INI Cambridge 12 June 2023

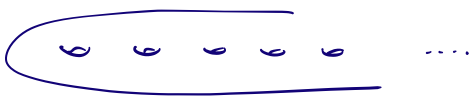
The homology of big MCGs

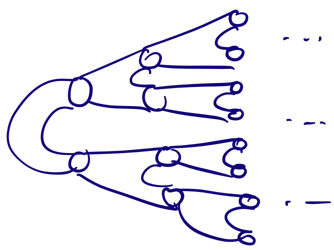
joint with Xiaolei Wu

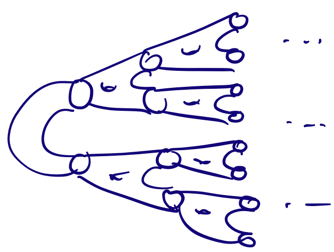
Big MCGs

$$\mathrm{MCG}(S) = \pi_0(\mathrm{Homeo}^+(S))$$

for an ∞-type surface S

Eg  $= \Sigma_\infty$ $\quad (\infty, *, *)$

 $\cong S^2 \smallsetminus \mathcal{C}$ $\quad (0, \mathcal{C}, \emptyset)$

 $\quad (\infty, \mathcal{C}, \mathcal{C})$

In general:

S — connected, orientable surface
second countable ($\equiv$ paracompact $\equiv$ metrisable)

∞-type $\overset{\text{def}}{\Longleftrightarrow}$ $\pi_1(S)$ is <u>not</u> fin. gen.

$\left(\Longleftrightarrow \pi_1(S) \cong F_\infty \right)$

$\left(\Longleftrightarrow \mathrm{MCG}(S) \text{ is uncountable} \right)$

<u>Classification</u> by
- genus $\in \mathbb{N} \cup \{\infty\}$
- space of ends (∂ of Freudenthal compact")
- subspace of non-planar ends
  (ends where all nbhds have ∞ genus)

[Kerékjártó 1923
Richards 1963]

Arise in dynamics (complements of attractors)
leaves of foliations of 3-manifolds
...

What is known about $H_*(MCG(S))$?

<u>Finite type</u>

- $S = \Sigma_{0,1}^n$ (braid groups) [Arnol'd, Cohen, Fuchs, '70s]
- $S = \Sigma_{g,b}$ $\rightsquigarrow$ $H_*(\Omega^\infty_0 MTSO(2))$ when $g \to \infty$.
  [Harer, Madsen-Weiss]
- various calc's for <u>small</u> g, b, n, i

Infinite type

Thm [Calegari–Chen '19 '20]

S compact $\Rightarrow$ $H_1(MCG(S \setminus \mathcal{C})) \cong H_1(MCG(S))$

Cantor

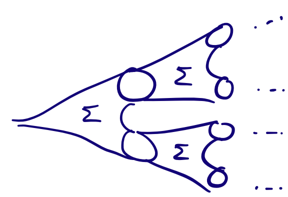
Q: $\boxed{H_*(MCG(S \setminus \mathcal{C})) \cong H_*(MCG(S)) \text{ in all degrees?}}$

$H_2(MCG(S^2 \setminus \mathcal{C})) \cong \mathbb{Z}/2$

$H_2(MCG(\mathbb{R}^2 \setminus \mathcal{C})) \supseteq \mathbb{Z}$

Thm A [P.–Wu '22]

$$H_i(MCG(\mathbb{R}^2 \setminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

More generally, same H_* for any 

For contrast...

Thm B [P.-Wu '22]

$$H_*(MCG(\Sigma_\infty)) \supseteq \Lambda^*_{\mathbb{Z}}\Big(\bigoplus_{c} \mathbb{Z}\Big)$$

c — continuum.

(Also generalises to an uncountable family of surfaces...)

