

# Mapping class group representations via Heisenberg, Schrödinger and Stone-von Neumann

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*Cobordisms, Strings, and Thom Spectra*, Oaxaca (and online)

Joint work with Christian Blanchet and Awais Shaukat

## Aims

Repr of  $B_n$

Repr of MCGs

– Moriyama

– abelian coeff

– Heisenberg

– Torelli

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– tautological

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- Linearity:  $B_n$  embeds into  $GL_N(\mathbb{R})$

[Bigelow, Krammer]

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[Burau] representation (1935):

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- Q: *Are the braid groups linear?*  
— Does  $B_n$  embed into some  $GL_N(\mathbb{F})$ ?

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Then  $H_*^{BM}(C_k(D_n); \mathbb{Z}[Q])$  is a free  $\mathbb{Z}[Q]$ -module  
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$$\text{Lawrence}_k: B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \text{Aut}_{\mathbb{Z}[Q]}(H_*^{BM}(C_k(D_n); \mathbb{Z}[Q]))$$

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## Theorem [Bigelow'00, Krammer'00]

$\text{Lawrence}_2$  is faithful (injective). Hence  $B_n$  embeds into  $GL_N(\mathbb{R})$ .

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- $\text{Map}(\Sigma_2) \subset GL_{64}(\mathbb{C})$  [Bigelow-Budney'01]

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## Main result [Blanchet-P.-Shaukat'21]

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(“Genuine” analogues of the Lawrence representations)



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$$\text{Map}(\Sigma) \cong H_k^{BM}(F_k(\Sigma'); \mathbb{Z})$$

- $F_k(\Sigma')$  = *ordered* configuration space

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### Theorem [Moriyama'07]

The kernel of this representation is  $\mathfrak{J}(k) \subset \text{Map}(\Sigma)$ .

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- $F_k(\Sigma) = \text{ordered configuration space}$
- $\Sigma' = \Sigma \setminus (\text{interval in } \partial\Sigma)$
- *untwisted*  $\mathbb{Z}$  coefficients
- $H_k^{BM}(F_k(\Sigma'); \mathbb{Z})$  is a free abelian group of finite rank

### Theorem [Moriyama'07]

The kernel of this representation is  $\mathfrak{J}(k) \subset \text{Map}(\Sigma)$ .

- $\mathfrak{J}(k)$  is the  $k$ -th term of the *Johnson filtration* of  $\text{Map}(\Sigma)$

Aims

Repr of  $B_n$

Repr of MCGs

– Moriyama

– abelian coeff

– Heisenberg

– Torelli

– Schrödinger

– tautological

Kernel

Summary

- Lower central series:  $\pi_1(\Sigma) = \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \Gamma_4 \supseteq \dots$
- $\Gamma_i = [\pi_1(\Sigma), \Gamma_{i-1}]$  (commutators of length  $i$ )



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### Definition [Johnson'81]

$\mathfrak{J}(k)$  = kernel of the action of  $\text{Map}(\Sigma)$  on  $\pi_1(\Sigma)/\Gamma_{k+1}$ .

- $\text{Map}(\Sigma) = \mathfrak{J}(0) \supset \mathfrak{J}(1) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \dots$

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### Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{BM}(F_k(\Sigma'); \mathbb{Z})$  is a faithful ( $\infty$ -rank)  $\text{Map}(\Sigma)$ -representation.

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- Idea: Enrich the representation by taking homology with *twisted coefficients*  $\mathbb{Z}[Q]$ , where  $\pi_1(C_k(\Sigma')) = B_k(\Sigma) \twoheadrightarrow Q$ .

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Fact ( $k \geq 2$ )

$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \left\{ \begin{array}{ll} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{array} \right\}$$



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- In  $\mathbb{Z}[B_k(S)^{ab}]$ , the corresp. variable  $t$  has order two:  $t^2 = 1$ .  
 $\rightsquigarrow$  we get a much “weaker” representation...

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## Theorem [Bellingeri'04]

$$B_k(\Sigma_{g,1}) \cong \left\langle \sigma_1, \dots, \sigma_{k-1}, \begin{array}{l} a_1, \dots, a_g \\ b_1, \dots, b_g \end{array} \mid \dots \text{ some relations } \dots \right\rangle$$

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$$\mathcal{H}_g = B_k(\Sigma_{g,1}) / \llbracket [\sigma_1, x] \rrbracket$$

This is the *genus-g discrete Heisenberg group*.

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This is the *genus-g discrete Heisenberg group*. Note that:

$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \frac{\mathbb{Z}}{2} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Q})$$



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The action  $\text{Map}(\Sigma) \curvearrowright B_k(\Sigma)$  descends to a well-defined action on the quotient  $\mathcal{H}_g$ .

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$$1 \rightarrow \mathbb{Z} \longrightarrow \mathcal{H}_g \longrightarrow H_1(\Sigma; \mathbb{Z}) \rightarrow 1$$

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and the  $\text{Map}(\Sigma)$ -action on  $\mathcal{H}_g$  lifts the natural action on  $H_1(\Sigma; \mathbb{Z})$ .

