The homotopy type of the cobordism category.

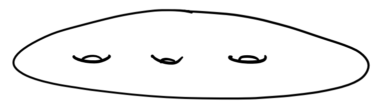
(after Galatius, Madsen, Tillmann and Weiss)

2 June 2022
GeMAT seminar
IMAR

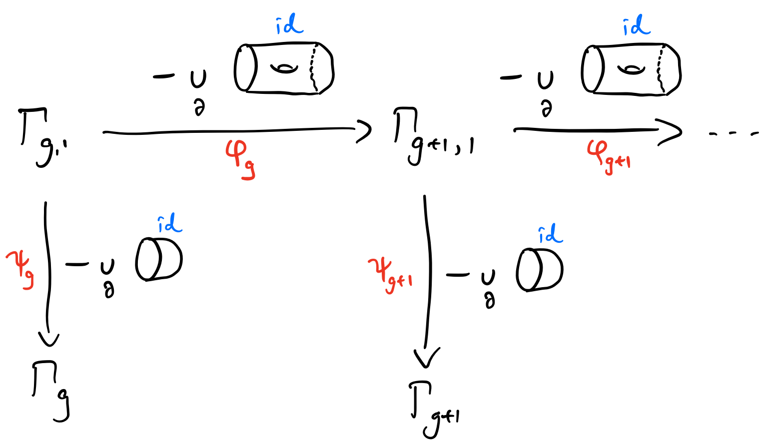
Plan

- Mumford conjecture
- Generalised —— " ——
- Cob. categories & the proof of GMTW modulo Theorem 1
- Idea of pf of Theorem 1.

The Mumford conjecture

$\Sigma_{g,1} =$  $\qquad \Sigma_g =$ 

$\Gamma_{g,1} = \pi_0 \mathrm{Diff}^+_\partial(\Sigma_{g,1}) \qquad \Gamma_g = \pi_0 \mathrm{Diff}^+_\partial(\Sigma_g)$



Thm [Harer '85] (Improvements by Ivanov, Boldsen, Randal-Williams)

$$\left.\begin{matrix}\varphi_g \\ \psi_g\end{matrix}\right\} \text{ induce } \cong \text{ on } H_*(-;\mathbb{Z}) \text{ up to degree } \begin{cases} \frac{2}{3}(g-1) \\ \frac{2}{3}g \end{cases}$$

Conj [Mumford '83]

Write $H^*(\Gamma_\infty) =$ inverse limit of $H^*(\Gamma_{g,1}) \leftarrow H^*(\Gamma_{g+1,1}) \leftarrow \cdots$

Then $H^*(\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$

for specific cohom. classes κ_i of degree $2i$.

For $g \geqslant 2$
$b \geqslant 0$

[Earle - Eells]

$$\Gamma_{g,b} \simeq \mathrm{Diff}_\partial^+(\Sigma_{g,b})$$ as topological groups

$$\underset{\text{classifying space}}{B\Gamma_{g,b}} \simeq \underbrace{B\mathrm{Diff}_\partial^+(\Sigma_{g,b})}$$

classifies smooth, oriented surface bundles with fibre $= \Sigma_{g,b}$ (and trivial boundary subbundle)

$$B\Gamma_g \simeq_{\mathbb{Q}} \underbrace{\mathcal{M}_g}_{\text{moduli space of alg. curves of genus } g}$$

$$H^*(\Gamma_{g,b}) \cong H^*(B\Gamma_{g,b}) \cong H^*(B\mathrm{Diff}_\partial^+(\Sigma_{g,b}))$$

$\cong$ ring of characteristic classes of smooth, oriented surface bundles with fibre $= \Sigma_{g,b}$ & trivial boundary subbundle.

Description of κ_i as a characteristic class

$$\begin{array}{c} E \\ {\scriptstyle\pi}\big\downarrow \\ B \end{array}$$ oriented surface bundle with fibre Σ

Assume that E, B are manifolds and π is smooth.

$$\begin{tikzcd} & TE \arrow[dl] \arrow[d, "D\pi"] \\ E \arrow[d, "\pi"'] & TB \arrow[dl] \\ B & \end{tikzcd}$$

$TE \supseteq \underbrace{\ker(D\pi)}$ vertical tangent bundle

(all tangent vectors on E that are parallel to fibres)

In fact, $TE \cong \pi^* TB \oplus \ker(D\pi)$.

