Lille workshop 9-6-23

Homology of big MCGs

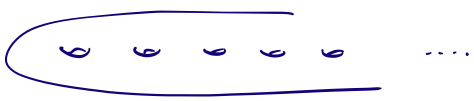
joint with Xiaolei Wu

Big MCGs

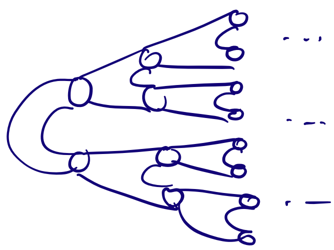
$$MCG(S) = \pi_0(\mathrm{Homeo}^+(S))$$

for an ∞-type surface S

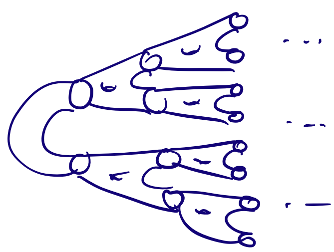
Eg



$= \Sigma_\infty \qquad (\infty, *, *)$



$\cong S^2 \smallsetminus \mathcal{C} \qquad (0, \mathcal{C}, \emptyset)$



$(\infty, \mathcal{C}, \mathcal{C})$

In general:

S — connected, orientable surface
second countable ($\equiv$ paracompact $\equiv$ metrisable)

∞-type $\overset{\text{def}}{\Longleftrightarrow}$ $\pi_1(S)$ is not fin. gen.

$(\Longleftrightarrow \pi_1(S) \cong F_\infty)$

$(\Longleftrightarrow MCG(S)$ is uncountable$)$

<u>Classification</u> by
- genus $\in \mathbb{N} \cup \{\infty\}$
- space of ends ("∂ of Freudenthal compact")
- subspace of non-planar ends
  (ends where all nbhds have ∞ genus)

[Kerékjártó 1923
Richards 1963]

Arise in dynamics (complements of attractors)
leaves of foliations
...

What is known about $H_*(MCG(S))$ for ∞-type S ?

<u>Thm</u> [Calegari - Chen '19 '20]

S compact $\Rightarrow$ $H_1(MCG(S \setminus \mathcal{C})) \cong H_1(MCG(S))$

$\uparrow$
Cantor

$H_2(MCG(S^2 \setminus \mathcal{C})) \cong \mathbb{Z}/2$

$H_2(MCG(\mathbb{R}^2 \setminus \mathcal{C})) \supseteq \mathbb{Z}$

<u>Thm</u> A [P.-Wu '22]

$$H_i(MCG(\mathbb{R}^2 \setminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

For contrast...

Thm B [P.-Wu '22]

$$H_*(\mathrm{MCG}(\Sigma_\infty)) \supseteq \Lambda^*_{\mathbb{Z}}\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big)$$

continuum.

In degree 1 : [Domat '20] $\quad H_1(\mathrm{MCG}(\Sigma_\infty)) \supseteq \bigoplus_{\mathfrak{c}} \mathbb{Q}$

Rmk $\quad \operatorname*{colim}_g \mathrm{MCG}(\Sigma_{g,1}) \hookrightarrow \mathrm{MCG}(\Sigma_\infty)$

$H_*(\Omega^\infty_0 MTSO(2)) \qquad H_*$ uncountable in all degrees

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Thm A' $\quad \widetilde{H}_*(\mathrm{MCG}(D^2 \setminus e)) = 0$

This implies Thm A:

$$1 \to \mathbb{Z} \hookrightarrow \mathrm{MCG}(D^2 \setminus e) \longrightarrow \mathrm{MCG}(\mathbb{R}^2 \setminus e) \to 1$$

$$E^2 = \begin{array}{|l}
\hline
H_*(\mathrm{MCG}(\mathbb{R}^2 \setminus e)) \\
\hline
H_*(\mathrm{MCG}(\mathbb{R}^2 \setminus e)) \\
\hline
\end{array} \longrightarrow \begin{array}{c|cc}
 & & 0 \\
0 & \boxed{\mathbb{Z}} & \\
\hline
 & 0 &
\end{array} = E^\infty$$

