31-5-22

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Homological stability for asymptotic monopole moduli spaces

(joint with U. Tillmann).

EPFL

Topology seminar.

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Plan
- "Physics" motivation (1)
- Monopole moduli spaces $\mathcal{M}_k$ (2)
- Asymptotic — " — (3)
- Homological stability. (4)

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(1) "Physics"

Electromagnetism : Maxwell's Equations

- E electric field , B magnetic field , etc.
- asymmetric under $E \longleftrightarrow B$
- but maybe extended to be symmetric :

$$\nabla \cdot E = 4\pi \rho_e$$

$$\nabla \cdot B = 4\pi \rho_m$$

$$-\nabla \times E = \frac{1}{c}\frac{\partial B}{\partial t} + \frac{4\pi}{c} j_m$$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} j_e$$

$$F = q_e\left(E + \frac{v}{c}\times B\right) + q_m\left(B - \frac{v}{c}\times E\right)$$

$\rho_{e/m}$ charge density

$q_{e/m}$ charge of particle

$j_{e/m}$ current density

F force on particle

classically : $\rho_m = j_m = q_m = 0$

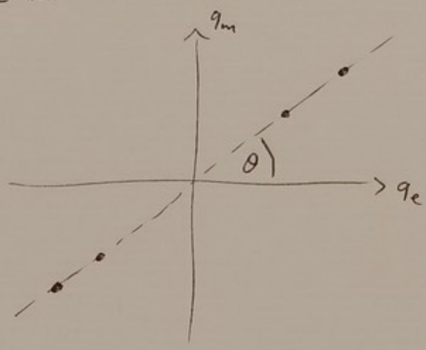
- invariant under duality transformations = action of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$

$\theta \in \mathbb{R}/2\pi\mathbb{Z}$ acts by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ on $\begin{pmatrix} q_e \\ q_m \end{pmatrix}$, $\begin{pmatrix} E \\ B \end{pmatrix}$, $\begin{pmatrix} j_e \\ j_m \end{pmatrix}$, etc.

<u>Obs</u> $\forall$ charged particles observed so far, the ratio

$$[\,q_m : q_e\,] \in \mathbb{RP}^1$$

is the <u>same</u>:



θ may be fixed at any value via <u>duality transformations</u>.

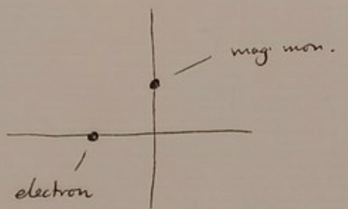
<u>Usual convention</u> : electron has $\begin{pmatrix} q_e \\ q_m \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so $\theta = 0$.

so all observed particles have $q_m = 0$

<u>Dirac</u> (1931) $\exists$ of magnetic monopoles — particles of charge $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Downarrow$

quantisation of electric (and magnetic) charges

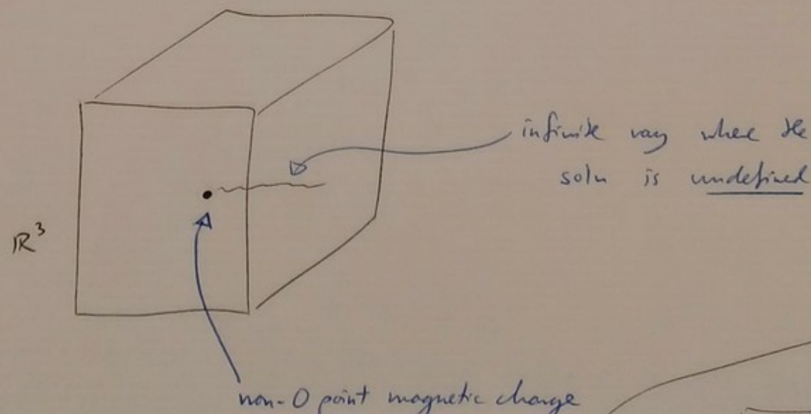


angular momentum of EM field $\leftarrow$ quantised by QM

$\|$

$q_m \cdot q_e$

<u>Dirac's solutions</u> :



"Dirac string".

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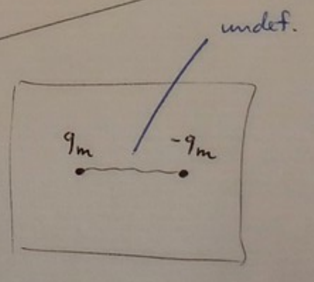
unobservable

"singular solns of M.E."