$$\begin{array}{c} \ker(D\pi) \\ \big\downarrow \\ E \\ {\scriptstyle \pi}\big\downarrow \\ B \end{array}$$

oriented 2-dim $\mathbb{R}$ vector bundle

Ring of char. classes $= H^*(BSO(2)) \cong H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[e]$

e Euler class

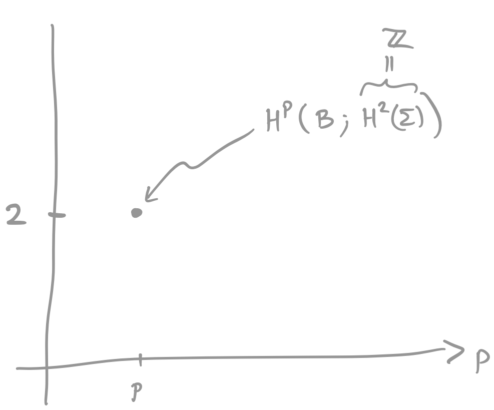
$|e| = 2$

$$e\begin{pmatrix} \ker(D\pi) \\ \downarrow \\ E \end{pmatrix} \in H^2(E)$$

$$e\begin{pmatrix} \ker(D\pi) \\ \downarrow \\ E \end{pmatrix}^{i+1} \in H^{2i+2}(E)$$

$$\int_\pi e\begin{pmatrix} \ker(D\pi) \\ \downarrow \\ E \end{pmatrix}^{i+1} \in H^{2i}(B)$$

$$\overset{!!}{=}\ K_i\begin{pmatrix} E \\ {\scriptstyle \pi}\downarrow \\ B \end{pmatrix}$$



$\Rightarrow H^*(E)$

$$\begin{tikzcd} H^{p+2}(E) \arrow[r, hook] \arrow[rrr, bend right, "\int_\pi -"'] & E_\infty^{p,2} \arrow[r, two heads] & E_2^{p,2} = & H^p(B) \end{tikzcd}$$

- κ_i characteristic class $\rightsquigarrow$ $\kappa_i \in H^{2i}(\Gamma_{g,b})$

  (or apply the construction to the "universal" $\Sigma_{g,b}$ - bundle over $B\mathrm{Diff}^+_\partial(\Sigma_{g,b})$)

- $(\varphi_g : \Gamma_{g,1} \longrightarrow \Gamma_{g+1,1})^*(\kappa_i) = \kappa_i$

  So this defines an element of the inverse limit

$$\underbrace{\kappa_i \in H^{2i}(\Gamma_\infty)}_{\text{Miller-Morita-Mumford classes}} = \varprojlim \left(H^{2i}(\Gamma_{g,1}) \longleftarrow H^{2i}(\Gamma_{g+1,1}) \longleftarrow \cdots \right).$$

Mumford conjecture

These freely generate $H^*(\Gamma_\infty; \mathbb{Q})$ as a graded $\mathbb{Q}$-algebra.

The generalised Mumford conjecture

Some constructions of Thom spectra

$$G(d,n) = \mathrm{Grass}_d^+(\mathbb{R}^{d+n})$$

$$\begin{array}{l} U_{d,n}^{\perp} = \{ (V,x) \in G(d,n) \times \mathbb{R}^{d+n} \mid x \perp V \} \\ \big\downarrow \\ G(d,n) \end{array}$$

oriented n-dim $\mathbb{R}$ v. bundle

fibre over V is $V^{\perp} \subseteq \mathbb{R}^{d+n}$

$$\underbrace{Th(U_{d,n}^{\perp})}_{\text{Thom space}} = \text{1-point compact}^n \text{ of } U_{d,n}^{\perp}$$

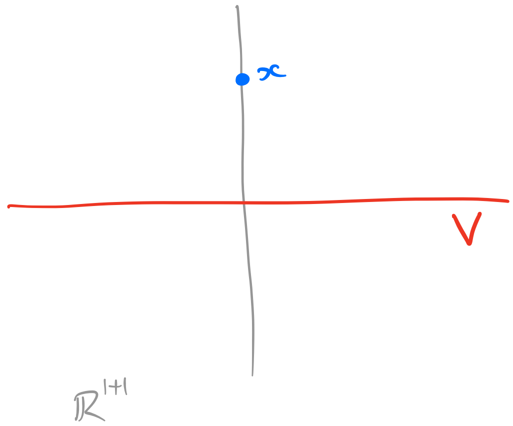
$$Th(U_{d,n}^{\perp}) \longrightarrow \Omega Th(U_{d,n+1}^{\perp}) = Map_*(S^1, Th(U_{d,n+1}^{\perp}))$$

