Spectrification of Khovanov homology I

GeMAT seminar
14 March 2022

Plan

(today) I The original construction of Lipshitz – Sarkar

(24 March) II A second, simpler construction (Lawson - Lipshitz – Sarkar, Hu - Kriz - Kriz)

(8 April) III An extension to $\underbrace{\text{tangles}}_{(0,1)-\text{TQFT}}$ and $\underbrace{\text{tangle cobordisms}}_{(0,1,2)-\text{TQFT}}$

1. What are Khovanov spectra?
2. Stable cohomology operations
3. Examples, alternating links
4. Spatial lifts
5. Lax spatial lifts
6. Framed flow categories
7. The Khovanov framed flow category } Next time, described in parallel with LLS's simpler construction.

1. Khovanov spectra

Khovanov (2000)

$$L \longmapsto Kh_R(L) = \bigoplus_{i,j} Kh_R^{\overset{\text{homological grading}}{i},\overset{\text{quantum grading}}{j}}(L)$$

bigraded abelian group (R-module)

such that

$$\chi(Kh_R^{*,j}(L)) = \text{coefficient of } q^j \text{ in } (q+q^{-1})\underbrace{J(L)}_{\text{Jones polynomial}}$$

$$\| $$

$$\sum_i (-1)^i \operatorname{rank}(Kh^{i,j}(L))$$

Lipshitz-Sarkar (2014)

$L \longmapsto$ sequence of $\underbrace{\text{suspension spectra}}_{\text{Def: soon!}}$ $X^j_{Kh}(L)$, $j \in \mathbb{Z}$

such that

$$\tilde{H}^i(X^j_{Kh}(L); R) \cong Kh_R^{i,j}(L)$$

Thus:

$$\chi(X^j_{Kh}(L)) = \text{coefficient of } q^j \text{ in } (q+q^{-1})J(L)$$

More precisely:

Notation:

$\mathbb{Z}[S]$ = sequences of objects of type S, indexed by $j \in \mathbb{Z}$, all but finitely many of them trivial.

e.g. $\mathbb{Z}[\mathbb{Z}]$ = Laurent polynomials in q

viewing the j^{th} integer as the coeff. of q^j

$$
\begin{tikzcd}
\{\text{link diagrams}\} \arrow[r, red] \arrow[dd] \arrow[rr, bend left, green] \arrow[drrr, bend left, blue] & \mathbb{Z}\left[\begin{array}{c}\text{finite}\\ \text{CW-complexes}\end{array}\right] \arrow[r, "\tilde{C}^*_{cell}"] \arrow[dd, "\Sigma^\infty"] & \mathbb{Z}\left[\begin{array}{c}\text{chain}\\ \text{complexes}\end{array}\right] \arrow[dd, "H_*"] \arrow[dr, "\chi"] & \\
 & & & \mathbb{Z}[\mathbb{Z}] \arrow[d, equal] \\
\{\text{links}\} \arrow[r, red, "\mathcal{X}_{Kh}"] \arrow[rr, bend right, green, "Kh"'] \arrow[urrr, bend right, blue, "J"'] & \mathbb{Z}\left[\begin{array}{c}\text{finite suspension}\\ \text{spectra up to} \simeq\end{array}\right] \arrow[r, "\tilde{H}^*"'] & \mathbb{Z}\left[\begin{array}{c}\text{graded}\\ \text{modules}\end{array}\right] \arrow[ur, "\chi"'] & \mathbb{Z}[q^{\pm 1}]
\end{tikzcd}
$$

J = Jones polynomial

Kh = Khovanov homology

$\mathcal{X}_{Kh}$ = Lipshitz-Sarkar-Khovanov spectrum

Suspension spectra

- $\text{Spaces} \xrightarrow{\Sigma^\infty} \text{Spectra}$

  complicated to define

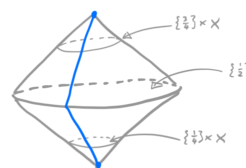
- The subcategory on the objects $\Sigma^\infty X$, X = finite CW-complex, has a simple description. This is the category of finite suspension spectra.

- Objects (X, i) for $\begin{cases} X \text{ finite based CW-complex} \\ i \in \mathbb{Z} \end{cases}$

  written $\Sigma^i X$

  identify $\Sigma^{i+1} X = \Sigma^i(\underbrace{\Sigma X})$

  (reduced) suspension of X $= \dfrac{[0,1] \times X}{(\{0,1\} \times X) \cup ([0,1] \times \{*\})}$

  $=$ 

  with | collapsed to a point

- Morphisms $\Sigma^i X \to \Sigma^j Y$ are $(k, \Sigma^{i+k} X \xrightarrow{f} \Sigma^{j+k} Y)$

  such that $i+k$ and $j+k \geq 0$ — usual based map of spaces

  identify $(k, f) \sim (k+1, \Sigma f)$

- Homotopies (2-morphisms) defined similarly.

- $\Sigma^\infty : (\text{finite CW-complexes}) \longrightarrow (\text{finite suspension spectra})$

  $$X \longmapsto \Sigma^0 X$$

- <u>Upshot</u>: $\Sigma^\infty X \simeq \Sigma^\infty Y$ if and only if $\exists k : \Sigma^k X \simeq \Sigma^k Y$.

2. Stable cohomology operations

- Spectra contain much more information than their cohomology groups $H^i(X;R)$.

- How to extract this information as algebraic invariants?

X spectrum $\rightsquigarrow$ $\tilde{H}^*(X;\mathbb{F}_p)$ is a module over the Steenrod algebra $\mathcal{A}_p$

- graded $\mathbb{F}_p$-algebra
- known explicit presentation

$\downarrow$

- other generalised cohomology theories

  e.g. (topological) K-theory $K^*(X)$

- which come with their own algebras of stable cohomology operations

  (e.g. Adams operations for K-theory)

$\mathcal{A}_2$

- generated by Sq^n for $n \geq 1$. $\quad Sq^n : \tilde{H}^i(X;\mathbb{F}_2) \longrightarrow \tilde{H}^{i+n}(X;\mathbb{F}_2)$
- subject to the Adem relations

- e.g. the subalgebra generated by $\{Sq^1, Sq^2\}$ is

$$\mathcal{A}_2(1) = \mathbb{F}_2\{Sq^1, Sq^2\} \Big/ \begin{pmatrix} Sq^1 Sq^1 = 0 \\ Sq^2 Sq^2 = -Sq^1 Sq^2 Sq^1 \end{pmatrix}$$

Rmks

(1) No nontrivial stable coh. operations on $\widetilde{H}^*(X;\mathbb{Q})$.

(2) Stable coh. operations on $\widetilde{H}^*(X;\mathbb{Z})$ are determined by the stable coh. operations on $\widetilde{H}^*(X;\mathbb{F}_p)$ for all primes p.

(3) The cohomology $\widetilde{H}^*(X)$ of spaces X also has

- a cup product
- Massey products, etc.

BUT these do not exist for spectra.

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(4) Atiyah-Hirzebruch spectral sequence

$$H^*(X; E^*(pt)) \Rightarrow E^*(X)$$

any generalised cohomology theory

Fact: d_2, d_3 differentials for $E^* =$ K-theory are determined by Sq^1 and Sq^2 operations.

So sometimes $Kh(L)$ plus Sq^1 and Sq^2 determine Khovanov K-theory

$$K^*(X^j_{Kh}(L))$$

[LS] compute this for all links with $\leqslant 11$ crossings, and it is just $Kh^{*,j}(L) \otimes \underbrace{K^*(pt)}_{\begin{cases}\mathbb{Z} & \text{even degrees}\\ 0 & \text{odd degrees}\end{cases}}$.

But
- there may be non-trivial Adams operations
- in general $K^*(X^j_{Kh}(L))$ is expected to be richer than $Kh^{*,j}(L)$.

3. Examples, alternating links

Unknot

quantum grading j	Khovanov spectrum X^j_{Kh}
1	S^0
-1	S^0

Hopf link

quantum grading j	Khovanov spectrum X^j_{Kh}
6	S^2
4	S^2
2	S^0
0	S^0

More generally:

Prop [LS] If L is an alternating link, then $X^j_{Kh}(L)$ is a wedge sum of Moore spaces. In particular:

- it is determined by its (co)homology
- all Steenrod operations are trivial.

Rmk. In fact, this is true of any spectral refinement of $Kh(L)$, independently of how it is defined.

- It follows from the fact* that $Kh^{i,j}(L) \neq 0$ only when $i = \frac{1}{2}(j + \underbrace{\sigma(L)}_{\text{signature}} \pm 1)$ and has torsion only when $i = \frac{1}{2}(j + \sigma(L) + 1)$.

*due to [Manolescu-Ozsváth, 2007]

BUT

$K = 10_{145}$

quantum grading j	Khovanov spectrum $\mathcal{X}^j_{Kh}$
-3	S^0
-5	S^0
-7	$\Sigma^{-3}(S^0 \vee S^1)$
-9	$\Sigma^{-6}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$
-11	$\Sigma^{-5}(S^0 \vee S^1 \vee S^1 \vee S^2)$
-13	$\Sigma^{-8}(\mathbb{RP}^4/\mathbb{RP}^1 \vee \Sigma\mathbb{RP}^2)$
-15	$\Sigma^{-10}(\mathbb{RP}^5/\mathbb{RP}^2 \vee S^4)$
-17	$\Sigma^{-8}(S^0 \vee S^1)$
-19	$\Sigma^{-10}\mathbb{RP}^2$
-21	$\Sigma^{-9} S^0$

Theorem [Seed] $\exists\, L_1, L_2$ with $Kh(L_1) \cong Kh(L_2)$

but $\mathcal{X}_{Kh}(L_1) \not\simeq \mathcal{X}_{Kh}(L_2)$

shown by computing Sq^2

$$\left(\begin{array}{l} L_1 = 11^n_{70} \\ L_2 = 13^n_{2566} \end{array} \right)$$

Theorem [LLS] $\forall\, n \geq 1$, $\exists$ knot L such that

$$Sq^n : Kh^{*,*}_{\mathbb{F}_2}(L) \longrightarrow Kh^{*+n,*}_{\mathbb{F}_2}(L)$$

is non-trivial.

