial when restricted to $\mathrm{Torelli}(\Sigma)$.

- Hence we obtain an untwisted representation of $\mathrm{Torelli}(\Sigma)$ on $H_n^{BM}(C_n(\Sigma), C_n^{\mp}(\Sigma); W)$ without passing to a central extension.

- In fact, using different methods, we obtain an untwisted representation of $\mathrm{Torelli}(\Sigma)$ on $H_n^{BM}(C_n(\Sigma), C_n^{\mp}(\Sigma); V)$ for any $\mathcal{H}(\Sigma)$-representation V, not just $V = W$.

Thank you for your attention!