<u>Also</u> :



't Hooft - Polyakov monopoles ('70s)

simplified model : $\underbrace{\text{B}}_{\text{Bogomolny}}\underbrace{\text{PS}}_{\text{Prasad - Sommerfield}}$

- $\underbrace{\text{"smooth"}}_{\text{def. on all of } \mathbb{R}^3}$ solns of a different set of equations
- At large distances, behave like Dirac monopoles.

Bogomolny equations

$$\underbrace{\phi}_{\text{Higgs field}}, \underbrace{A}_{\text{connection}} : \mathbb{R}^3 \longrightarrow su(2) \cong \mathbb{R}^3 \qquad \text{smooth}$$

- $\underbrace{D\phi}_{\text{cov. derivative}} = \underbrace{* F_A}_{\text{curvature of } A}$
- $|\phi|_2 = 1 - \frac{k}{2r} + O(r^{-2}) \qquad (r \gg 0)$

<u>Ref:</u> [Atiyah - Hitchin '82]

$\mathcal{N}$ = monopole moduli space

$:= \left\{ \phi, A : \mathbb{R}^3 \to \mathbb{R}^3 \;\middle|\; \text{eq}^{\text{n}}\text{s above} \right\} \big/ \cong$

$$\phi(x) \neq 0 \quad \text{for} \quad |x|_2 \geq R \gg 0$$

$$\phi : \mathbb{R}^3 \setminus B_R(0) \longrightarrow \mathbb{R}^3 \setminus \{0\}$$

$\deg(\phi) =:$ "total charge" of the solution (ϕ, A). $\quad \in \mathbb{Z}$

(always > 0).

$$\mathcal{N} = \coprod_{k \geq 1} \underbrace{\mathcal{N}_k}_{\text{path-connected}}$$

(2) Monopole moduli space

<u>Aim:</u> Understand $\mathcal{N}_k$ topologically.

What is known:

- $\mathcal{N}_k$ admits a principal S^1-bundle $\mathcal{M}_k \longrightarrow \mathcal{N}_k$

  quotient only by gauge transf's that fix a direction in $\mathbb{R}^3$.

- translation in $\mathbb{R}^3$ defines $\mathcal{N}_k \longrightarrow \mathcal{N}_k/_{\mathbb{R}^3} =: \mathcal{N}_k^\circ$ principal $\mathbb{R}^3$-bundle.

- $\mathcal{M}_k$ is a $4k$-dim hyperKähler mfld
  $\mathcal{N}_k^\circ$ — $(4k-4)$ ——

- $\mathcal{M}_1 = S^1 \times \mathbb{R}^3$
  $\mathcal{M}_2$ studied by [Atiyah-Hitchin]

- $\pi_1(\mathcal{M}_k) \cong \mathbb{Z}$ $\quad \pi_1(\mathcal{N}_k) \cong \mathbb{Z}/_k$

- [Donaldson '84] + [Boyer-Mann '88] $\mathcal{M}_k \cong \mathrm{Rat}_k^*(\mathbb{CP}^1) = \left\{ \mathbb{CP}^1 \xrightarrow{f} \mathbb{CP}^1 \mid \text{rational, degree} = k,\ f(\infty) = 1 \right\}$
  $\subseteq SP^k(\mathbb{C}) \times SP^k(\mathbb{C})$

  (if instead we set $f(\infty) = 0$ then the S^1-action is evident)

- [Segal '79] - $\exists$ maps $\mathcal{M}_k \to \mathcal{M}_{k+1}$ inducing <u>isoms</u> on π_i for $i \leq k$
  - $\mathrm{hocolim}(\mathcal{M}_k) \simeq \Omega_0^2 S^2$

  Hence $H_i(\mathcal{M}_\infty) \cong H_i(B_\infty)$

  stable H_* of braid groups

- [Cohen-Cohen-Mann-Milgram '91]

$$\boxed{\mathcal{M}_k \simeq_s B(B_{2k})}$$

$$\underbrace{\mathcal{M}_k \simeq_s}_{\text{[CCMM]}} \bigvee_{j=1}^{k} D_j(S^1) \underbrace{\simeq_s}_{\text{[Brown-Peterson '78]}} B(B_{2k})$$

$$D_j(A) = F_j(\mathbb{R}^2)_+ \underset{\Sigma_j}{\wedge} (A^{\wedge j})$$

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③ Asymptotic monopole moduli spaces