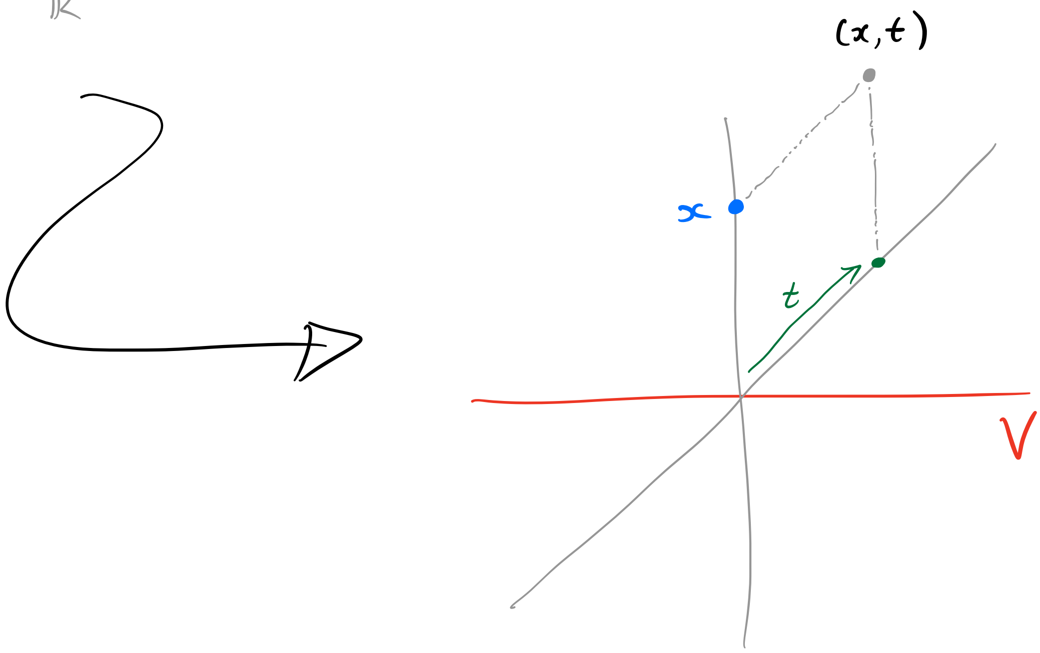
$$\left.\begin{array}{l} V \subseteq \mathbb{R}^{d+n} \\ x \in V^{\perp} \\ t \in S^1 = \mathbb{R} \cup \{\infty\} \end{array}\right\} \longmapsto \left\{\begin{array}{l} V \subseteq \mathbb{R}^{d+n} \subseteq \mathbb{R}^{d+n+1} \\ (x,t) \in \mathbb{R}^{d+n} \times \mathbb{R} \end{array}\right.$$

OR $\quad \infty \quad$ if $\quad t = \infty$

Picture:

$t \in \mathbb{R} \cup \{\infty\}$





A *spectrum* $\mathbb{X}$ is a sequence of spaces X_n and maps $X_n \to \Omega X_{n+1}$.

The *Thom spectrum* $MTSO(d)$ consists of

$$X_{d+n} = Th(U_{d,n}^{\perp})$$

maps as above.

$$\Omega^\infty \mathbb{X} = \operatorname{hocolim}\left(X_0 \to \Omega X_1 \longrightarrow \Omega^2 X_2 \longrightarrow \Omega^3 X_3 \longrightarrow \cdots\right)$$

Thm [Tillmann '97]

$\exists$ spectrum $\mathbb{X}$: $H_*(\Gamma_\infty) \cong \underbrace{H_*(\Omega^\infty \mathbb{X})}$

has a lot of structure
(Dyer-Lashof operations, ...)
$\Rightarrow$ in theory computable

Conj [Madsen-Tillmann '01]

(Generalised Mumford conjecture)

$$B\Gamma_\infty \xrightarrow{H_* \cong} \Omega^\infty MTSO(2)$$

Note:
In each case, $\pi_0(\Omega^\infty \ldots) \cong \mathbb{Z}$ and we just look at one component.

