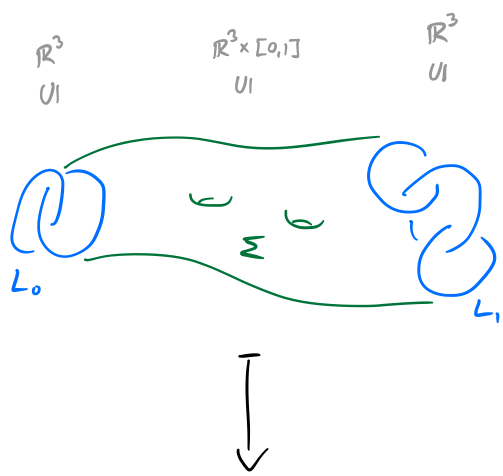
Spectrification of Khovanov homology III

GeMAT seminar, IMAR
8 April 2022

Extension to tangles & cobordisms

Aim:



$$X_{Kh}(L_0) \xrightarrow[X_{Kh}(\Sigma)]{} X_{Kh}(L_1)$$

(up to $\pm$)

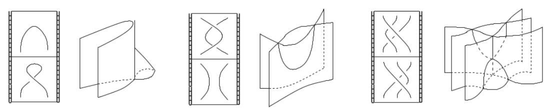
Plan :

- How this is done at the level of $Kh(-)$ ↳ incl. extension to tangles } [Jacobsson, Khovanov, Bar-Natan]
- Why the argument does not lift directly to $X_{Kh}(-)$
- How to fix this } [Lawson-Lipshitz-Sarkar]

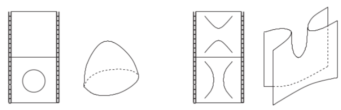
[Carter-Saito '93 / Carter-Rieger-Saito '97]

Presentation of the embedded cobordism category via

- elementary cobordisms

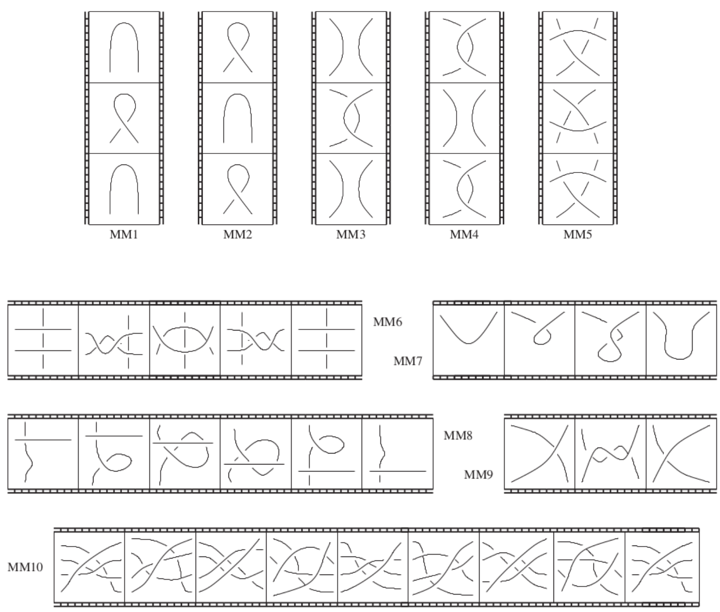


(Reidemeister)