In degree 1 : [Domat '20] $H_1(MCG(\Sigma_\infty)) \supseteq \bigoplus_c \mathbb{Q}$

Rmk

$$\begin{array}{ccc} \operatorname{colim}_g MCG(\Sigma_{g,1}) & \xrightarrow{\text{dense}} & MCG(\Sigma_\infty) \\ \| & & \\ MCG_c(\Sigma_\infty) & & \end{array}$$

$$\Big\downarrow H_*$$

$$H_*(\Omega^\infty_0 MTSO(2)) \xrightarrow{?} \begin{pmatrix}\text{uncountable in} \\ \text{all degrees}\end{pmatrix}$$

Thm A' $\tilde{H}_*(MCG(D^2 \setminus e)) = 0$ (answer to Q for the disc)

This implies Thm A via:

$$1 \to \mathbb{Z} \hookrightarrow MCG(D^2 \setminus e) \to MCG(\mathbb{R}^2 \setminus e) \to 1$$

$E^2 =$

$H_*(MCG(\mathbb{R}^2 \setminus e))$
$H_*(MCG(\mathbb{R}^2 \setminus e))$

$\rightsquigarrow$

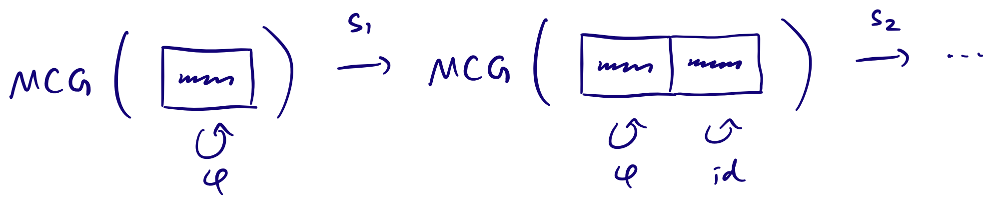
		0
0	$\mathbb{Z}$	
	0	

$= E^\infty$

Plan

- Proof of Thm A' — 2 steps
- Proof of Thm B

① Consider the sequence



Thm: This is homologically stable.

proof Apply the machine of [RWW] to:

$\mathcal{M}_2^\infty$ = groupoid of conn., orientable surfaces S with $I \hookrightarrow \partial S$ & homeomorphisms

braided monoidal under $\natural$

fails cancellation $\longrightarrow$ fix formally by passing to another category with objects = words on the alphabet of surfaces

Set $A = D^2$

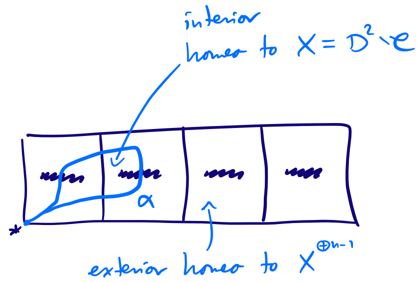
$X = D^2 \setminus e$

Machine of [RWW] + tricks

$\hookrightarrow$ reduce to proving high-conn. of:

$TC_n(A,X)$

vertex:



$\{\alpha_0, \dots, \alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X^{\oplus n-p-1}$

(n-1)-dim.

$TC_\infty(A,X)$

same vertices

$\{\alpha_0, \dots, \alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X$

∞-dim.

same $(n-2)$-skeleton as $TC_n(A,X)$

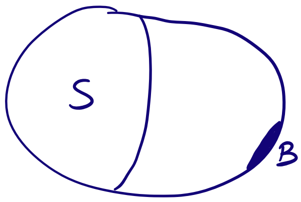
(only clopen subsets of $\mathcal{C}$ are $\emptyset$ and $\mathcal{C}$)

Prop $TC_\infty(A,X)$ is contractible. $\square$.

Rmk Follows ideas of proof of [Szymik–Wahl], who proved $\tilde{H}_*(V) = 0$ via hom. stab. for $V \to V \to \cdots$.

V ← Thompson group

(2) <u>Lemma</u> (adapted from [Mather '71], related to "dissipated" groups, "binate" groups ....)