Explicit combinatorial formulas for $Sq^n : Kh^{*,*}_{\mathbb{F}_2}(L) \longrightarrow Kh^{*+n,*}_{\mathbb{F}_2}(L)$

- Lipshitz - Sarkar $n = 1, 2, 3$
- Cantero all n

4. Spatial lifts

Aim: Idea of how to lift to

$$
\begin{tikzcd}
\{\text{link diagrams}\} \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbb{Z}\left[\begin{smallmatrix}\text{finite}\\ \text{CW-complexes}\end{smallmatrix}\right] \arrow[r, "\widetilde{C}^*_{\text{cell}}"] \arrow[d, "\Sigma^\infty"] & \mathbb{Z}\left[\begin{smallmatrix}\text{chain}\\ \text{complexes}\end{smallmatrix}\right] \arrow[d, "H_*"] \\
\{\text{links}\} \arrow[r, "\mathcal{X}_{Kh}"] \arrow[rr, bend right, "Kh"'] & \mathbb{Z}\left[\begin{smallmatrix}\text{finite suspension}\\ \text{spectra up to } \simeq\end{smallmatrix}\right] \arrow[r, "\widetilde{H}^*"'] & \mathbb{Z}\left[\begin{smallmatrix}\text{graded}\\ \text{modules}\end{smallmatrix}\right]
\end{tikzcd}
$$

Then one has to check that the (sequence of) CW-complexes associated to a link diagram is invariant <u>up to stable $\simeq$</u> under Reidemeister moves.

Fix a quantum grading j. Then we want to lift

$$
\begin{tikzcd}
 & \{\text{CW-complexes}\} \arrow[d, "\widetilde{C}^*_{\text{cell}}"] \\
\{\text{link diagrams}\} \arrow[ur, dashed] \arrow[r, "KhC^{*,j}"'] & \{\text{chain complexes}\}
\end{tikzcd}
$$

Def $\mathcal{B}(\mathbb{Z})$ $\begin{cases} \text{objects} = \text{finite sets} \\ \text{maps } X \to Y = \text{homomorphisms } \bigoplus_X \mathbb{Z} \to \bigoplus_Y \mathbb{Z} \end{cases}$

$\mathcal{B}(S^N)$ $\begin{cases} \text{objects} = \text{finite sets} \\ \text{maps } X \to Y = \text{continuous maps } \bigvee_X S^N \to \bigvee_Y S^N \end{cases}$

- A $\mathbb{Z}$-chain complex is a linear diagram in $\mathcal{B}(\mathbb{Z})$:

$$F(n) \to \cdots \to F(1) \to F(0)$$

such that any pair composes to the zero homomorphism.

Note: By abuse of notation, we conflate a finite set X with the free abelian group $\bigoplus_X \mathbb{Z}$ that it generates or the wedge of spheres $\bigvee_X S^N$ that it indexes, depending on the context.

- An S^N-chain complex is a linear diagram in $\mathcal{B}(S^N)$:

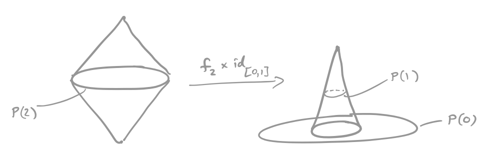
$$P(n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} P(1) \xrightarrow{f_1} P(0)$$

such that any pair composes to the constant map.

A strict spatial lift of a chain complex F is an S^N-chain complex P such that $\underbrace{H_N}_{\text{degree-}N \text{ homology}} \circ P = F$.

From this we can construct a CW-complex X_P with $\tilde{C}^*(X_P) \cong F$ as follows.

The fact that $f_1 \circ f_2 = *$ means that $\exists$ map $g_1 : \Sigma P(2) \longrightarrow \mathrm{Cone}(f_1)$



Thus we obtain a new sequence of CW-complexes with each pair of maps composing to the constant map:

$$\Sigma P(n) \xrightarrow{\Sigma f_n} \cdots \longrightarrow \Sigma P(3) \xrightarrow{\Sigma f_3} \Sigma P(2) \xrightarrow{g_1} \mathrm{Cone}(f_1)$$

Iterating, we eventually obtain a CW-complex X_P whose reduced cellular cochain complex is the original chain complex F.

<u>Summary</u>:

$$\begin{tikzcd} \{S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"'] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\ & \{\mathbb{Z}\text{-chain complexes}\} \end{tikzcd}$$

So it would be enough to refine the Khovanov chain complex to an S^N-chain complex.

<u>BUT</u>: • $\mathcal{B}(\mathbb{Z})(X,Y) = \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_X \mathbb{Z}, \bigoplus_Y \mathbb{Z}\right) = (Y \times X)$-matrices over $\mathbb{Z}$

(because $\oplus$ is a product and coproduct)

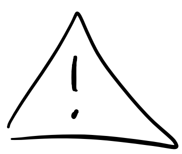
• $\mathcal{B}(S^N)(X,Y) = \mathrm{Map}\left(\bigvee_X S^N, \bigvee_Y S^N\right)$

$$= \prod_X \mathrm{Map}\left(S^N, \bigvee_Y S^N\right) \quad \text{(because } \vee \text{ is a coproduct)}$$

$$\pi_0 B(S^N)(X,Y) = \bigoplus_X \pi_N \underbrace{\Big(\bigvee_Y S^N \Big)}_{\bigoplus_Y \pi_N(S^N)} \cong \bigoplus_X \bigoplus_Y \pi_N(S^N) \cong B(\mathbb{Z})(X,Y).$$

exercise, or
Hilton-Milnor theorem

So a strict spatial lift P of F does not contain any more information (up to $\simeq$) than F. Hence the homotopy type of X_P also cannot contain any more information.



5. Lax spatial lifts

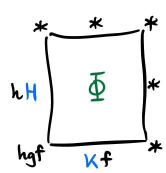
A **lax S^N-chain complex** is a linear diagram

$$P(n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} P(1) \xrightarrow{f_1} P(0),$$

where $P(n) = \bigvee_{F(n)} S^N$, together with

- for each pair
$$\begin{tikzcd} \bigvee S^N \arrow[r, "f"] \arrow[rr, bend right, "H"', "*"] & \bigvee S^N \arrow[r, "g"] & \bigvee S^N \end{tikzcd}$$
a homotopy

- for each triple
$$\begin{tikzcd} \bigvee S^N \arrow[r, "f"] \arrow[rr, bend right, "H"', "*"] & \bigvee S^N \arrow[r, "g"] \arrow[rr, bend right, "K"', "*"] & \bigvee S^N \arrow[r, "h"] & \bigvee S^N \end{tikzcd}$$
a homotopy of homotopies:



$\times \bigvee S^N \longrightarrow \bigvee S^N$

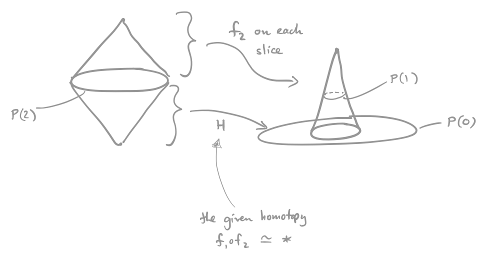
- etc.

Emphasis: The maps f_i, homotopies $H, K, \ldots$, higher homotopies $\Phi, \ldots$, etc. are all choices — i.e. they are part of the data of a lax S^N-chain complex.

A **lax spatial lift** of a chain complex F is a lax S^N-chain complex P such that $H_N \circ P = F$.

A lax spatial lift P also determines a CW-complex X_P whose reduced cellular chain complex is the original chain complex F.

- Similar construction to above (iterative), except now the map $g_1 : \Sigma P(2) \longrightarrow \mathrm{Cone}(f_1)$ is defined by



- Formally it can be defined as the homotopy colimit of a certain diagram.

Thus we have:

$$\begin{tikzcd}
\{\text{lax } S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"'] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\
& \{\mathbb{Z}\text{-chain complexes}\}
\end{tikzcd}$$

So we need to construct a lax spatial lift of the Khovanov chain complex.

6. Framed flow categories

$$
\begin{tikzcd}
\{\text{framed flow categories}\} \arrow[r, "\text{Cohen-Jones-Segal}"] \arrow[drr] & \{\text{lax } S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\
\{\text{link diagrams}\} \arrow[u, "\text{Lipshitz-Sarkar}"] \arrow[rr, "KhC^{*,j}"'] & & \{\mathbb{Z}\text{-chain complexes}\}
\end{tikzcd}
$$

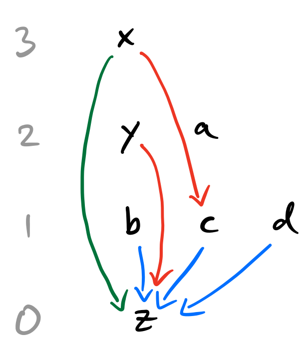
A framed flow category has a finite set of objects, graded by $\{0, \dots, n\}$, and for objects x, y:

- $\mathcal{M}(x,y)$ k-dimensional manifold with corners,
$$k = gr(x) - gr(y) - 1$$

- composition is an inclusion
$$\mathcal{M}(x,y) \times \mathcal{M}(y,z) \hookrightarrow \partial \mathcal{M}(x,z).$$

- framed embeddings of all of these manifolds into high-dim. Euclidean space.

Picture:



$\longrightarrow$ 0-dim space of morphisms

$\longrightarrow$ 1-dim space of morphisms

$\longrightarrow$ 2-dim space of morphisms

Idea:

- $\{\text{framed flow categories}\} \longrightarrow \{\mathbb{Z}\text{-chain complexes}\}$
  - Objects $\rightsquigarrow$ generators
  - Forget all higher-dim. morphism spaces
  - #'s of 0-dim. morphism spaces $\rightsquigarrow$ coefficients of the differential.

