Mapping class group representations via Heisenberg, Schrödinger and Stone-von Neumann

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$$\mathrm{Lawrence}_k \colon B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \mathrm{Aut}_{\mathbb{Z}[Q]}\left(H_*^{BM}(C_k(D_n);\mathbb{Z}[Q])\right)$$

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Theorem [Bigelow'00,Krammer'00]

Lawrence_2 is faithful (injective). Hence B_n embeds into $GL_N(\mathbb{R})$.

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("Genuine" analogues of the Lawrence representations)

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- $\mathfrak{J}(k)$ is the k-th term of the *Johnson filtration* of $\mathrm{Map}(\Sigma)$

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Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{BM}(F_k(\Sigma'); \mathbb{Z})$ is a faithful (∞-rank) $\mathrm{Map}(\Sigma)$-representation.

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- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresp. variable t has order two: $t^2 = 1$.
  $\rightsquigarrow$ we get a much "weaker" representation...

Rep. of MCGs – Heisenberg twisted coefficients

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Theorem [Bellingeri'04]

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$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \frac{\mathbb{Z}}{2} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Q})$$

Rep. of MCGs – Heisenberg twisted coefficients

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The action $\mathrm{Map}(\Sigma) \circlearrowleft B_k(\Sigma)$ descends to a well-defined action on the quotient $\mathcal{H}_g$.

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and the $\mathrm{Map}(\Sigma)$-action on $\mathcal{H}_g$ lifts the natural action on $H_1(\Sigma;\mathbb{Z})$.

Rep. of MCGs – Heisenberg twisted coefficients

Corollary

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We obtain a *twisted* representation, defined over $\mathbb{Z}[\mathcal{H}_g]$:

$$\mathrm{Map}(\Sigma) \quad \circlearrowleft \quad H_k^{BM}\left(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g]\right) = \mathcal{V}$$

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Note

Replace the coefficients $\mathbb{Z}[\mathcal{H}_g]$ with any $\mathcal{H}_g$-representation W over R to get a twisted $\mathrm{Map}(\Sigma)$-representation $\mathcal{V}(W)$ over R.

Rep. of MCGs – Heisenberg twisted coefficients

Problem

How to untwist this representation?

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Rep. of MCGs – restricting to the Torelli group

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The pre-image of $\mathrm{Inn}(\mathcal{H}_g) \subset \mathrm{Aut}(\mathcal{H}_g)$ under $\alpha \colon \mathrm{Map}(\Sigma) \to \mathrm{Aut}(\mathcal{H}_g)$ is the Torelli group.

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\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
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Lemma

The central extension $\widetilde{\mathrm{Tor}}(\Sigma)$ of $\mathrm{Tor}(\Sigma)$ turns out to be trivial.

Rep. of MCGs – untwisting via Schrödinger $\mathcal{H}_g$-rep

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For $\varphi \in \mathrm{Aut}(\mathcal{H}_g^{\mathrm{Re}})$ there is a unique inner automorphism $T(\varphi)$ of $U(W_{\mathrm{Sch}})$ such that the following square commutes:

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and $\widetilde{\alpha}$ allows us to untwist the representation:

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Corollary

The kernel of $H_2^{BM}\left(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g]\right)$ is **strictly smaller** than $\mathfrak{J}(2)$.

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Exercise: this reduces to the identity if we set $a = b = \sigma^2 = 1$.

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Q: linearity? $\rightsquigarrow$ study $\mathcal{V}(W)$ for well-chosen W

extensions to TQFTs?

Thank you for your attention!