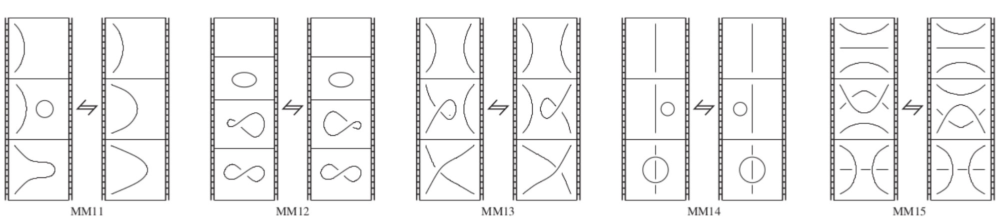


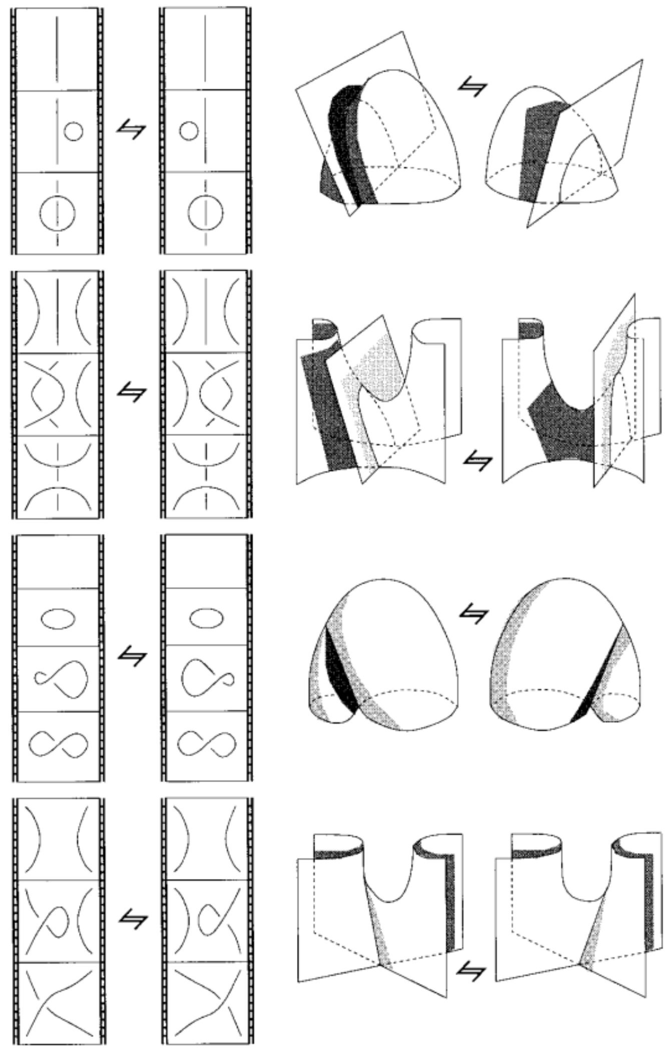
(Morse)

- movie moves — relations between finite sequences of elementary cobordisms



The movie depicted is equivalent to the identity movie.





3-dim interpretation of the last 4 moves

- There are "obvious" candidates for how to define $Kh(-)$ on elementary cobordisms.

- $\rightsquigarrow$ "Just" need to verify that the movie move relations are satisfied. (up to $\pm$)

  (15 in the formulation of [CS] — smooth movies
  31 in the formulation of [CRS]) — movies with finitely many frames

[Jacobsson '04]

Explicit verification by hand.

[Khovanov '05]

Extended $Kh(-)$ to tangles (composable by stacking)

$\hookrightarrow$ Suffices to check the movie moves on "local" (tangle) pieces

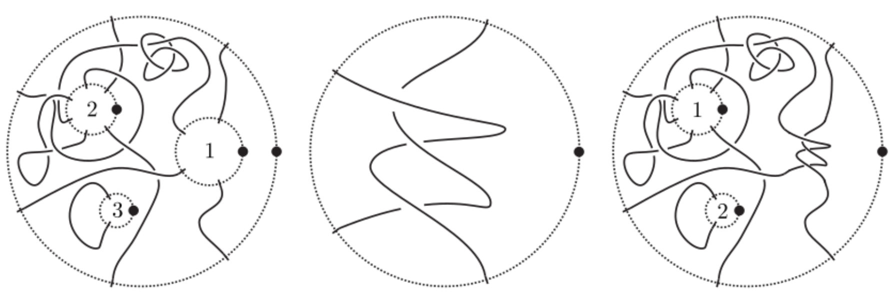
$\rightsquigarrow$ less brute-force computation required

[Bar-Natan '05]

Extended $Kh(-)$ to tangles (composable "in all directions")

i.e. respecting their structure as a planar algebra

[V. Jones]



T $\qquad$ S $\qquad$ $T \circ_1 S$

↳ Even more localised

(almost) no brute-force computation required!

Instead

- Most movie moves (MM1 — MM10) are of the form:

movie (reversible)

$$\begin{tikzcd} T \arrow[r, bend left, "M"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\rightleftarrows"] & T \end{tikzcd}$$

tangle with no closed components

Proposition [BN] If T has no closed components, then

$$\mathrm{Aut}(\underbrace{\mathrm{KhC}}(T)) = \{\pm 1\}$$

Khovanov complex

Corollary $\mathrm{Kh}(M) = \pm\, id.$

Idea of proof

First : where does $\mathrm{KhC}(T)$ live?