$T =$ 

S, B disjoint, closed $\subseteq T$

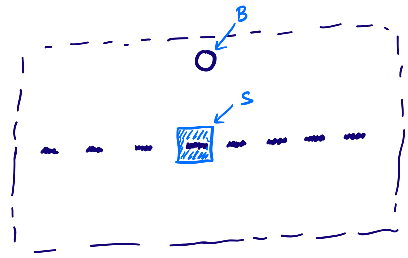
$$\pi_0 \mathrm{Homeo}_{\partial S}(S) \xrightarrow{(*)} \pi_0 \mathrm{Homeo}_B(T)$$

Suppose (a) $(*)$ is a H_*-isomorphism

(b) $\exists\, \varphi \in \mathrm{Homeo}_B(T)$: $\forall\, k \geq 1$ $\quad \varphi^k(S) \cap S = \phi$

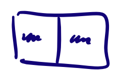
$\bigcup_{i=k}^{\infty} \varphi^i(S)$ is <u>closed</u>

Then $\widetilde{H}_*\left(\pi_0 \mathrm{Homeo}_B(T)\right) = 0$

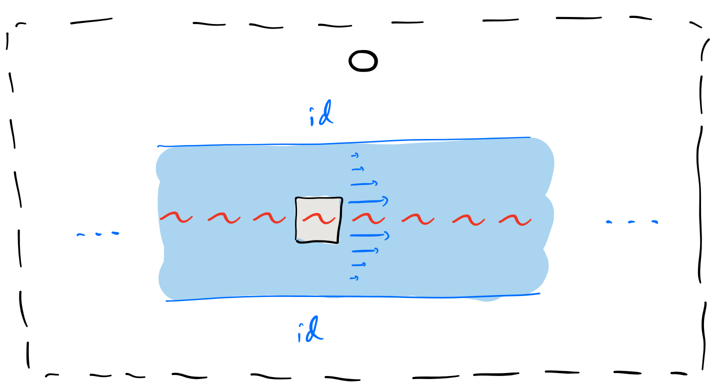
Apply to 

$= T \cong D^2 \setminus \left(\coprod_{\infty} \mathcal{C}\right)^+ \cong D^2 \setminus \mathcal{C}.$

So Thm A' $\Longleftarrow$ (a) + (b).

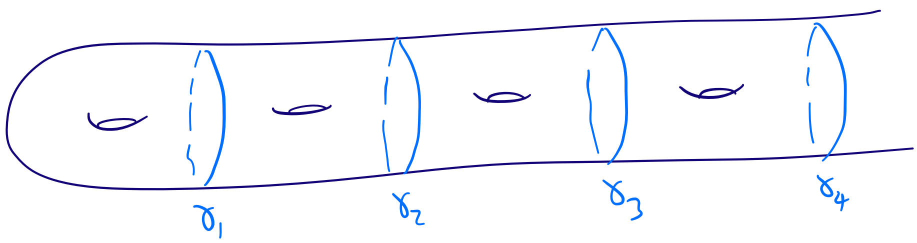
Step 1 $\rightsquigarrow$ MCG() $\xrightarrow{H_* \cong}$ MCG() $\rightsquigarrow$ (a)

(b) : φ = shift 1 step to the right.



$\square$.

Idea of proof of Thm B



Idea: Factor $\bigoplus_c \mathbb{Z} \hookrightarrow \bigoplus_c \mathbb{Q}$ through $MCG(\Sigma_\infty)$.

Then:

$$\begin{array}{ccccc} H_*(\bigoplus_c \mathbb{Z}) & \longrightarrow & H_*(MCG(\Sigma_\infty)) & \longrightarrow & H_*(\bigoplus_c \mathbb{Q}) \\ \cong & & & & \cong \\ \Lambda^*_{\mathbb{Z}}(\bigoplus_c \mathbb{Z}) & & \xrightarrow{\text{injective}} & & \Lambda^*_{\mathbb{Z}}(\bigoplus_c \mathbb{Q}) \end{array}$$

$$X \subseteq \mathbb{N} \rightsquigarrow f(X) := \prod_{i \in X} (T_{\gamma_i})^{i!}$$

$$f_n(X) := \prod_{\substack{i \in X \\ i \geq n}} (T_{\gamma_i})^{i!/n}$$

Exercise

(*) There exists a continuum! collection of infinite $X \subseteq \mathbb{N}$ so that any two of them intersect in a finite set.

Construction

$$1 \longmapsto f(x)$$

$$\begin{tikzcd} \bigoplus_c \mathbb{Z} \arrow[r] \arrow[d, hook] & MCG(\Sigma_\infty) \arrow[d] \\ \bigoplus_c \mathbb{Q} \arrow[r, "\theta"] & MCG(\Sigma_\infty)^{ab} \end{tikzcd}$$

$$1/n \longmapsto f_n(x).$$

$$\left[(f_n(x))^n \right] = \left[f(x) \right]$$

[Birman, Powell '70s] : Every mapping class on a <u>compact</u> surface $(g \geq 3)$ is a product of commutators.