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## Corollary



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We obtain a *twisted* representation, defined over  $\mathbb{Z}[\mathcal{H}_g]$ :

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In other words a functor  $\text{Ac}(\text{Map}(\Sigma) \circlearrowleft \mathcal{H}_g) \rightarrow \text{Mod}_{\mathbb{Z}[\mathcal{H}_g]}$ .

(where  $\text{Ac}(\text{Map}(\Sigma) \circlearrowleft \mathcal{H}_g)$  is the *action groupoid*)

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## Note

Replace the coefficients  $\mathbb{Z}[\mathcal{H}_g]$  with any  $\mathcal{H}_g$ -representation  $W$  over  $R$  to get a twisted  $\text{Map}(\Sigma)$ -representation  $\mathcal{V}(W)$  over  $R$ .

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How to untwist this representation?

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Three methods:

(1) On the Torelli group  $\text{Tor}(\Sigma) \subset \text{Map}(\Sigma)$

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Aims

Repr of  $B_n$

Repr of MCGs

– Moriyama

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## Problem

How to untwist this representation?

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## Proposition

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We obtain a central extension of the Torelli group:

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## Lemma

The central extension  $\widetilde{\text{Tor}}(\Sigma)$  of  $\text{Tor}(\Sigma)$  turns out to be trivial.

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## Definition (Schrödinger representation)

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$$\mathcal{H}_g \cong \mathbb{Z}\langle \sigma, a_1, \dots, a_g \rangle \rtimes \mathbb{Z}\langle b_1, \dots, b_g \rangle \quad (b_i \cdot a_i = a_i + 2\sigma)$$

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and  $\tilde{\alpha}$  allows us to untwist the representation:

$$\widetilde{\text{Map}}(\Sigma)^{\text{univ}} \longrightarrow U(H_k^{\text{BM}}(C_k(\Sigma'); W_{\text{Sch}})) = U(\mathcal{V}(W_{\text{Sch}}))$$

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$$\mathcal{H}_g = \mathbb{Z}^{2g+1} = \mathbb{Z} \times H_1(\Sigma) \quad \text{with } (k, x)(l, y) = (k + l + x.y, x + y)$$

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## Lemma

This untwists the representation with coefficients in  $W_{\text{lin}}$ :

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For  $k \geq 2$  and  $W = \mathcal{H}_g$ -representation over  $R$ , we obtain:



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- a unitary  $\widetilde{\text{Map}}(\Sigma)^{\text{univ}}$ -representation  $\mathcal{V}(W)$  if  $W = W_{\text{Sch}}$ ;

Aims

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Kernel

Summary

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For  $k \geq 2$  and  $W = \mathcal{H}_g$ -representation over  $R$ , we obtain:

- a  $\text{Tor}(\Sigma)$ -representation  $\mathcal{V}(W)$  over  $R$ ;
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## Lemma

As an  $R$ -module,  $\mathcal{V}(W) \cong \bigoplus_{\binom{k+2g-1}{k}} W$ .

For example,  $\mathcal{V}(W_{\text{in}})$  is a free  $\mathbb{Z}$ -module of rank  $(2g+2)\binom{k+2g-1}{k}$ .

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**Q':** Is it smaller than  $\mathfrak{J}(k) = \ker(\text{Moriyama}_k)$ ?

$(k = 2)$

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$$H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g]) \twoheadrightarrow H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2])$$



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## Corollary

The kernel of  $H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$  is **strictly smaller** than  $\mathfrak{J}(2)$ .

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Set  $k = 2$  and  $g = 1$ . In this case the representation

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Exercise: this reduces to the identity if we set  $a = b = \sigma^2 = 1$ .

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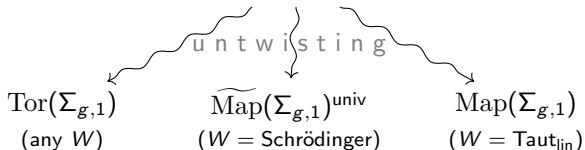
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Map( $\Sigma_{g,1}$ ): Moriyama $_k \rightsquigarrow$  kernel =  $\mathfrak{J}(k)$

twisted representations  $\mathcal{V}(W) \cong \bigoplus_{\text{fin}} W$



kernel  $\subseteq \mathfrak{J}(k)$  (when  $W = \mathbb{Z}[\mathcal{H}_g]$ )

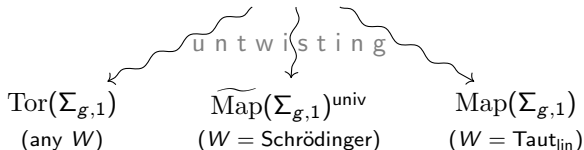
kernel  $\subsetneq \mathfrak{J}(2)$  (for  $k = 2$ )

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**Thank you for your attention!**