- [CJS] construction refines this to a lax S^N-chain complex using the higher-dim. morphism spaces and the Pontrjagin-Thom collapse construction.

- [LS] construct a framed flow category from the Khovanov cube of resolutions...

Spectrification of Khovanov homology II

GeMAT seminar, IMAR
24 March 2022

Plan

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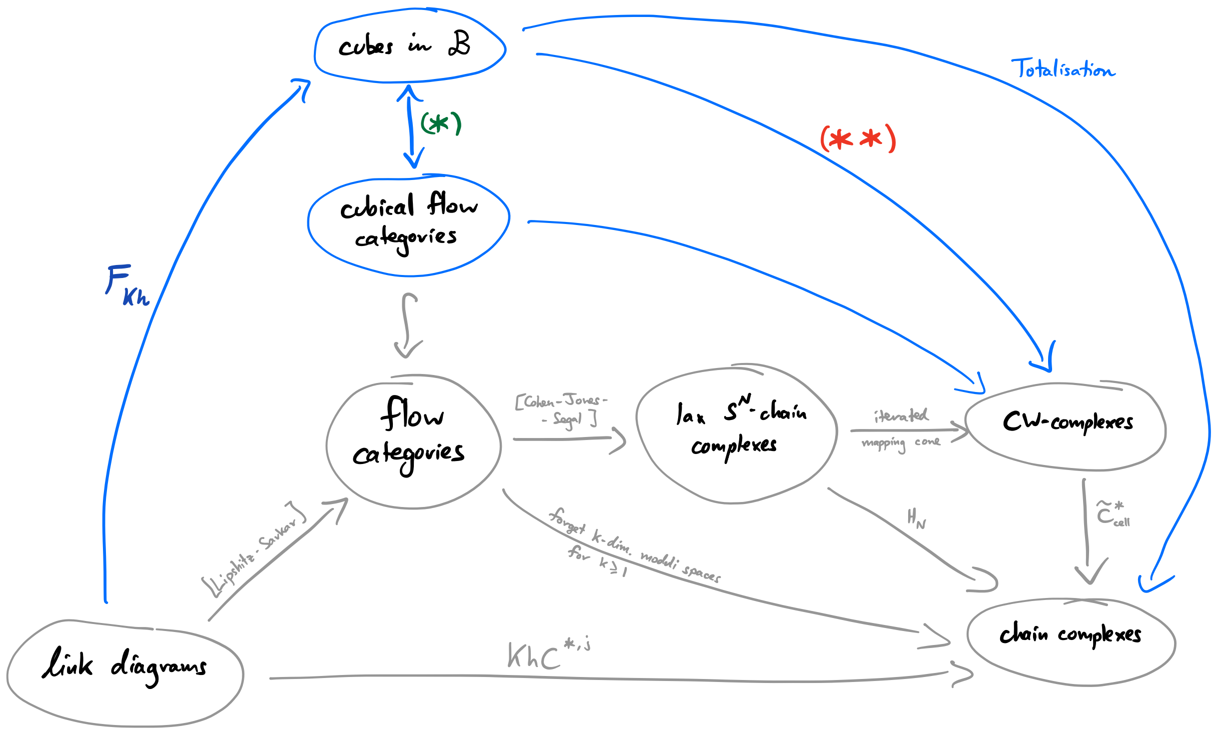


Plan

Setup
1. Flow categories & cubical flow categories
2. The Burnside category $\mathcal{B}$, cubical diagrams
3. The correspondence (*)

Construction
4. The Khovanov functor $F_{Kh}(D)$ of a link diagram D
5. Invariance of F_{Kh}
6. The realisation (**)

Applications
7. Corollary : $\amalg$, $\#$, mirrors of links
8. Corollary : non-trivial Sq^n
9. Universal Khovanov homology (deformations of Kh) & other questions

1. Flow categories

A *flow category* $\mathcal{C}$ is a category where

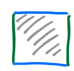
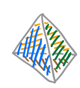
- its object set $ob(\mathcal{C})$ is graded by
  $gr : ob(\mathcal{C}) \longrightarrow \mathbb{Z}$
- $Hom(x,x) = \{id_x\}$
- if $x \neq y$ and $gr(x) - gr(y) = k+1$
  then $Hom(x,y)$ is a compact k-dimensional $\langle k \rangle$-manifold
- the disjoint union of composition maps

$$\coprod_{\substack{z \neq x \\ z \neq y \\ gr(z)-gr(y)=i}} Hom(z,y) \times Hom(x,z) \longrightarrow Hom(x,y)$$

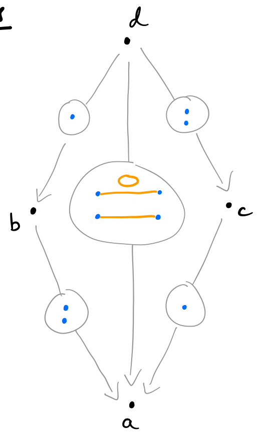
is a diffeomorphism onto $\partial_i Hom(x,y)$

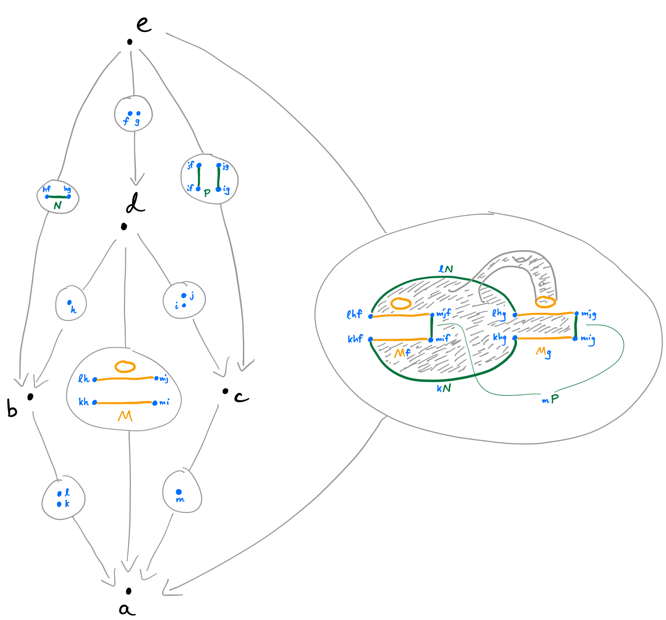
a d-dimensional $\langle c \rangle$-manifold is a manifold with corners M (modelled on $\mathbb{R}^d_+ = [0,\infty)^d$), equipped with a decomposition of ∂M into $\partial_1 M, \dots, \partial_c M$, where each $\partial_i M$ is a disjoint union of codimension-1 faces.

Eg.

-  2-dim $\langle 2 \rangle$-mfld
-  2-dim $\langle 3 \rangle$-mfld
- $[0,1]^n$ n-dim $\langle n \rangle$-mfld
-  3-dim $\langle 4 \rangle$-mfld
- Δ^n n-dim $\langle n+1 \rangle$-mfld

Examples





A framed flow category is a flow category $\mathcal{C}$ together with framed embeddings of each $\langle k \rangle$-manifold $\mathrm{Hom}(x,y)$ into some Euclidean space with corners, and these should be compatible with composition in $\mathcal{C}$.

Note:

When $gr(x) - gr(y) = 1$, $\mathrm{Hom}(x,y)$ is a compact 0-dim. $\langle 0 \rangle$-manifold, i.e. a finite set.

A framed embedding of $\mathrm{Hom}(x,y)$ into Euclidean space determines a sign for each point of $\mathrm{Hom}(x,y)$.

framed flow categories $\longrightarrow$ chain complexes

objects $\longrightarrow$ generators

$\left.\begin{array}{l} \mathrm{Hom}(x,y) \\ gr(x) - gr(y) = 1 \end{array}\right\} \longrightarrow$ differential $\qquad \delta(x) = \sum_y \underbrace{\# \mathrm{Hom}(x,y)}_{\text{counted with sign}} . y$

Def

Consider the $n!$ points $(\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n$ for $\sigma \in \Sigma_n$.

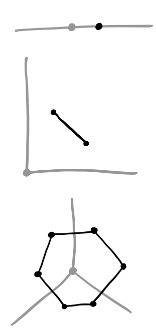
Note: they all lie on the affine subspace $\{ \underline{x} \in \mathbb{R}^n \mid \sum x_i = \frac{1}{2} n(n+1) \}$.

The $(n-1)$-dimensional permutohedron Π_{n-1} is the convex hull of these points.

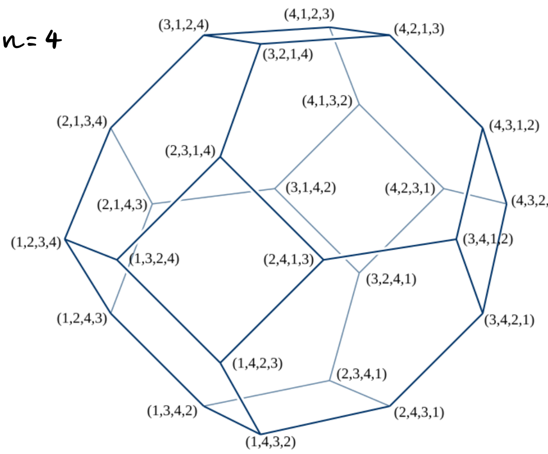
E.g. $n=1$

$n=2$

$n=3$

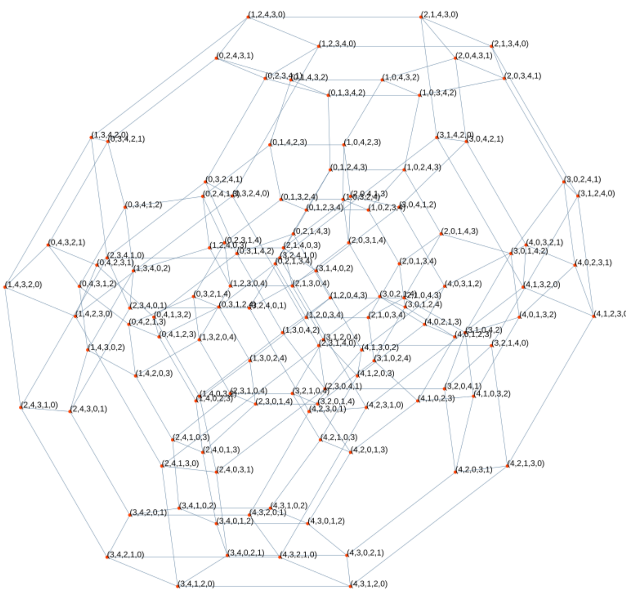


$n=4$



(source : Wikipedia)