Using $(*)$ and [Domat '20]

$\Uparrow$

machinery of [Bestvina - Bromberg - Fujiwara '15]

The map θ is <u>injective</u>.

<u>Fact</u> : Any injective map $\bigoplus_I \mathbb{Q} \longrightarrow A$ admits a <u>retraction</u>.

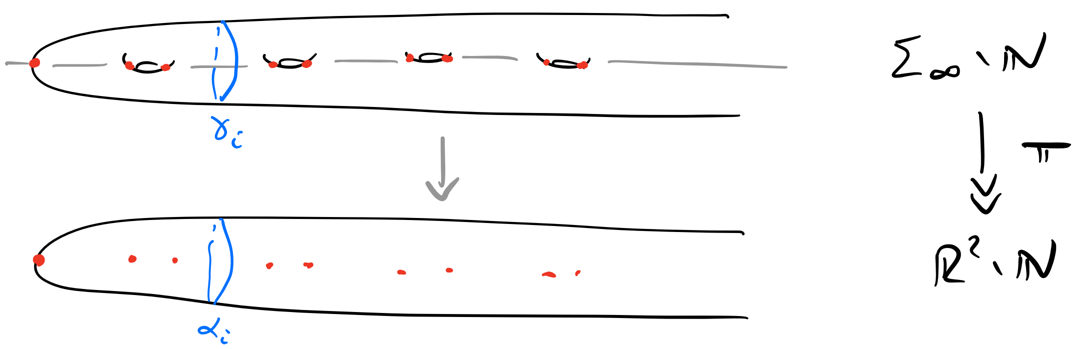
$$\begin{tikzcd} \bigoplus_c \mathbb{Z} \arrow[r] \arrow[d, hook] & MCG(\Sigma_\infty) \arrow[d] \\ \bigoplus_c \mathbb{Q} & MCG(\Sigma_\infty)^{ab} \arrow[l] \end{tikzcd}$$

Rmk

$$\begin{array}{ccc} H^i\left(\bigoplus_c \mathbb{Z}\right) & \xleftarrow{\quad 0 \quad} & H^i\left(\bigoplus_c \mathbb{Q}\right) \\ \cong & & \cong \\ \prod_c \mathbb{Z} & & \begin{cases} 0 & i=1 \\ \bigoplus_{2^c} \mathbb{Q} & i \geq 2 \end{cases} \end{array}.$$

Extra notes: — for $\Sigma = \mathbb{R}^2 \setminus \mathbb{N}$:

Adapt [Malestein–Tao '21]:



Lemma: Every based homeo of $\mathbb{R}^2 \setminus \mathbb{N}$ preserves $\pi_1(\Sigma_\infty \setminus \mathbb{N}) \triangleleft \pi_1(\mathbb{R}^2 \setminus \mathbb{N})$ — hence lifts uniquely to a homeo of $\Sigma_\infty \setminus \mathbb{N}$.

$$
\begin{tikzcd}
 & (T_{\alpha_i})^2 \in \arrow[ddd, mapsto] & \mathrm{MCG}_*(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[d, "\text{lift}"] & \\
 & & \mathrm{MCG}_*(\Sigma_\infty \setminus \mathbb{N}) \arrow[d, "\text{forget basept}"] & \\
 & & \mathrm{MCG}(\Sigma_\infty \setminus \mathbb{N}) \arrow[d, "\text{fill in punctures}"] & \\
 & T_{\gamma_i} \in & \mathrm{MCG}(\Sigma_\infty) \arrow[dr] & \\
\bigoplus_i \mathbb{Z} \arrow[uuuurr, bend left] \arrow[urr] \arrow[rrr, hook] & & & \bigoplus_i \mathbb{Q}
\end{tikzcd}
$$

Then pass to $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ via

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{MCG}_*(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[r, two heads] & \mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[r] & 1 \\
 & F_\infty & \underline{\text{countable}} \arrow[l] & &
\end{tikzcd}
$$

//