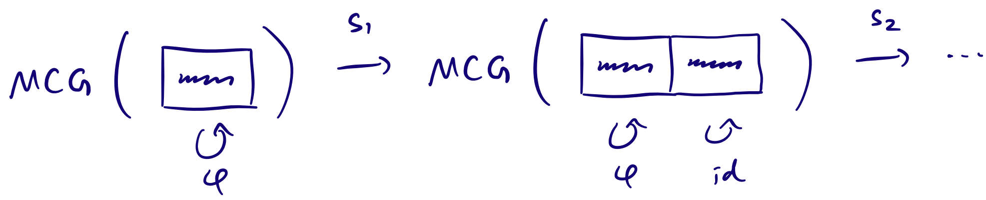
cf. Exercise session on Weds.

Plan

- Proof of Thm A' — 2 steps

(• Proof of Thm B)

① Consider the sequence



Thm : This is hom. stable.

proof $\mathcal{M}_2^\infty$ = groupoid of conn., orientable surfaces S with $I \hookrightarrow \partial S$

& homeomorphisms

braided monoidal ✓

injectivity ✓ ⇒ (H2)

fails cancellation ⟶ fix formally by passing to another category with objects = words on the alphabet of surfaces

Set $A = D^2$

$X = D^2 \setminus e$

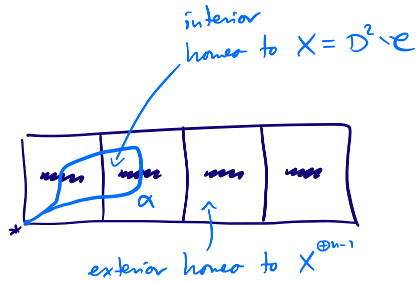
Machine of [RWW] ⇝ reduces to proving high-connectivity of $S_n(A,X)$.

"trick" (complete join argument from [Hatcher-Wahl])

⇝ reduce to high-conn. of $TC_n(A,X)$

$TC_n(A,X)$

vertex:



$\{\alpha_0, \dots, \alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X^{\oplus n-p-1}$

$(n-1)$-dim.

$TC_\infty(A,X)$

same vertices

$\{\alpha_0, \dots, \alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X$

∞-dim.

<u>same</u> $(n-2)$-skeleton as $TC_n(A,X)$

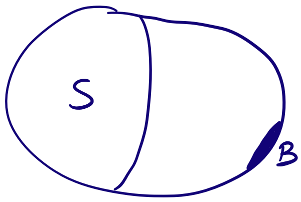
(only clopen subsets of $\mathcal{C}$ are $\emptyset$ and $\mathcal{C}$)

<u>Prop</u> $TC_\infty(A,X)$ is contractible. $\square$.

<u>Rmk</u> Follows ideas of proof of [Szymik–Wahl], who proved $\tilde{H}_*(V) = 0$ via hom. stab. for $V \to V \to \cdots$.

V: Thompson group

② <u>Lemma</u> (adapted from [Mather '71], related to "dissipated" groups, "binate" groups ....)

$T =$ 

S, B disjoint, closed $\subseteq T$

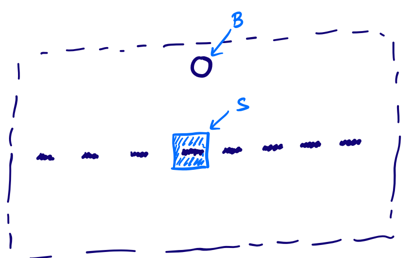
$$\pi_0 \operatorname{Homeo}_{\partial S}(S) \xrightarrow{(*)} \pi_0 \operatorname{Homeo}_B(T)$$

Suppose (a) $(*)$ is a H_*-isomorphism

(b) $\exists\, \varphi \in \operatorname{Homeo}_B(T)$: $\forall k \geq 1$ $\quad \varphi^k(S) \cap S = \phi$

$\bigcup_{i=k}^{\infty} \varphi^i(S)$ is <u>closed</u>

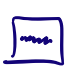
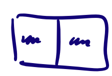
Then $\tilde{H}_*\big(\pi_0 \operatorname{Homeo}_B(T)\big) = 0$

Apply to 

$= T$ $\quad \cong D^2 \setminus \big(\coprod_\infty \mathcal{C}\big)^+$

$\cong D^2 \setminus \mathcal{C}$.

