Martin Palmer-Anghel | MPIM, 28 June 2022

The homology of configuration-section spaces & asymptotic monopole moduli spaces

(1)

joint with Ulrike Tillmann

$$\begin{array}{c} X \hookrightarrow E \\ \downarrow \xi \\ M^d \end{array}$$

$\partial M \neq \emptyset$

$$C_n(M) = \{(x_1,\dots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\} / \Sigma_n$$

$$C\Gamma_n(\xi) = \{\underline{x} \in C_n(\mathring{M}),\ \text{section of } \xi \text{ on } M \smallsetminus \underline{x}\}$$

$$= \frac{\Gamma(\xi|_{M \smallsetminus \underline{x}}) \times \mathrm{Aut}(\xi)}{\mathrm{Aut}(\xi, \underline{x})}$$

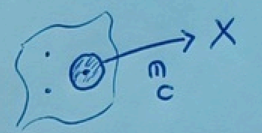
(2)

Local behaviour near points:

Case $\begin{array}{c} E = M \times X \\ \downarrow \\ M \end{array}$

fix $c \in [S^{d-1}, X]$

$C\Gamma_n^c(\xi)$ = subspace where



In general $\exists$ covering space $\Sigma(\xi)$ with fibres $[S^{d-1}, E_x]$

$$\begin{array}{c} \Sigma(\xi) \\ \downarrow \\ M \ni x \end{array}$$

fix $c \in \pi_0(\Sigma(\xi)) \twoheadleftarrow [S^{d-1}, X]$

(3)

Ex $\quad c \in [S^1, BG] = \mathrm{Conj}(G)$

(1) $\begin{array}{c} \mathbb{R}^2 \times BG \\ \downarrow \\ \mathbb{R}^2 \end{array}$ Hurwitz space $\left\{ \text{branched } G\text{-coverings of } \mathbb{R}^2 \text{ with monodromy in } c \right\}$

(2) $\begin{array}{c} TM \smallsetminus 0 \\ \downarrow \\ M \end{array}$ $\left\{ \text{configuration + non-vanishing v. field on the complement} \right\}$

(3) $\begin{array}{ccc} \tau_M^* \theta & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \theta \\ M & \xrightarrow{\tau_M} & BO(d) \end{array}$ $\left\{ \text{configuration + } \theta\text{-structure on the complement} \right\}$

(4)

$$c \in \pi_0(\Sigma(\xi)) \twoheadleftarrow [S^{d-1}, X] \twoheadleftarrow \pi_{d-1}(X)$$

Thm (P.-Tillmann '21) If the pre-image of c in $\pi_{d-1}(X)$ is one element, then $\exists$

$$C\Gamma_n^c(\xi) \longrightarrow C\Gamma_{n+1}^c(\xi)$$

inducing $\cong$ on H_* for $* \leq n/2 - 2$.

(5)

(1) condition $\longleftrightarrow$ $c \in Z(G)$

(2) $c \in [S^{d-1}, S^{d-1}] \cong \mathbb{Z}$ $\Big\}$ M parallelisable

(3) e.g. $B = BSpin(d) \rightsquigarrow \pi_{d-1}(X) = 0$ if $d \geq 3$ $\Big\}$ M parallelisable

[Ellenberg-Venkatesh-Westerland '16]

Periodic rational homological stability for Hurwitz spaces when $G = A \rtimes \{\pm 1\}$ and $c = A \times \{-1\}$

(A finite ab. group of odd order)

(6)

Idea of proof

- Reduce via Serre SS to twisted hom. stab^y for $C_n(M)$
- Prove that the coefficients form a polynomial coeff system
- Apply [Kranhold '19] or [P. '18]

Monopole moduli space

(7)

$$\mathcal{M} = \{\text{soln's of Bogomolny equations}\}/\cong$$
$$= \{\phi, A : \mathbb{R}^3 \longrightarrow su(2) = \mathbb{R}^3 \mid \cdots\cdots\cdots\}/\cong$$
$$= \coprod_{k \geq 1} \mathcal{M}_k \circlearrowleft S^1$$

space of magnetic monopoles in $\mathbb{R}^3$ of total charge k.

[Donaldson '84] $\mathcal{M}_k \cong Rat_k^*(\mathbb{CP}^1) \subseteq SP^k(\mathbb{C}) \times SP^k(\mathbb{C})$

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[Segal '79] $\exists$ maps $\mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}$ inducing $\cong$ on π_i for $i \leq k$.

$$\underset{\overrightarrow{k}}{\operatorname{hocolim}}(\mathcal{M}_k) \simeq \Omega^2_0 S^2$$

[Cohen-Cohen-Mann-Milgram '91]

$$\mathcal{M}_k \simeq_s \bigvee_{j=1}^{k} D_j(S^1) \simeq_s C_{2k}(\mathbb{R}^2)$$

[Brown-Peterson '78]

Asymptotic monopoles:

(9)

(replace $\mathcal{M}_k$ with $\mathcal{M}_k/\mathbb{R}^3$)

[Kottke-Singer '15]: Partial compactification

$$\overline{\mathcal{M}}_k = \bigcup_{\substack{\lambda \\ \text{partition of } k}} \mathcal{M}_\lambda$$

"widely separated clusters of magnetic charges"

int. $= \mathcal{M}_{(k)} = \mathcal{M}_k$

(10)

Thm (P. - Tillmann '22)

Fix $c \geq 1$ and for $\lambda = \{k_1, \dots, k_r\}$ write $\lambda[n] = \{k_1, \dots, k_r, \underbrace{c, \dots, c}_{n}\}$

$\exists$ maps $\mathcal{M}_{\lambda[n]} \longrightarrow \mathcal{M}_{\lambda[n+1]}$ inducing $\cong$ on H_i for $i \leq n/2$.

Idea:
- Use Kottke-Singer's description of $\mathcal{M}_\lambda$ as a non-local configuration space.
- Prove a gen. hom. stab^y result for non-local config. spaces via twisted hom. stab^y as before.

(11)

Principal S^1-bundles on $F_r(\mathbb{R}^3) \xleftrightarrow{1:1} H^2(F_r(\mathbb{R}^3); \mathbb{Z})$

$$= \mathbb{Z}\{\alpha_{ij} \mid 1 \leq i < j \leq r\}$$

α_{ij} pulled back along $F_r(\mathbb{R}^3) \longrightarrow S^2$

$$\underline{x} \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

$\lambda = \{k_1, \dots, k_r\}$

$$\begin{array}{c} T_\lambda = \bigoplus_{j=1}^{r} S_{\lambda,j} \longleftrightarrow \sum_{\substack{i=1 \\ i \neq j}}^{r} k_i \cdot \alpha_{ij} \\ \downarrow \\ F_r(\mathbb{R}^3) \end{array}$$

"Gibbons-Manton torus bundle"

(12)

$$\begin{array}{c} \left(T_\lambda \times_{(T^r)} (\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r})\right)/\Sigma_\lambda \cong \mathcal{M}_\lambda \\ \downarrow \\ F_r(\mathbb{R}^3)/\Sigma_\lambda \end{array}$$

Thm [Kottke-Singer]

$\rightsquigarrow$ "non-local configuration space"