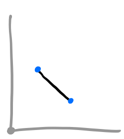
$n=5$



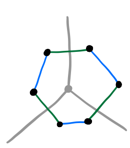
(source : Wikipedia)

Lemma: Π_k is a k-dimensional $\langle k \rangle$-manifold

Eg: Π_1



Π_2



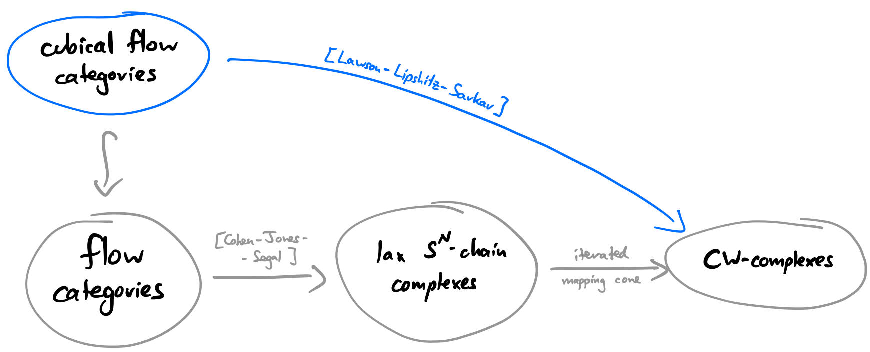
Proposition: There is a flow category $\mathcal{C}(n)$ with

- objects $= \{0,1\}^n$
- grading $gr(v_1,\dots,v_n) = \sum_i v_i$
- when $gr(u) - gr(v) = k > 0$, $\mathrm{Hom}(u,v) = \begin{cases} \Pi_{k-1} & \text{if } u_i \geq v_i \\ \emptyset & \text{otherwise} \end{cases}$

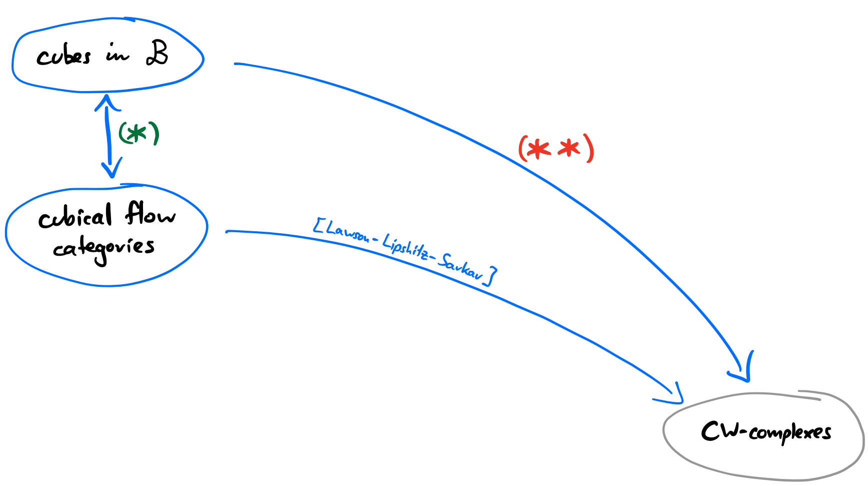
Rmk: The morphism spaces $\mathrm{Hom}(u,v)$ in $\mathscr{C}(n)$ may be thought of as moduli spaces of "broken Morse flows" on the n-cube $[0,1]^n$.

Def: A <u>cubical flow category</u> is a flow category $\mathscr{C}$ and a functor $f: \mathscr{C} \longrightarrow \mathscr{C}(n)$ that preserves the grading up to a global shift, such that each $f: \mathrm{Hom}_{\mathscr{C}}(x,y) \longrightarrow \underbrace{\mathrm{Hom}_{\mathscr{C}(n)}(f(x), f(y))}_{\text{permutohedron or empty}}$ is a trivial covering.

Rmk
- Cubical flow categories are very rigid compared with general flow categories.
- This allows [LLS] to give a construction of the <u>realisation</u> of a cubical flow category that is much simpler than the one of [CJS], which works for all flow categories:



- We'll see this construction later, via <u>cubes in $\mathcal{B}$</u>:



2. The Burnside category $\mathcal{B}$ and cubical diagrams

Def $\mathcal{B}$ has objects $=$ finite sets

$\mathrm{Hom}(A,B)$ $=$ diagrams of functions of the form
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$
$=$ correspondences

2-morphisms from
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$
to
$$\begin{tikzcd} A & & B \\ & X' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$
are bijections $X \cong X'$ such that
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, phantom, "\cong" rotate=90] & \\ A & & B \\ & X' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$
commutes.

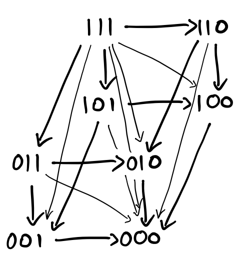
Composition is given by
$$\begin{tikzcd} & & X \times_B Y = \{(x,y) \in X \times Y \mid t(x) = s'(y)\} \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl, "s"'] \arrow[dr, "t"] & & Y \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\ A & & B & & C \end{tikzcd}$$

The cube category has objects $\{0,1\}^n$
$$\mathrm{Hom}(u,v) = \begin{cases} \{\varphi_{u,v}\} & \text{if } u_i \geq v_i \\ \emptyset & \text{otherwise} \end{cases}$$

Notation: $\underline{2}^n$

E.g.: $n=2$
$$\begin{tikzcd} 11 \arrow[r] \arrow[d] \arrow[dr] & 10 \arrow[d] \\ 01 \arrow[r] & 00 \end{tikzcd}$$

$n=3$



A **cube in $\mathcal{B}$** consists of

- for each object $v \in \{0,1\}^n$, a finite set $F(v)$
- for each pair $u > v$, a correspondence
$$\begin{tikzcd} & F(u>v) \arrow[dl] \arrow[dr] & \\ F(u) & & F(v) \end{tikzcd}$$
- for each triple $u > v > w$, a bijection
$$\begin{tikzcd} & & F(u>w) \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\cong" {rotate=90}] & & \\ & & F(u>v) \times_{F(v)} F(v>w) \arrow[dl] \arrow[dr] & & \\ & F(u>v) \arrow[dl] \arrow[dr] & & F(v>w) \arrow[dl] \arrow[dr] & \\ F(u) & & F(v) & & F(w) \end{tikzcd}$$
- satisfying a certain condition for any quadruple $u > v > w > x$

(Formally, this is a strictly unitary lax 2-functor $\underline{2}^n \longrightarrow \mathcal{B}$.)

Lemma: It is enough to specify:
- $F(v)$ for each vertex u,
- $F(u>v)$ for each edge $u \longrightarrow v$,
- the composite $F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \cong F(u>v') \times_{F(v')} F(v'>w)$ for each 2-face $u \begin{smallmatrix} \nearrow v \searrow \\ \searrow v' \nearrow \end{smallmatrix} w$.

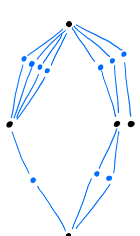
The **underlying chain complex** of a cube in $\mathcal{B}$: (also called its **totalisation**)

- First construct a cube of free abelian groups by

$$\begin{tikzcd} & F(u) \\ F(u>v) \arrow[ur, "s"] \arrow[dr, "t"'] & \\ & F(v) \end{tikzcd} \longmapsto \begin{tikzcd} \mathbb{Z}\langle F(u) \rangle \arrow[d] \\ \mathbb{Z}\langle F(v) \rangle \end{tikzcd} \quad \begin{tikzcd} x \arrow[d, mapsto] \\ \sum_{y \in F(v)} \#(s^{-1}(x) \cap t^{-1}(y)) \cdot y \end{tikzcd}$$

- Then add signs to certain edges of the cube and sum.

E.g.



$$\longmapsto \begin{tikzcd} & \mathbb{Z} \arrow[dl, "4"'] \arrow[dr, "\binom{2}{1}"] & \\ \mathbb{Z} \arrow[dr, "-1"'] & \oplus & \mathbb{Z}^2 \arrow[dl, "(2\ 0)"] \\ & \mathbb{Z} & \end{tikzcd}$$

3. The correspondence cubes in $\mathcal{B}$ $\longleftrightarrow$ cubical flow categories

Recall

- In cubical flow categories, morphism spaces are disjoint unions of permutohedra.
- So most of the complication is taken care of by the combinatorics of permutohedra.
- The additional information is exactly encoded by a cube in $\mathcal{B}$.

cubes in $\mathcal{B}$ $\longleftarrow$ cubical flow categories

$$\begin{array}{l} \mathcal{C} \xrightarrow{f} \underbrace{\mathcal{C}(n)}_{\text{objects } = \{0,1\}^n} \quad \text{cubical flow category} \\ \quad \downarrow \\ F : \underline{2}^n \longrightarrow \mathcal{B} \end{array}$$

$$F(v) = f^{-1}(v)$$

$$F(u>v) = \coprod_{\substack{x \in f^{-1}(u) \\ y \in f^{-1}(v)}} \pi_0 \underbrace{\mathrm{Hom}_{\mathcal{C}}(x,y)}_{\text{disjoint union of permutohedra}}$$