Exercise: rational coh. $\cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$

[Galatius '04] computed mod-p coh.
(much more complicated)

Thm (Madsen-Weiss '07) This is true!

[Galatius-Madsen-Tillmann-Weiss '09] Another proof & more general results along the way.

The proof of GMTW :

$\mathcal{C}_d^+$ = cobordism category of oriented d-manifolds

space of objects = space of closed, oriented $(d-1)$ - submanifolds of $\mathbb{R}^\infty$

$$= \coprod_{\text{diff. types } M} \operatorname{colim}_{n \to \infty} \left(\mathrm{Emb}(M, \mathbb{R}^n) / \mathrm{Diff}(M) \right)$$

space of morphisms = space of compact, oriented d-submanifolds of $[0,1] \times \mathbb{R}^\infty$

$W \subseteq [0,1] \times \mathbb{R}^n$

$\partial W \subseteq \{0,1\} \times \mathbb{R}^n$

equal to $\partial_0 W \times [0, \varepsilon)$ in $[0,\varepsilon) \times \mathbb{R}^n$

$\partial_1 W \times (1-\varepsilon, 1]$ in $(1-\varepsilon, 1] \times \mathbb{R}^n$

Why this is relevant:

$$\mathcal{C}_d^+(\emptyset, M) \simeq \coprod_{\partial W = M} B\mathrm{Diff}_\partial^+(W)$$

$$\mathcal{C}_2^+(\emptyset, S^1) \simeq \coprod_{\partial \Sigma = S^1} B\mathrm{Diff}_\partial^+(\Sigma)$$

many such Σ, most disconnected.

$$\mathcal{C}_{d,\partial}^{+} \subseteq \mathcal{C}_{d}^{+}$$

- same objects
- morphisms: only consider those $W \subseteq [0,1] \times \mathbb{R}^n$ such that each component of W intersects $\{1\} \times \mathbb{R}^n$

$$\mathcal{C}_{2,\partial}^{+}(\emptyset, S^1) \simeq \coprod_{g=0}^{\infty} B\mathrm{Diff}_{\partial}^{+}(\Sigma_{g,1})$$

Every category $\mathcal{C}$ has a classifying space $B\mathcal{C}$

- vertices = objects
- edges = morphisms
- $\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd}$ commutes $\Rightarrow$ fill with a 2-simplex
- etc.

Similar for topological categories $\sim$ space of $\begin{cases} \text{vertices} \\ \text{edges} \\ \text{etc.} \end{cases}$

Theorem 1 (GMTW)

$$\Omega B\mathcal{C}_d^+ \simeq \Omega^\infty MTSO(d)$$

Theorem 2 (GMTW)

The inclusion $\mathcal{C}_{d,\partial}^+ \hookrightarrow \mathcal{C}_d^+$ induces

$$B\mathcal{C}_{d,\partial}^+ \simeq B\mathcal{C}_d^+ .$$

How does this imply the gen. Mumford Conj?

Prop (analogue of the group-completion theorem of [McDuff - Segal])

$\mathcal{C}^{op} \xrightarrow{F}$ Spaces functor such that every $x \to y$ in $\mathcal{C}$ is sent to $H_*(F(x)) \xleftarrow{\cong} H_*(F(y))$

(+ technical conditions)

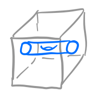
Then $\forall x \in \mathcal{C}$

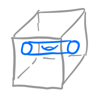
$\exists \quad F(x) \longrightarrow \Omega_x B\mathcal{C}$ inducing $\cong$ on H_*.

Proof of GMC

$$\mathcal{C} = \mathcal{C}^+_{2,\partial}$$

$$(\mathcal{C}^+_{2,\partial})^{op} \longrightarrow \text{Spaces}$$





$$\underset{\text{1-manifold}}{M} \longmapsto \operatorname{hocolim}\left(\mathcal{C}^+_{2,\partial}(M, S^1) \xrightarrow{\ \circ -\ } \mathcal{C}^+_{2,\partial}(M, S^1) \xrightarrow{\ \circ -\ } \cdots \right)$$

$$\begin{tikzcd}
M \arrow[d, "\Sigma"'] & & \mathcal{C}^+_{2,\partial}(M, S^1) \arrow[r, "\Sigma_{1,2} \circ -"] & \mathcal{C}^+_{2,\partial}(M, S^1) \arrow[r, "\Sigma_{1,2} \circ -"] & \cdots \\
N & \longmapsto & \mathcal{C}^+_{2,\partial}(N, S^1) \arrow[r, "\Sigma_{1,2} \circ -"'] \arrow[u, "- \circ \Sigma"'] & \mathcal{C}^+_{2,\partial}(N, S^1) \arrow[r, "\Sigma_{1,2} \circ -"'] \arrow[u, "- \circ \Sigma"'] & \cdots
\end{tikzcd}$$