[Kottke-Singer '15 / '22]
arXiv MAMS

- Partial compact^n of $\mathcal{M}_k$ to a manifold with boundary
- Extended to a full compact^n to a cpt manifold with corners

by [Fritzsch-Kottke-Singer '18 + future]

$$\overline{\mathcal{M}}_k = \bigcup_{\substack{\lambda \text{ unordered} \\ \text{partition of } k \\ \lambda = \{k_1, \dots, k_r\}}} \mathcal{M}_\lambda$$

- $\mathcal{M}_\lambda$ = moduli space of "asymptotic" monopoles with r "clusters" of magnetic charges $k_1, \dots, k_r$ that are "very far apart" from each other.
- $\mathcal{M}_{(k)} = \mathcal{M}_k = \mathrm{int}(\overline{\mathcal{M}}_k)$
- for λ nontrivial, $\mathcal{M}_\lambda$ is a boundary face of $\overline{\mathcal{M}}_k$.

[KS] · construct the space $\overline{\mathcal{M}}_k$

· give a topological description of each stratum $\mathcal{M}_\lambda$.

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4 Homological stability for $\mathcal{M}_\lambda$.

Thm (P.-Tillmann '22)

Fix $c \geq 1$ and for $\lambda = \{k_1, \dots, k_r\}$

$$\lambda[n] = \{k_1, \dots, k_r, \underbrace{c, \dots, c}_{n}\}$$

$\exists$ map $\mathcal{M}_{\lambda[n]} \to \mathcal{M}_{\lambda[n+1]}$

inducing isoms on $H_i(-)$ for $i \leq n/2$.

Idea of proof

- Description of $\mathcal{M}_\lambda$ as "non-local" config. spaces
- General HS thm for non-local config spaces
  - ↳ Proof uses twisted hom. stab. for (ordinary) config. spaces
    - ↳ [P. '18] or [Krannich '19]
- Why the stab^n maps for $C_n(\mathbb{R}^3)$ lift to $\mathcal{M}_{[n]}$. //.

Description of $\mathcal{M}_\lambda$

$$H^2(F_n(\mathbb{R}^3);\mathbb{Z}) \cong \mathbb{Z}\{\alpha_{ij} : 1 \leq i < j \leq n\}$$

α_{ij} = pullback of (fixed) generator of $H^2(S^2)$ along

$$\begin{aligned} F_n(\mathbb{R}^3) &\longrightarrow S^2 \\ (x_1, \dots, x_n) &\longmapsto \frac{x_i - x_j}{|x_i - x_j|} \end{aligned} \qquad (\alpha_{ji} = -\alpha_{ij})$$

principal S^1-bundles on $X \xleftrightarrow{1:1} H^2(X;\mathbb{Z})$

Def $\lambda = \{k_1, \dots, k_n\}$
$1 \leq j \leq n$

$$\begin{array}{c} S^j_\lambda \\ \downarrow \\ F_n(\mathbb{R}^3) \end{array} := \text{the prin. } S^1\text{-bundle corresponding to } \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i \cdot \alpha_{ij}$$

The S^1 parameter encodes the interaction of the j^{th} point with all other points (weighted by λ).

$$\begin{array}{c} T_\lambda := \bigoplus_{j=1}^{n} S^j_\lambda \\ \downarrow \\ F_n(\mathbb{R}^3) \end{array}$$

principal $(S^1)^n$-bundle whose n circle parameters encode the pairwise interactions of config. points (weighted by λ).

Terminology: $T_{(1,\dots,1)}$ = the Gibbons-Manton torus bundle

T_λ = the generalised —"— for λ.

Recall that $\mathcal{M}_K$ has a (free) S^1-action.