$\longrightarrow f^{-1}(u) = F(u)$

$\longrightarrow f^{-1}(v) = F(v)$

$$F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \quad \text{induced from composition in } \mathcal{C}.$$

cubes in $\mathcal{B}$ $\longrightarrow$ cubical flow categories

$$F: 2^n \longrightarrow \mathcal{B} \qquad \longrightarrow \qquad \mathcal{C} \xrightarrow{f} \mathcal{C}(n)$$

- $ob(\mathcal{C}) = \coprod_{v \in \{0,1\}^n} F(v)$

- For $u > v$ and $x \in F(u)$, $y \in F(v)$, consider
$$\begin{tikzcd} & F(u>v) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ F(u) & & F(v) \end{tikzcd}$$

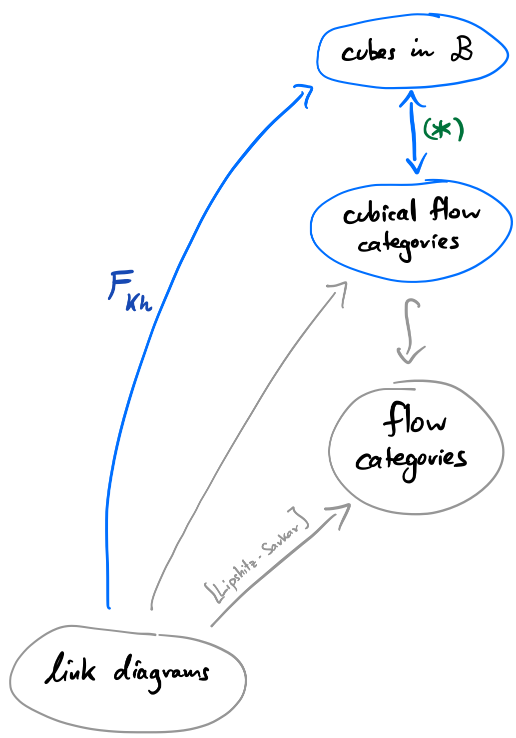
  and let $\mathrm{Hom}_{\mathcal{C}}(x,y) = \left(s^{-1}(x) \cap t^{-1}(y)\right) \times \underbrace{\mathrm{Hom}_{\mathcal{C}(n)}(u,v)}_{\text{permutohedron}}$

- Composition is defined using $\begin{cases} F(u>v) \times_{F(v)} F(v>w) \cong F(u>w) \text{ from } F \\ \text{composition in } \mathcal{C}(n) \end{cases}$

<u>Lemma</u>: These constructions are inverse to each other.
In other words:

$$\{\text{cubical flow categories}\} = \begin{matrix} (\text{the standard permutohedron flow category } \mathcal{C}(n)) \\ + \\ \{\text{cubes in the Burnside category } \mathcal{B}\} \end{matrix}$$

The flow categories constructed by [Lipshitz-Sarkar] are <u>cubical</u>, so we may describe them equivalently as cubes in $\mathcal{B}$:



Next: describe F_{Kh} ...

4. The Khovanov functor of a link diagram

Warm-up /recollection : link diagram $D \rightsquigarrow$ Khovanov cube $F^{Ab}_{Kh}(D)$ of abelian groups

(• add signs to certain edges
• take $\oplus$ within each homological grading
$\rightsquigarrow$ Khovanov chain complex)

- label crossings by $\{1, 2, \dots, n\}$
- $v \in \{0,1\}^n \longmapsto D_v =$ perform 0-resolution $\diagup\!\!\!\diagdown \rightsquigarrow)\,($ at the i^{th} crossing if $v_i = 0$

  perform 1-resolution $\diagup\!\!\!\diagdown \rightsquigarrow \smile\frown$ at the i^{th} crossing if $v_i = 1$

  This is an embedded disjoint union of circles in $\mathbb{R}^2$.

$$F^{Ab}_{Kh}(D)(v) = \bigotimes_{\pi_0(D_v)} \mathbb{Z}\langle x_+, x_- \rangle$$

- $u \to v$ edge of the cube
  - if $D_u \rightsquigarrow D_v$ merges two circles, apply $m: \begin{array}{l} x_+ \otimes x_+ \longmapsto x_+ \\ x_+ \otimes x_- \longmapsto x_- \\ x_- \otimes x_+ \longmapsto x_- \\ x_- \otimes x_- \longmapsto 0 \end{array}$ (and identity on other components)
  - if $D_u \rightsquigarrow D_v$ splits a circle in two, apply $\Delta: \begin{array}{l} x_+ \longmapsto x_+ \otimes x_- + x_- \otimes x_+ \\ x_- \longmapsto x_- \otimes x_- \end{array}$ (and identity on other components)

Aim : link diagram $D \rightsquigarrow$ Khovanov cube $F_{Kh}(D)$ in $\mathcal{B}$

- $F_{Kh}(D)(v) = \{x_+, x_-\}^{\pi_0(D_v)}$
- $u \to v$ edge of the cube

$$F^{Ab}_{Kh}(D)(u \to v) : \mathbb{Z}\langle F_{Kh}(D)(u) \rangle \longrightarrow \mathbb{Z}\langle F_{Kh}(D)(v) \rangle$$

all coefficients are 0 or 1 by construction, so there is no choice :

$$\begin{tikzcd}[column sep=tiny] & F_{Kh}(D)(u \to v) = \{(x,y) \in F_{Kh}(D)(u) \times F_{Kh}(D)(v) \mid \text{coeff of } y \text{ in the image of } x \text{ is } 1\} \arrow[dl] \arrow[dr] & \\ F_{Kh}(D)(u) & & F_{Kh}(D)(v) \end{tikzcd}$$

Rmk So far, this is the same information as $F_{Kh}^{Ab}(D)$, repackaged.

The difference lies in the last part of the construction of $F_{Kh}(D)$.

- $\begin{tikzcd}[row sep=small] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}$ 2-face of the cube

We need to choose a bijection:

$$\begin{tikzcd}[column sep=small]
& & F_{Kh}(D)(u\to v)\times_{F_{Kh}(D)(v)} F_{Kh}(D)(v\to w) \arrow[dl] \arrow[dr] \arrow[ddddd, leftrightarrow] \arrow[ddll, bend right, "s"'] \arrow[ddrr, bend left, "t"] & & \\
& F_{Kh}(D)(u\to v) \arrow[dl] \arrow[dr] & & F_{Kh}(D)(v\to w) \arrow[dl] \arrow[dr] & \\
x\in F_{Kh}(D)(u) & & F_{Kh}(D)(v) & & F_{Kh}(D)(w)\ni y \\
& & F_{Kh}(D)(v') & & \\
& F_{Kh}(D)(u\to v') \arrow[uul] \arrow[ur] & & F_{Kh}(D)(v'\to w) \arrow[uur] \arrow[ul] & \\
& & F_{Kh}(D)(u\to v')\times_{F_{Kh}(D)(v')} F_{Kh}(D)(v'\to w) \arrow[ul] \arrow[ur] \arrow[uuull, bend left, "\bar{s}"] \arrow[uuurr, bend right, "\bar{t}"'] & &
\end{tikzcd}$$

Unpacking this, for each x (labelling of $\pi_0(D_u)$ by $\{x_+, x_-\}$)

y (labelling of $\pi_0(D_w)$ by $\{x_+, x_-\}$)

we need to choose a bijection:

$$s^{-1}(x)\cap t^{-1}(y) = \{\text{labellings of } \pi_0(D_v), \text{ compatible with } F_{Kh}(D)(\underset{x}{u}\to v\to \underset{y}{w})\}$$

$$\updownarrow$$

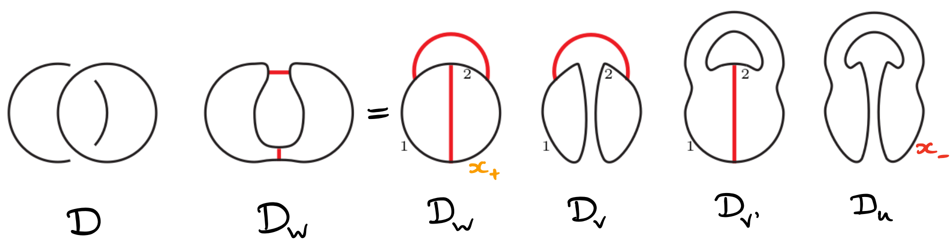
$$\bar{s}^{-1}(x)\cap \bar{t}^{-1}(y) = \{\text{labellings of } \pi_0(D_{v'}), \text{ compatible with } F_{Kh}(D)(\underset{x}{u}\to v'\to \underset{y}{w})\}$$

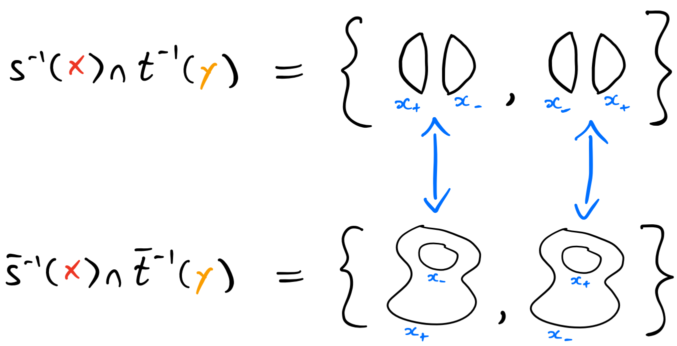
In all cases, these sets:

- are both empty, or
- both have size 1, or
- both have size 2.

$\Big\}$ no choice!

The last case occurs when:





- This is called the "ladybug matching"
- It is the only extra information about the link diagram D that is remembered by $\underbrace{F_{Kh}(D): \underline{2}^n \to \mathcal{B}}_{\text{Khovanov spectrum}}$ compared with $\underbrace{F^{Ab}_{Kh}(D): \underline{2}^n \longrightarrow Ab}_{\text{Khovanov homology}}$.

