Stability in the topology of configuration spaces

IMAR Monthly Lecture
19 Apr 2023

M — connected manifold

Def $C_n(M) = \{ S \subseteq M \mid |S| = n \}$
n^{th} configuration space on M

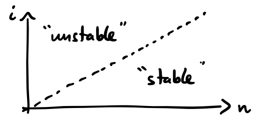
Ex $M = \mathbb{R}^2$
$\pi_1(C_n(\mathbb{R}^2)) =: B_n$ braid group $\rightsquigarrow$ knot theory

Two kinds of stability:

1. Homological: $H_*(C_n(M))$ as $n \to \infty$
2. Group-theoretical: $\underbrace{\Gamma_i}_{\text{lower central series}}(\pi_1(C_n(M)))$ as $i \to \infty$

Thm [McDuff, Topology, '75] [Segal, Acta.Math., '79]
If M is non-compact, $\exists$ maps
$$C_n(M) \xrightarrow{(*)} C_{n+1}(M)$$
that induce isomorphisms on $H_i(-;\mathbb{Z})$ for $i \leq \frac{n}{2}$.

$(*)$ defined using a non-surjective $M \hookrightarrow M$



Rmk

- $\exists$ similar result for moduli spaces of surfaces (algebraic curves) [Harer, Ann.Math., '85]
- [Madsen-Weiss, Ann.Math., '07] calculated the limit as $g \to \infty$ $\rightsquigarrow$ proving the Mumford Conj.

Q: What if M is closed?
$\hookrightarrow$ $\nexists$ non-surj. $M \hookrightarrow M$
$\hookrightarrow$ Counter-ex: $H_1(C_n(S^2)) \cong \mathbb{Z}/_{(2n-2)}$

Thm [Cantero-P., Doc.Math, '15]
Fix $i \geq 0$, prime p. For $n \geq 2i$:

dim(M) odd: $H_i(C_n(M);\mathbb{Z})$ is 2-periodic

dim(M) even: $H_i(C_n(M);\mathbb{F}_p)$ is $p^{1+v_p(i)}$-periodic
& depends only on $v_p(2n-\chi)$

$\chi = \chi(M)$

Idea:

- [McDuff]: stable H_* of $C_n(M)$
$\cong H_*$ of a path-cpt of a section space $\Gamma(M)$
- We construct operations $\Gamma(M) \to \Gamma(M)$ and study
  (1) their effect on π_0
  (2) when they are hty equivalences after localising at p.

Rmk:

Similar ideas used by [Galatius-Randal-Williams, Res.Math.Sci., '20] to study moduli spaces of surfaces embedded in an ambient closed manifold M.

"Non-local" configuration spaces

Def X space
$$C_n(M,X) = \left\{ \begin{matrix} S \subseteq M \\ f: S \to X \end{matrix} \;\middle|\; |S| = n \right\}$$
"local" data

Ex of "non-local" data:

- Equip S with a total ordering modulo even permutations.
$\rightsquigarrow$ "oriented" configuration spaces $C_n^+(M,X)$
- Replace $f: S \to X$ with $f: M \setminus S \to X$
$\rightsquigarrow$ "configuration-mapping spaces" $CMap_n(M,X)$

($M = \mathbb{R}^2$, $X = BG = K(G,1)$ $\rightsquigarrow$ Hurwitz spaces)

Thm [P., TAMS, '13] If M is non-compact and X is path-connected, $\exists$ maps $C_n^+(M,X) \to C_{n+1}^+(M,X)$ inducing $\cong$ on $H_i(-;\mathbb{Z})$ for $i \leq \frac{n-5}{3}$.

Rmk: The slope of $1/3$ is optimal.

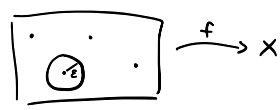
Configuration-mapping spaces

$d = \dim(M)$
Choose $c \subseteq [S^{d-1}, X] = \{\text{maps } S^{d-1} \to X\}/\simeq$

Def $CMap_n^c(M,X) =$ subspace of (S,f) where $\forall p \in S$, $[f|_{\partial B_\varepsilon(p)}] \in c$

Interpretation

- f "field" on M
- S "singularities" of f
- c "charges" of f at S



For Hurwitz spaces:

- $c \subseteq [S^1, BG] = \mathrm{Conj}(G)$
- points $\longleftrightarrow$ branched G-coverings of $\mathbb{R}^2$ with monodromy in c

Thm [Ellenberg-Venkatesh-Westerland, Ann. Math., '16]