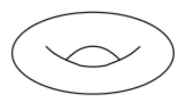
Tangle $\rightsquigarrow$ formal chain complex of matrices with entries in $\mathbb{Z}[\mathrm{Cob}_{1+1}(D^2)] / \sim$

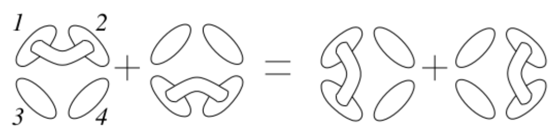
take the Khovanov cube of smoothings, but do not apply the TQFT / Frobenius algebra yet

obj : 1-manifolds in D^2
mor : 2-manifolds in $D^2 \times [0,1]$

three local relations :

 $= 0$ (S)

 $= 2$ (T)



(4Tu)

All arguments take place at this level.

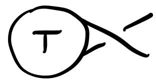
Afterwards, one may apply the construction

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{Morphisms}(\emptyset, \quad) \Big/ \left(\begin{array}{l} S = 0 \text{ for any closed} \\ \text{surface } S \text{ of genus} \geq 2 \end{array} \right)^{*}$$

see pages 10 – 11 ...

which recovers the usual Khovanov complex (over $\mathbb{Z}_{(2)}$) when T is a link, and take homology.

Lemma If $\mathrm{Aut}(\mathrm{KhC}(T)) = \{\pm 1\}$, then the same is true for

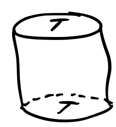


Idea: Formal, using the planar algebra structure of tangles.

Lemma If T has no closed components and no crossings (T is a <u>matching</u>), then $\mathrm{Aut}(\mathrm{KhC}(T)) = \{\pm 1\}$

Sketch
- The cube of smoothings is 0-dim, i.e. just one vertex
- So $\mathrm{KhC}(T) = T$ ← in degree 0, no differential.

  $\mathrm{Aut}(\mathrm{KhC}(T))$ = formal $\mathbb{Z}$-linear combinations of self cobordisms

  (that are equal to $\partial T \times [0,1]$ on $\partial D^2 \times [0,1]$) (**)

- In order to preserve homological degree (o/w it cannot be invertible since $\mathrm{KhC}(T) = \emptyset$ in all other degrees)

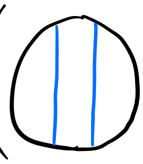
  each cobordism C must satisfy

  $$\chi(C) = |\pi_0(\partial C)| \qquad (*)$$

- We may assume by (T) that C has no torus components.
- There are no sphere components, by (S).
- If there is a Σ_g component for $g \geq 2$, then by $(*)$ there would also have to be a sphere component. ↯
- Hence each component of C is a disc ⇝ C is the identity cobordism.
- ⇒ Each automorphism is a $\mathbb{Z}$ multiple of the identity.

  Since it is invertible it must be $\pm$ id. //

Why does this not lift to $\mathcal{X}_{Kh}$?

$\mathrm{KhC}\Bigg($  $\Bigg)$ has a non-trivial automorphism!

i.e. not $\pm \mathrm{id}$.

sphere spectrum

$\longleftrightarrow \quad \pi_1(\mathbb{S}) \neq 0$

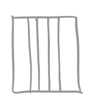
$\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ generated by the Hopf map

$\Sigma^\infty(\eta : S^3 \to S^2)$

How to bypass this problem?

(0,m)-tangles

Prop (LLS) $\quad \mathrm{Hom}_{\mathrm{KhC}(\mathrm{id}_m)}\big(\mathrm{KhC}(T_1), \mathrm{KhC}(T_2)\big) \cong \mathrm{KhC}(\underbrace{\hat{T}_1, T_2})$

identity (0,m)-tangle 

link formed by gluing
$\hat{T}_1 = T_1$ upside-down
to T_2.

<u>Idea</u> Most movie moves are of the form $M \leftrightarrows \mathrm{id}$ for

$$T \xrightarrow{M} T.$$

"bridge tangle"

By the above proposition,

$$\mathrm{KhC}(M) \in \mathrm{Hom}_{\mathrm{KhC}(\mathrm{id}_m)}\big(\mathrm{KhC}(T), \mathrm{KhC}(T)\big) \cong \mathrm{KhC}(\underbrace{\hat{T}\,T}_{\text{unlink}})$$

and the right-hand side is $\cong \mathbb{Z}$
(in the bidegree under consideration)
hence $\mathrm{KhC}(M)$ is a scalar multiple of the identity.

This re-proves functoriality of $\mathrm{KhC}(-)$.

Moreover, this alternative strategy <u>does</u> lift directly to $\mathcal{X}_{Kh}(-)$!