Consider the Borel construction:

$$\begin{array}{c} (\Upsilon_\lambda) \times_{(S^1)^n} \left(\mathcal{M}_{K_1} \times \cdots \times \mathcal{M}_{K_n}\right) \\ \downarrow \\ F_n(\mathbb{R}^3) \end{array}$$

Let $\Sigma_\lambda \subseteq \Sigma_n$ be the stabiliser of $\lambda \in \mathbb{N}^n$ (under the permutation action)

Then the action of Σ_λ on $F_n(\mathbb{R}^3)$ lifts to the total space, and:

<u>Thm</u> [Kotthe-Singer]

$$\begin{array}{ccc} \mathcal{M}_\lambda & \cong & \left[(\Upsilon_\lambda) \times_{(S^1)^n} \left(\mathcal{M}_{K_1} \times \cdots \times \mathcal{M}_{K_n}\right)\right] \big/ \Sigma_\lambda \\ \downarrow & & \\ C_\lambda(\mathbb{R}^3) & = & F_n(\mathbb{R}^3) \big/ \Sigma_\lambda \end{array}$$

In particular, we have

$$\begin{array}{ccc} \mathcal{M}_{[n]} & \cong & \left[(\Upsilon_{[n]}) \times_{(S^1)^n} \left(\mathcal{M}_c \times \cdots \times \mathcal{M}_c\right)\right] \big/ \Sigma_n \\ \downarrow & & \\ C_n(\mathbb{R}^3) & & \end{array}$$

$$\left([n] = \{\underbrace{c, \ldots, c}_{n}\} \right)$$

<u>Thm</u> [P.-Tillmann '22] (Homological stability for config. spaces with "non-local labels")

M connected, open manifold

$$\begin{tikzcd} E_n \arrow[r, "\tilde{s}_n"] \arrow[d, two heads, "f_n"] & E_{n+1} \arrow[d, two heads, "f_{n+1}"] \\ C_n(M) \arrow[r, "s_n"'] & C_{n+1}(M) \end{tikzcd}$$

fibrations

classical stabilisation map

basepoint $c_n \in C_n(M)$

Z based, connected space

Suppose (a) $f_n^{-1}(c_n) = Z^n$

(b) monodromy of f_n :
$$\begin{tikzcd} \pi_1 C_n(M) \arrow[rr] \arrow[dr, two heads] & & hAut(Z^n) \\ & \Sigma_n \arrow[ur] & \end{tikzcd}$$

"obvious" maps

(c) $\tilde{s}_n\big|_{Z^n}$:
$$\begin{tikzcd} Z^n \arrow[rr] \arrow[dr, "\cong"'] & & Z^{n+1} \\ & \{*\} \times Z^n \arrow[ur, hook] & \end{tikzcd}$$

inclusion.

Then $E_n \xrightarrow{\tilde{s}_n} E_{n+1} \xrightarrow{\tilde{s}_{n+1}} E_{n+2} \longrightarrow \cdots$ is <u>homologically stable</u>.

<u>Eg</u> (<u>local</u> labels) — fix a fibration $\begin{matrix} E' \\ \downarrow \pi \\ M \end{matrix}$

and set $E_n = \left\{ \{y_1, \ldots, y_n\} \in C_n(E') \;\middle|\; \pi(y_i) \neq \pi(y_j) \text{ for } i \neq j \right\}$

$$\begin{matrix} E_n \\ \downarrow \\ C_n(M) \end{matrix}$$

(monopole labels) — $\begin{matrix} \mathcal{M}_{[n]} \\ \downarrow \\ C_n(\mathbb{R}^3) \end{matrix}$ (Lemma)

non-local!

(Gibbons-Manton labels) ⟿ More generally, for any S^1-space A:

$$\begin{array}{c} \left[\left(\Upsilon_{[n]} \right) \times_{(S^1)^n} \left(A^n \right) \right] / \Sigma_n \\ \Big\downarrow \\ C_n(\mathbb{R}^3) \end{array}$$

[+ analogues of $\Upsilon_{[n]}$ for $C_n(\mathbb{R}^d)$ with fibre $K(\mathbb{Z}, d-2)^n$]

Sketch of proof (of HS for non-local config spaces)

- Serre spectral sequence

$$H_p(C_n(M); H_q(Z^n)) \Rightarrow H_*(E_n)$$

- $n \mapsto H_q(Z^n)$ extends to a polynomial coeff. system of degree q on $\{C_n(M)\}$.

- $\underbrace{[P. '18] \text{ or } [\text{Krannich } '19]}_{\text{twisted homological stability for (ordinary) configuration spaces.}}$ $\Rightarrow$ stability on each E^2_{pq} for $p \leq \frac{1}{2}(n-q)$

- $\Rightarrow$ stability for $\{E_n\}$.