G finite group

c conj. class $\rightsquigarrow$ "nonsplitting condition"

Then $H_i(\mathrm{CMap}_n^c(\mathbb{R}^2, BG); \mathbb{Q})$ is periodic for $n \gg i$

$\longrightarrow$ Coro [EVW] Cohen-Lenstra conjecture over $\mathbb{F}_q(t)$ as $q \to \infty$

Thm [P.-Tillmann, Res Math Sci, '21]

Generalise: any non-compact M, path-connected X

But: stronger hypothesis on c

$$c \subseteq [S^{d-1}, X] \overset{p}{\twoheadleftarrow} \pi_{d-1}(X)$$

If $|p^{-1}(c)| = 1$,

then $H_i(\mathrm{CMap}_n^c(M, X); \mathbb{Z})$ is stable for $n \gg i$.

Rmk

- True more generally for "config-section spaces"
- Ex non-0 vector field on $M \setminus$ config.
  flat connection — n —.

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Motivating question:

M smooth, orientable manifold.

Q: Is $\mathrm{MCG}(M)$ linear?

$\mathrm{MCG}(M) = \pi_0(\mathrm{Diff}^+_\partial(M))$

linear: embeds into $GL_n(\mathbb{F})$

Spheres:

- $\mathrm{MCG}(S^d) = 0$ $(d \leq 3)$ [Smale, Cerf] (Proc. AMS '59; Sém. Cartan '62/63)
- $\mathrm{MCG}(S^d)$ is finite abelian $(d \geq 5)$
  [Smale + Cerf + Kervaire-Milnor] ('62 AJM; '70 IHES; '63 Ann. Math.)

In dim ≥ 3, typically no:

Ex For $g \geq 3$,

$\mathrm{MCG}(\#^g(S^1 \times S^2) \setminus D^3) \cong \mathrm{Aut}(F_g)$

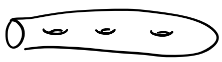
is not linear [Formanek-Procesi, J. Algebra, '92]

$\mathrm{MCG}(D^2 \setminus n) \cong B_n$ is linear [Bigelow, JAMS, '01]
[Krammer, Ann. Math, '02]

Idea: B_n acts on $H_i(C_k(D^2 - n); \mathcal{L})$

"Lawrence representation"

For $i = k = 2$ it is faithful.

$M = \Sigma_{g,1} =$ 

Is $\mathrm{MCG}(\Sigma_{g,1})$ linear?

- $g = 0, 1$ ✓
- $g \geq 2$ open...

Thm [Blanchet-P.-Shaukat, arXiv, '21]

For each $k \geq 2$:

1. V repr. of "discrete Heisenberg group" $\mathcal{H}_g$
   $\rightsquigarrow$ $W(V)$ twisted repr. of $\mathrm{MCG}(\Sigma_{g,1})$
2. $W(V)$ may be untwisted on
   - Torelli — for all V
   - MCG — for $V =$ Schrödinger.
3. $\mathrm{Ker}(W(V)) \to 0$ as $k \to \infty$.

The construction of the local system on $C_k(\Sigma_{g,1})$ involves:

$$\pi_1(C_k(\Sigma_{g,1})) = B_k(\Sigma_{g,1}) \twoheadrightarrow \mathcal{H}_g$$

When $k \geq 3$ this is the quotient by $\Gamma_3(B_k(\Sigma_{g,1}))$.

Def (lower central series $\Gamma_*(G)$)

G group

$\Gamma_1(G) := G$

$\Gamma_{i+1}(G) := [G, \Gamma_i(G)]$

$\rightsquigarrow$ descending filtration of G.

Problem: Study $\Gamma_*(B_k(S))$ for fixed S and k.

Q: Does it stop?

$\exists ?$ i such that $\Gamma_i(\cdot\cdot) = \Gamma_{i+1}(\cdot\cdot)$.

Thm [Darné-P.-Soulié, Memo. AMS, '23]

The lower central series $\Gamma_*(B_k(S))$

$(k \geq 3)$ stops at $i=2$ if $S \subseteq S^2$ or non-or.
stops at $i=3$ if $S \nsubseteq S^2$ & or.

(!) $\begin{cases} (k=2) & \text{does not stop if } S \neq D^2, S^2, P^2 \\ (k=1) & \text{does not stop if } S \neq (6 \text{ exceptions}) \end{cases}$

Rmk We also answer the question for virtual braid groups, welded braid groups and partitioned versions of all of the above.