<u>Why?</u> Roughly:

- The first strategy fundamentally depends on KhC() having no non-trivial automorphisms. But the existence of non-trivial elements of $\pi_i(\mathbb{S})$, $i>0$, means that this becomes false when lifted to $\mathcal{X}_{Kh}$.

- The second strategy fundamentally depends on KhC(unlink) being (essentially) trivial. When lifting to $\mathcal{X}_{Kh}$, we instead need $\pi_0(\mathcal{X}_{Kh}(\text{unlink}))$ to be trivial. But $\mathcal{X}_{Kh}(\text{unlink}) \simeq \mathbb{S}$ (in the appropriate quantum degree), so this remains true.

Addendum

Recall that after taking the Khovanov chain complex $KhC(T)$ — a formal chain complex over $\mathrm{Cob}_{n+1}(D^2)/\sim$

1-mflds in D^2
2-mflds in $D^2 \times [0,1]$

(S)
(T) relations
(4Tu)

we then apply the operation

$$\underbrace{\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}}}_{\text{invert 2 in the coefficients}} \mathrm{Morphisms}(\emptyset, \quad) \Big/ \left(\begin{array}{l} S=0 \text{ for any closed} \\ \text{surface } S \text{ of genus} \geq 2 \end{array}\right)$$

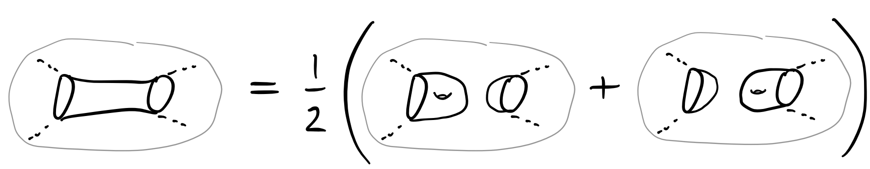
to obtain a cube where each vertex v is sent to a $\mathbb{Z}$-linear combination of nullbordisms of the smoothing of T corresponding to v.

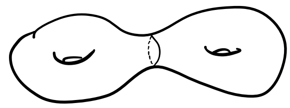
modulo the relations
- $S^2 = 0$
- $T^2 = 2$
- $4Tu$
- $\Sigma_g = 0 \quad (g \geq 2)$

(This turns out to identify with the usual Khovanov complex when $T = L$ is a link, so this extends $Kh(-)$ to tangles.)

Rmk. Above, the relation $\Sigma_g = 0 \quad (g \geq 2)$ is equivalent to $\Sigma_3 = 0$.

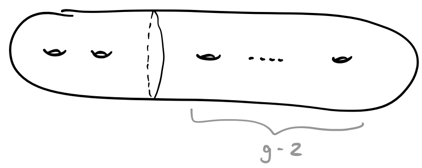
Proof: Since 2 is invertible, a special case of the 4Tu relation says:



Applying this to $\Sigma_2 =$  we obtain

$$\begin{aligned}\Sigma_2 &= \tfrac{1}{2}\left(\Sigma_2 \sqcup \Sigma_1 + \Sigma_1 \sqcup \Sigma_2\right)\\ &= \Sigma_2 \sqcup \Sigma_1\\ &= 2.\,\Sigma_2 \qquad \text{(by relation (T))}\end{aligned}$$

Hence $\underline{\Sigma_2 = 0}$.

Applying the relation to $\Sigma_g =$ 

$(g \geqslant 2)$

we obtain

$$\begin{aligned}\Sigma_g &= \tfrac{1}{2}\Big(\Sigma_3 \sqcup \Sigma_{g-2} + \underbrace{\cancel{\Sigma_2 \sqcup \Sigma_{g-1}}}_{0 \text{ by above}}\Big)\\ &= \tfrac{1}{2}\left(\Sigma_3 \sqcup \Sigma_{g-2}\right).\end{aligned}$$

Inductively:

$$\Sigma_{2g} = \frac{1}{2^g}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g} \sqcup \overbrace{\cancel{\Sigma_0}}^{0 \text{ by relation (S)}}\Big) = 0$$

$$\Sigma_{2g+1} = \frac{1}{2^g}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g} \sqcup \Sigma_1\Big) = \frac{1}{2^{g-1}}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g}\Big)$$

by relation (T)

In particular, adding the relation $\Sigma_3 = 0$ automatically implies that $\Sigma_g = 0$ for all $g \geqslant 2$.

//.

Rmk. We could instead set $\Sigma_3 = k$ for any other $k \in \mathbb{Z}$.

Then $\Sigma_{2g+1} = \dfrac{k^g}{2^{g-1}}$.

When $k = 8$, this recovers Lee homology.