In the colimit, $g \to \infty$, so Harer stability implies that the colimit of the vertical maps induces $\cong$ on H_* in all degrees.

Thus
$$\underset{\mathbb{Z} \times B\Gamma_\infty}{\overset{\text{IS}}{F(\emptyset)}} \xrightarrow{H_* \cong} \Omega B \mathcal{C}^+_{2,\partial} \overset{(2)}{\simeq} \Omega B \mathcal{C}^+_{2} \overset{(1)}{\simeq} \Omega^\infty MTSO(2).$$

$\square$.

$$\mathcal{C}^+_{2,\partial}(\emptyset, S^1) \simeq \coprod_{g=0}^{\infty} BDiff^+_\partial(\Sigma_{g,1})$$

Idea of proof of Theorem 1

$$\Omega B \mathcal{C}_d^{+} \simeq \Omega^{\infty} MTSO(d).$$

Sheaf models

$\mathcal{X}$ = category of $\begin{cases} \text{smooth manifolds (no } \partial) \\ \text{smooth maps} \end{cases}$

Consider sheaves

$$\mathcal{X}^{op} \xrightarrow{F} \begin{cases} \text{Sets} \\ \text{Posets} \\ \text{Categories} \end{cases}$$

Geometric realisation of $F : \mathcal{X}^{op} \longrightarrow \text{Sets}$ is the usual geometric realisation of the simplicial set $\Delta^{op} \longrightarrow \mathcal{X}^{op} \xrightarrow{F} \text{Sets}$, where $\Delta^{op} \longrightarrow \mathcal{X}^{op}$ sends the standard n-simplex to the standard extended n-simplex $\Delta^n_e = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} \mid \Sigma x_i = 1\}$ and the face/degeneracy maps to the evident analogues for extended simplices.

For a sheaf of categories

$$F : \mathcal{X}^{op} \longrightarrow \text{Categories},$$

its geometric realisation $|F|$ is naturally a topological category, and therefore has a classifying space $B(|F|)$

Step 1 [GMTW] construct sheaf models for $\mathcal{C}_d^+$ and $\Omega^\infty MTSO(d)$, i.e. sheaves C_d^+ and D_d^+ such that

- $B(|C_d^+|) \simeq B\mathcal{C}_d^+$

by construction

- $\Omega B(|D_d^+|) \simeq \Omega^\infty MTSO(d)$

uses the Pontrjagin-Thom construction & Phillips' submersion theorem.

Step 2 They then construct a zig-zag of maps of sheaves

$$D_d^+ \longleftarrow D_d^{+,\pitchfork} \longrightarrow C_d^{+,\pitchfork} \longleftarrow C_d^+$$

and prove that they all induce weak equivalences on $B(|-|)$.

$C_d^+(X) \approx$

ignoring some details

- $a_0 \leq a_1 : X \longrightarrow \mathbb{R}$ smooth
- $W \subseteq X \times [a_0, a_1] \times \mathbb{R}^\infty$

submanifold

such that

(i) $W \longrightarrow X$ is a submersion with d-dimensional fibres

(ii) $W \longrightarrow X \times [a_0, a_1]$ is a proper map.

(iii) W is a product near $X \times \{a_0\} \times \mathbb{R}^\infty$ and $X \times \{a_1\} \times \mathbb{R}^\infty$

Composition = concatenation of intervals in the 2nd coordinate.

$D_d^+(X) \approx$ $W \subseteq X \times \mathbb{R} \times \mathbb{R}^\infty$

such that

(i) $W \longrightarrow X$ is a submersion with d-dimensional fibres

(ii) $W \longrightarrow X \times \mathbb{R}$ is a proper map.

(iii) $\forall$ compact $K \subseteq X$, $\pi^{-1}(K) \subseteq K \times \mathbb{R} \times \mathbb{R}^n$ for some $n < \infty$.