5. Invariance of $\mathcal{F}_{Kh}$

Thm (Lawson-Lipshitz-Sarkar)

$$\begin{tikzcd}
\{\text{link diagrams}\}/_{\text{isotopy}} \arrow[r, "\mathcal{F}_{Kh}"] \arrow[d, two heads] & \{\text{cubes in } \mathscr{B}\}/_{\text{isomorphism}} \arrow[d, two heads] \\
\{\text{links}\}/_{\text{isotopy}} \arrow[r] & \{\text{cubes in } \mathscr{B}\}/_{\text{stable equivalence}}
\end{tikzcd}$$

Def "stable equivalence" is the equivalence relation on cubes in $\mathscr{B}$ generated by:

- natural transformations $\underline{2}^n \overset{\Downarrow}{\rightrightarrows} \mathscr{B}$ whose induced map on underlying chain complexes is a chain homotopy equivalence.

- if $\underline{2}^N \longrightarrow \mathscr{B}$ is obtained from $\underline{2}^n \longrightarrow \mathscr{B}$, $n < N$, by sending all vertices of $\underline{2}^N \setminus \underline{2}^n$ to $\emptyset$.

Idea of proof: Copy the proof of invariance of Khovanov homology under the Reidemeister moves and upgrade to a stable equivalence at each stage.

Rmk: We will next construct $\{\text{cubes in } \mathscr{B}\} \xrightarrow{(**)} \{\text{CW-complexes}\}$, which will send stable equivalences to stable homotopy equivalences.

$$\{\text{links}\}/_{\text{isotopy}} \longrightarrow \{\text{cubes in } \mathscr{B}\}/_{\text{stable equivalence}} \xrightarrow{\overline{(**)}} \{\text{spectra}\}$$

Question: How much information about a link does $\overline{(**)}$ forget ??

6. The realisation of a cube in the Burnside category

Box maps

Fix
$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$

"box" $= \prod_{i=1}^{k} [a_i, b_i] \subseteq \mathbb{R}^k$

and "boxes"

$B_x \quad x \in X$
$B_y \quad y \in Y$] pairwise disjoint

$B_a \subseteq B_{s(a)} \quad a \in A$] pairwise disjoint

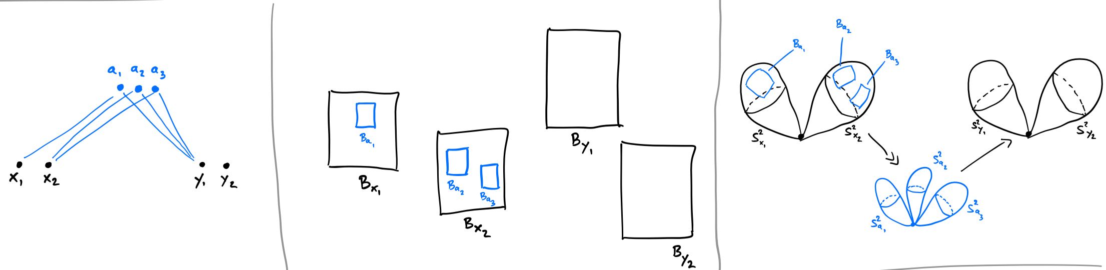
The corresponding box map is

$$\bigvee_{x\in X} S^k = \coprod_{x\in X} B_x \Big/ \partial \longrightarrow \bigvee_{a\in A} B_a/\partial B_a = \bigvee_{a\in A} S^k \longrightarrow \bigvee_{y\in Y} S^k$$

collapse the complement of $\coprod_{a\in A} \mathring{B}_a$ to a point

determined by t

Example



Lemma For fixed
$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd},$$
the space of box maps

$$\bigvee_{x\in X} S^k \longrightarrow \bigvee_{y\in Y} S^k$$

is <u>$(k-2)$-connected</u>.

Def For a cube $\underline{2}^n \xrightarrow{F} \mathcal{B}$, a <u>k-dim. spatial refinement</u> is

a <u>homotopy coherent diagram</u> $\underline{2}^n \xrightarrow{\widetilde{F}} \mathrm{Top}$.

- for each object $v \in \underline{2}^n$, a space $\widetilde{F}(v)$
- for each sequence of morphisms

$$v_0 \xrightarrow{f_1} v_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} v_n$$

a continuous map

$$\widetilde{F}(f_n, \dots, f_1) : [0,1]^{n-1} \times \widetilde{F}(v_0) \longrightarrow \widetilde{F}(v_n)$$

- satisfying some conditions

where $\widetilde{F}(v) = \bigvee_{F(v)} S^k$

$$\widetilde{F}(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) : \bigvee_{F(v_0)} S^k \longrightarrow \bigvee_{F(v_n)} S^k$$

is a box map associated to the correspondence

$$\begin{tikzcd} & F(f_n \circ \cdots \circ f_1) \arrow[dl] \arrow[dr] & \\ F(v_0) & & F(v_n) \end{tikzcd}$$

Proposition If $k \geq n+1$, k-dim. spatial refinements exist and are unique up to hty equivalence of hty coherent diagrams.

→ <u>Idea:</u> construct it recursively, using the fact that the space of box maps is highly-connected.

Rmk Homotopy coherent diagrams $\underline{2}^n \xrightarrow{\widetilde{F}} \mathrm{Top}_\bullet$ have well-defined iterated mapping cones $|\widetilde{F}| \in \mathrm{Top}_\bullet$.

([Vogt, 1973])

Construction

$$\underline{2}^n \xrightarrow{F} \mathcal{B} \rightsquigarrow \underline{2}^n \xrightarrow{\widetilde{F}} \mathrm{Top}_\bullet \rightsquigarrow |\widetilde{F}| \in \mathrm{Top}_\bullet$$

Rmk
- Both steps depend on the higher faces of the cube.
- This is essential for preserving the extra info that $\underline{2}^n \longrightarrow \mathcal{B}$ captures!

Proposition If F, G are stably equivalent cubes in $\mathcal{B}$, then $|\widetilde{F}|, |\widetilde{G}|$ are stably homotopy equivalent (pointed) spaces, i.e. they determine homotopy equivalent (suspension) spectra.

Proposition The composition

$$\{\text{links}\}/_{\text{isotopy}} \xrightarrow{F_{Kh}} \{\text{cubes in } \mathcal{B}\}/_{\text{stable equivalence}} \xrightarrow{(**)} \{\text{spectra}\}$$

$$L \longmapsto X_{Kh}(L)$$

- comes equipped with a decomposition $X_{Kh}(L) = \bigvee_{j \in \mathbb{Z}} X^j_{Kh}(L)$
- recovers the Khovanov spectrum of [Lipshitz-Sarkar]
- in particular, $\widetilde{H}^*(X^j_{Kh}(L)) \cong Kh^{*,j}(L)$.

7. Corollary : $\amalg$, $\#$, mirrors of links

This new, much simpler construction of $\mathcal{X}_{Kh}(L)$ allows [LLS] to prove:

Thm [LLS]

- $\mathcal{X}_{Kh}(L_1 \amalg L_2) \simeq \mathcal{X}_{Kh}(L_1) \wedge \mathcal{X}_{Kh}(L_2)$

  smash product

- $\mathcal{X}_{Kh}(L_1 \# L_2) \simeq \mathcal{X}_{Kh}(L_1) \underset{\mathbb{S} \vee \mathbb{S}}{\otimes} \mathcal{X}_{Kh}(L_2)$

  tensor product over $\mathbb{S} \vee \mathbb{S} = \mathcal{X}_{Kh}(\text{unknot})$

- $\mathcal{X}_{Kh}^{j}(\underbrace{m(L)}_{\text{mirror of } L}) \simeq \mathcal{X}_{Kh}^{-j}(L)^{\vee}$

  Spanier-Whitehead dual

Rmk In each case, the proof reduces to a statement about cubes in $\mathcal{B}$ instead of flow categories, which is what makes it tractable.

8. Corollary^2: non-trivial Sq^n

Thm [LLS] $\forall n \ \exists$ link L_n such that its Khovanov homology has a non-trivial $Sq^n: Kh^{i,j}(L_n) \longrightarrow Kh^{i+n,j}(L_n)$ operation.

Proof
- We need to find L_n such that the spectrum $X_{Kh}(L_n)$ has a non-trivial Sq^n in its cohomology.
- The space $\underbrace{\mathbb{RP}^2 \wedge \cdots \wedge \mathbb{RP}^2}_{n \text{ copies}}$ has a non-trivial Sq^n.
- $\Rightarrow$ Enough to find L_n with $X_{Kh}(L_n) \simeq Z \vee \Sigma^k(\mathbb{RP}^2 \wedge \cdots \wedge \mathbb{RP}^2)$
- Calculation of Lipshitz-Sarkar:

$$X_{Kh}(\text{left trefoil}) \simeq Z' \vee \Sigma^{-4}\mathbb{RP}^2$$

- By the previous corollary we may take

$$L_n = \text{disjoint union of } n \text{ left trefoils.} \qquad \square$$

9. Universal Khovanov homology (deformations of Kh) & other questions

Khovanov homology may be upgraded to <u>universal Khovanov homology</u>:

- label crossings by $\{1, 2, \dots, n\}$
- $v \in \{0,1\}^n \longrightarrow D_v$ = perform 0-resolution $\times \rightsquigarrow)($ at the i^{th} crossing if $v_i = 0$

  perform 1-resolution $\times \rightsquigarrow \asymp$ at the i^{th} crossing if $v_i = 1$

  This is an embedded disjoint union of circles in $\mathbb{R}^2$.

$$F^{Ab}_{Kh}(D)(v) = \bigotimes_{\pi_0(D_v)} \mathbb{Z}[h,t]\langle x_+, x_- \rangle$$

- $u \longrightarrow v$ edge of the cube
  - if $D_u \rightsquigarrow D_v$ merges two circles, apply m: $\begin{array}{l} x_+ \otimes x_+ \longmapsto x_+ \\ x_+ \otimes x_- \longmapsto x_- \\ x_- \otimes x_+ \longmapsto x_- \\ x_- \otimes x_- \longmapsto h x_- + t x_+ \end{array}$ (and identity on other components)
  - if $D_u \rightsquigarrow D_v$ splits a circle in two, apply Δ: $\begin{array}{l} x_+ \longmapsto x_+ \otimes x_- + x_- \otimes x_+ - h.x_+ \otimes x_+ \\ x_- \longmapsto x_- \otimes x_- + t.x_+ \otimes x_+ \end{array}$ (and identity on other components)
- add signs to certain edges
- take $\bigoplus$ within each homological grading

  $\rightsquigarrow$ universal Khovanov chain complex

Quantum gradings:

x_+	$+1$
x_-	-1
h	-2
t	-4

universal Khovanov chain complex $\xrightarrow{\text{set } (h,t)=(0,0) \text{ and take homology}}$ Khovanov homology (over $\mathbb{Z}$) (bigraded)

universal Khovanov chain complex $\xrightarrow{\text{set } (h,t)=(0,1) \text{ and take homology}}$ Lee homology (over $\mathbb{Z}$) (graded + filtered)

universal Khovanov chain complex $\xrightarrow{\text{set } t=0 \text{ and take homology}}$ Bar-Natan homology (over $\mathbb{Z}[h]$) (bigraded)

if we set: (homological, quantum) grading of $h = (0,-2)$

universal Khovanov chain complex $\xrightarrow{\text{take homology}}$ universal Khovanov homology (over $\mathbb{Z}[h,t]$) (bigraded)

if we set:
(homological, quantum) grading of $h = (0,-2)$
(homological, quantum) grading of $t = (0,-4$

Question Can any of these deformations of Khovanov homology be spectrified?