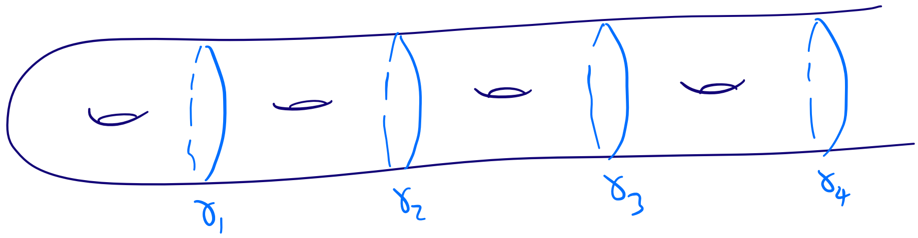
So Thm A' $\Longleftarrow$ (a) + (b).

Step ① $\rightsquigarrow$ MCG() $\xrightarrow{H_* \cong}$ MCG() $\rightsquigarrow$ (a)

(b) : φ = shift 1 step to the right.

$\square$.

Idea of proof of Thm B



$$X \subseteq \mathbb{N} \rightsquigarrow f(X) := \prod_{i \in X} (T_{\gamma_i})^{i!}$$

$$f_n(X) := \prod_{\substack{i \in X \\ i \geq n}} (T_{\gamma_i})^{i!/n}$$

Exercise

(*) There exists a ~~continuum!~~ collection of infinite $X \subseteq \mathbb{N}$ so that any two of them intersect in a finite set.

Construction

$$\begin{array}{ccc}
1 & \longmapsto & f(X) \\
\bigoplus_{c} \mathbb{Z} & \longrightarrow & MCG(\Sigma_\infty) \\
\big\downarrow & & \big\downarrow \\
\bigoplus_{c} \mathbb{Q} & \xrightarrow{\ \theta\ } & MCG(\Sigma_\infty)^{ab} \\
1/n & \longmapsto & f_n(X).
\end{array}$$

$$\left[(f_n(X))^n \right] = \left[f(X) \right]$$

[Birman, Powell '70s]: Every mapping class on a compact surface ($g \geq 3$) is a product of commutators.

Using (*) and [Domat '20]

machinery of [Bestvina-Bromberg-Fujiwara '15]

The map Θ is <u>injective</u>.

<u>Fact</u> : Any injective map $\bigoplus_I \mathbb{Q} \longrightarrow A$ admits a <u>retraction</u>.

$$
\begin{tikzcd}
\bigoplus_c \mathbb{Z} \arrow[r] \arrow[d, hook] & MCG(\Sigma_\infty) \arrow[d] \\
\bigoplus_c \mathbb{Q} & MCG(\Sigma_\infty)^{ab} \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_*\left(\bigoplus_c \mathbb{Z}\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_*(MCG(\Sigma_\infty)) \arrow[r] & H_*\left(\bigoplus_c \mathbb{Q}\right) \arrow[d, phantom, "\cong" rotate=-90] \\
\Lambda^*_{\mathbb{Z}}\left(\bigoplus_c \mathbb{Z}\right) \arrow[rrd, "\text{injective}"'] & & \Lambda^*_{\mathbb{Z}}\left(\bigoplus_c \mathbb{Q}\right) \arrow[d, phantom, "\cong" rotate=-90] \\
& & \Lambda^*_{\mathbb{Q}}\left(\bigoplus_c \mathbb{Q}\right)
\end{tikzcd}
$$

$\square$.