Homological mapping class group representations and lower central series

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- $\mathfrak{J}(k)$ is the k-th term of the *Johnson filtration* of $\mathrm{MCG}(\Sigma_{g,1})$

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Fact (because $\mathrm{MCG}(\Sigma_{g,1}) \subset \mathrm{Aut}(\mathbf{F}_{2g})$ and $\mathbf{F}_{2g}$ is residually nilpotent)

$$\bigcap_{k=1}^{\infty} \mathfrak{J}(k) = \{\mathrm{id}\}$$

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$\mathfrak{J}(k) =$ kernel of the action of $\mathrm{MCG}(\Sigma_{g,1})$ on $\mathbf{F}_{2g}/\Gamma_{k+1}$.

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Corollary [Moriyama'07]

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The Johnson filtration

- Lower central series: $\pi_1(\Sigma_{g,1}) = \mathbf{F}_{2g} = \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \cdots$
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$\rightsquigarrow$ a much "weaker" representation...

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$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \frac{\mathbb{Z}}{2} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Q})$$

Reps of MCGs – Heisenberg twisted coefficients

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Reps of MCGs – Heisenberg twisted coefficients

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We obtain a *twisted* representation, defined over $\mathbb{Z}[\mathcal{H}_g]$:

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More generally

Replace the coefficients $\mathbb{Z}[\mathcal{H}_g]$ with any $\mathcal{H}_g$-representation W over R to get a twisted $\mathrm{MCG}(\Sigma_{g,1})$-representation $\mathcal{V}(W)$ over R.

Reps of MCGs – Untwisting and kernel

Summary: for each $k \geqslant 2$:

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Lower central series of surface braid groups

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Fundamental question

Given Σ and $\lambda \vdash k$, when does $\Gamma_*(\mathbf{B}_\lambda(\Sigma))$ stop? ($\exists i : \Gamma_i = \Gamma_{i+1}$?)

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Theorem [Darné-P.-Soulié, to appear, Memo. AMS]

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