↳ **Question** If we specialize (h,t) to integers, can the Khovanov cube in Ab be lifted to a cube in the Burnside category?

Rmk Clearly <u>no</u> unless $h = 0$ and $t \geq 0$ (because sets cannot have negative cardinality).

[LLS] Also impossible for $(h,t) = (0,1)$

Question

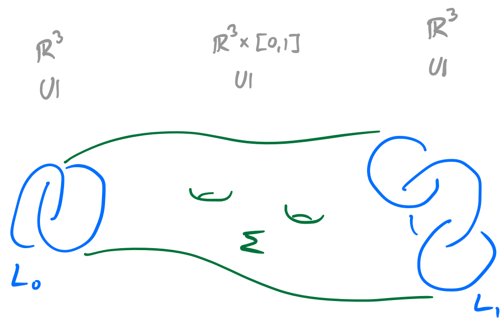
- Can Seidel-Smith's description of Khovanov homology via Floer theory be upgraded to produce a flow category refining the Khovanov chain complex and then a Khovanov spectrum via the construction of Cohen-Jones-Segal ?
- If yes, is it homotopy equivalent to $\underbrace{\mathcal{X}_{Kh}(-)}_{\text{(Lawson-)Lipshitz-Sarkar}}$?

Spectrification of Khovanov homology III

GeMAT seminar, IMAR
8 April 2022

Extension to tangles & cobordisms

Aim:



$$\Big\downarrow$$

$$X_{Kh}(L_0) \xrightarrow[X_{Kh}(\Sigma)]{} X_{Kh}(L_1) \qquad \text{(up to } \pm)$$

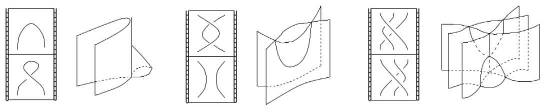
Plan:

- How this is done at the level of $Kh(-)$
  ↳ incl. extension to tangles
  [Jacobsson, Khovanov, Bar-Natan]
- Why the argument does not lift directly to $X_{Kh}(-)$
- How to fix this [Lawson-Lipshitz-Sarkar]

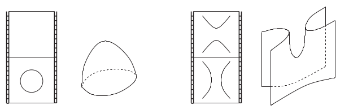
[Carter - Saito '93 / Carter - Rieger - Saito '97]

Presentation of the embedded cobordism category via

- elementary cobordisms

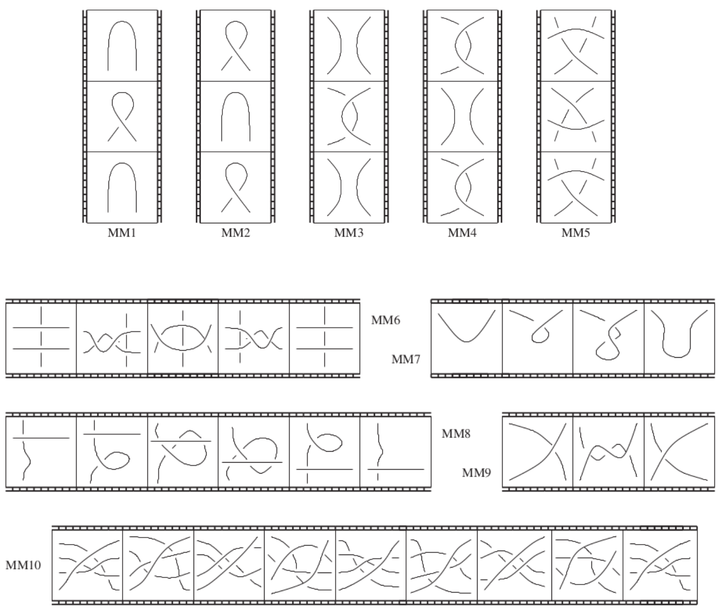


(Reidemeister)

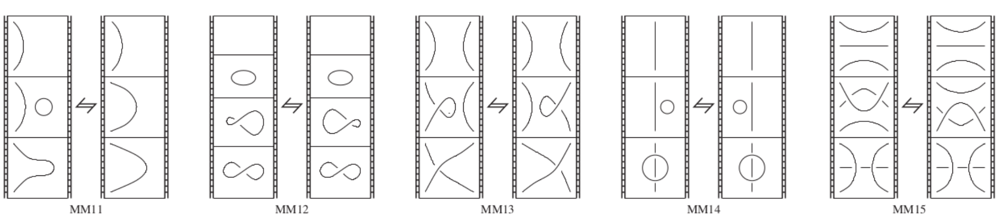


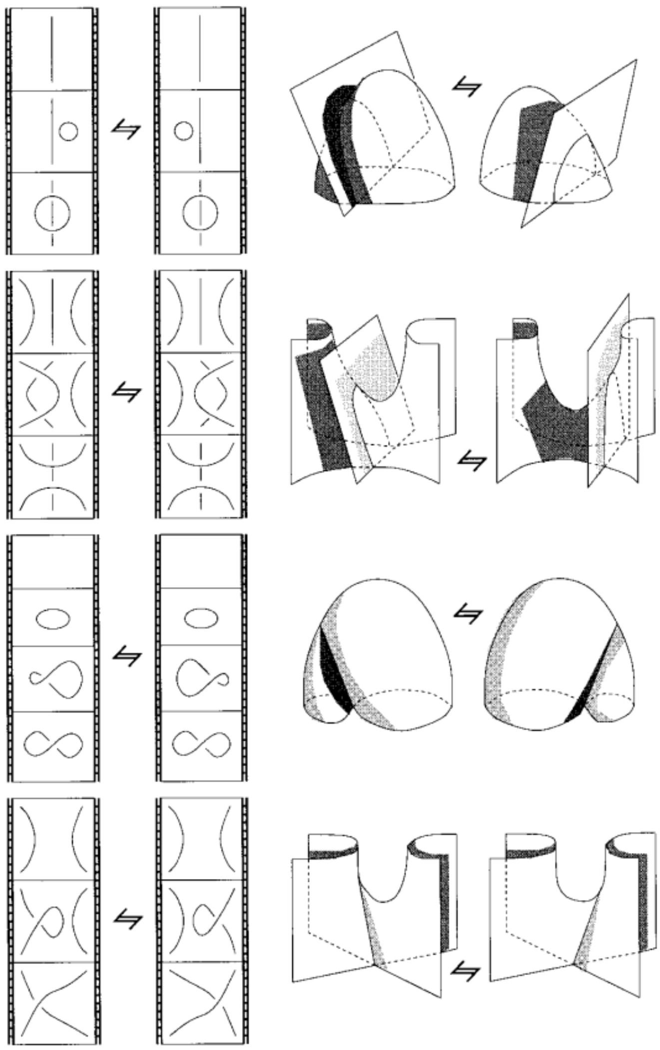
(Morse)

- movie moves — relations between finite sequences of elementary cobordisms



The movie depicted is equivalent to the identity movie.





3-dim interpretation of the last 4 moves

- There are "obvious" candidates for how to define $Kh(-)$ on elementary cobordisms.

- $\rightsquigarrow$ "Just" need to verify that the movie move relations are satisfied. (up to $\pm$)

  (15 in the formulation of [CS] — smooth movies
  31 in the formulation of [CRS]) — movies with finitely many frames

[Jacobsson '04]

Explicit verification by hand.

[Khovanov '05]

Extended $Kh(-)$ to tangles (composable by stacking)

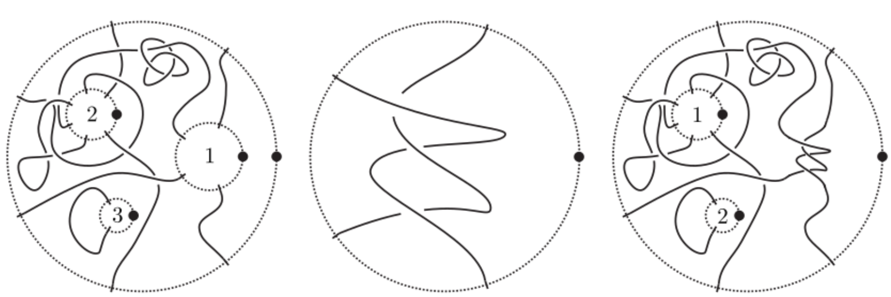
$\hookrightarrow$ Suffices to check the movie moves on "local" (tangle) pieces

$\rightsquigarrow$ less brute-force computation required

[Bar-Natan '05]

Extended $Kh(-)$ to tangles (composable "in all directions")

i.e. respecting their structure as a planar algebra [V. Jones]



T $\qquad$ S $\qquad$ $T \circ_1 S$

$\hookrightarrow$ Even more localised

(almost) no brute-force computation required!

Instead

- Most movie moves (MM1 — MM10) are of the form:

movie (reversible)

$$\begin{tikzcd} T \arrow[r, bend left, "M"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\Uparrow"] & T \end{tikzcd}$$

tangle with no closed components

<u>Proposition</u> [BN] If T has no closed components, then

$$\mathrm{Aut}(\underbrace{KhC}_{\text{Khovanov complex}}(T)) = \{\pm 1\}$$

<u>Corollary</u> $Kh(M) = \pm\, id.$

Idea of proof

First: where does $KhC(T)$ live?

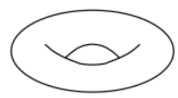
Tangle $\rightsquigarrow$ formal chain complex of matrices with entries in $\mathbb{Z}[Cob_{1+1}(D^2)] / \sim$

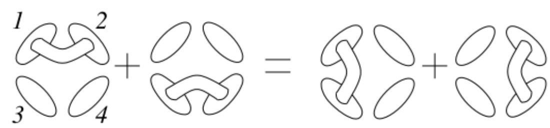
take the Khovanov cube of smoothings, but do not apply the TQFT / Frobenius algebra yet

obj: 1-manifolds in D^2
mor: 2-manifolds in $D^2 \times [0,1]$

three local relations:

 $= 0$ (S)

 $= 2$ (T)



(4Tu)

All arguments take place at this level.

Afterwards, one may apply the construction

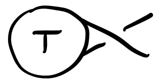
$$\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{Morphisms}(\phi, \quad) \Big/ \left(\begin{array}{l} S = 0 \text{ for any closed} \\ \text{surface } S \text{ of genus} \geq 2 \end{array} \right)^{*}$$

see pages 10 – 11 ...

which recovers the usual Khovanov complex (over $\mathbb{Z}_{(2)}$) when T is a link, and take homology.

Lemma If $\mathrm{Aut}(\mathrm{KhC}(T)) = \{\pm 1\}$,

then the same is true for



Idea: Formal, using the planar algebra structure of tangles.

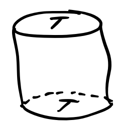
Lemma If T has no closed components and no crossings (T is a <u>matching</u>),

then $\mathrm{Aut}(\mathrm{KhC}(T)) = \{\pm 1\}$

Sketch

- The cube of smoothings is 0-dim, i.e. just one vertex
- So $\mathrm{KhC}(T) = T$ ← in degree 0, no differential.

$\mathrm{Aut}(\mathrm{KhC}(T))$ = formal $\mathbb{Z}$-linear combinations of self cobordisms



(that are equal to $\partial T \times [0,1]$ on $\partial D^2 \times [0,1]$) $(**)$

- In order to preserve homological degree (o/w it cannot be invertible since $\mathrm{KhC}(T) = \emptyset$ in all other degrees)

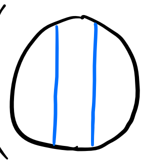
  each cobordism C must satisfy

$$\chi(C) = |\pi_0(\partial C)| \qquad (*)$$

- We may assume by (T) that C has no torus components.
- There are no sphere components, by (S).
- If there is a Σ_g component for $g \geq 2$, then by $(*)$ there would also have to be a sphere component. ⨳
- Hence each component of C is a disc ⇝ C is the identity cobordism.
- ⇒ Each automorphism is a $\mathbb{Z}$ multiple of the identity.

  Since it is invertible it must be $\pm\mathrm{id}$. //

Why does this not lift to $\mathcal{X}_{Kh}$?

$KhC\Big($  $\Big)$ has a non-trivial automorphism!

i.e. not $\pm id$.

sphere spectrum

$\longleftrightarrow \quad \pi_1(\mathbb{S}) \neq 0$

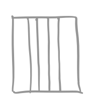
$\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ generated by the Hopf map

$\Sigma^\infty(\eta : S^3 \to S^2)$

How to bypass this problem?

(0,m)-tangles

Prop (LLS) $\quad \mathrm{Hom}_{KhC(id_m)}\big(KhC(T_1), KhC(T_2)\big) \cong KhC(\underbrace{\hat{T}_1, T_2})$

identity (0,m)-tangle 

link formed by gluing
$\hat{T}_1 = T_1$ upside-down
to T_2.

Idea Most movie moves are of the form $\textcolor{red}{M \leftrightarrows \mathrm{id}}$ for

$$T \xrightarrow{\textcolor{red}{M}} T.$$

"bridge tangle"

By the above proposition,

$$\mathrm{KhC}(\textcolor{red}{M}) \in \mathrm{Hom}_{\mathrm{KhC}(\mathrm{id}_m)}\big(\mathrm{KhC}(T), \mathrm{KhC}(T)\big) \cong \mathrm{KhC}(\underbrace{\hat{T}\,T}_{\text{unlink}})$$

and the right-hand side is $\cong \mathbb{Z}$

(in the bidegree under consideration)

hence $\mathrm{KhC}(\textcolor{red}{M})$ is a scalar multiple of the identity.

This re-proves functoriality of $\mathrm{KhC}(-)$.

Moreover, this alternative strategy <u>does</u> lift directly to $\mathcal{X}_{Kh}(-)$!

<u>Why?</u> Roughly:

- The first strategy fundamentally depends on $\mathrm{KhC}($  $)$ having no non-trivial automorphisms. But the existence of non-trivial elements of $\pi_i(\mathbb{S})$, $i>0$, means that this becomes false when lifted to $\mathcal{X}_{Kh}$.

- The second strategy fundamentally depends on $\mathrm{KhC}(\textcolor{blue}{\text{unlink}})$ being (essentially) trivial. When lifting to $\mathcal{X}_{Kh}$, we instead need $\pi_0(\mathcal{X}_{Kh}(\textcolor{blue}{\text{unlink}}))$ to be trivial. But $\mathcal{X}_{Kh}(\textcolor{blue}{\text{unlink}}) \simeq \mathbb{S}$ (in the appropriate quantum degree), so this remains true.

Addendum

Recall that after taking the Khovanov chain complex $KhC(T)$ — a formal chain complex over $Cob_{n+1}(D^2)/\sim$

1-mflds in D^2
2-mflds in $D^2\times[0,1]$

(S)
(T) relations
(4Tu)

we then apply the operation

$$\underbrace{\mathbb{Z}_{(2)}\otimes_{\mathbb{Z}}}\mathrm{Morphisms}(\phi,\quad)\Big/\left(\begin{array}{l}S=0 \text{ for any closed}\\ \text{surface } S \text{ of genus} \geq 2\end{array}\right)$$

invert 2 in the coefficients

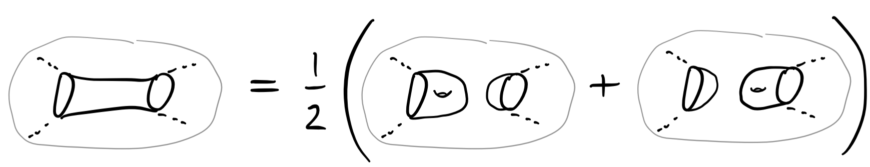
to obtain a cube where each vertex v is sent to a $\mathbb{Z}$-linear combination of nullbordisms of the smoothing of T corresponding to v.

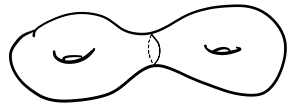
modulo the relations
- $S^2=0$
- $T^2=2$
- $4Tu$
- $\Sigma_g=0 \quad (g\geq 2)$

(This turns out to identify with the usual Khovanov complex when $T=L$ is a <u>link</u>, so this extends $Kh(-)$ to tangles.)

Rmk. Above, the relation $\Sigma_g=0 \quad (g\geq 2)$ is equivalent to $\Sigma_3=0$.

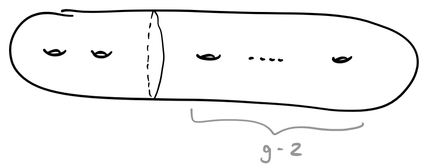
Proof: Since 2 is invertible, a special case of the 4Tu relation says:



Applying this to $\Sigma_2 =$  we obtain

$$\begin{aligned}\Sigma_2 &= \tfrac{1}{2}\left(\Sigma_2 \sqcup \Sigma_1 + \Sigma_1 \sqcup \Sigma_2\right)\\ &= \Sigma_2 \sqcup \Sigma_1 \\ &= 2 . \Sigma_2 \qquad \text{(by relation (T))}\end{aligned}$$

Hence $\underline{\Sigma_2 = 0}$.

Applying the relation to $\Sigma_g =$ 

$(g \geq 2)$

we obtain

$$\begin{aligned}\Sigma_g &= \tfrac{1}{2}\Big(\Sigma_3 \sqcup \Sigma_{g-2} + \underbrace{\cancel{\Sigma_2 \sqcup \Sigma_{g-1}}}_{0 \text{ by above}}\Big)\\ &= \tfrac{1}{2}\left(\Sigma_3 \sqcup \Sigma_{g-2}\right).\end{aligned}$$

Inductively:

$$\Sigma_{2g} = \frac{1}{2^g}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g} \sqcup \overset{0 \text{ by relation (S)}}{\cancel{\Sigma_0}}\Big) = 0$$

$$\Sigma_{2g+1} = \frac{1}{2^g}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g} \sqcup \Sigma_1\Big) = \frac{1}{2^{g-1}}\Big(\underbrace{\Sigma_3 \sqcup \cdots \sqcup \Sigma_3}_{g}\Big)$$

by relation (T)

In particular, adding the relation $\Sigma_3 = 0$ automatically implies that $\Sigma_g = 0$ for all $g \geq 2$. //.

Rmk. We could instead set $\Sigma_3 = k$ for any other $k \in \mathbb{Z}$.

Then $\Sigma_{2g+1} = \dfrac{k^g}{2^{g-1}}$.

When $k = 8$, this recovers Lee homology.