Untwisting Heisenberg homological representations of mapping class groups, I

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Outline

- Motivation
- Constructing twisted rep's of $MCG(\Sigma_{g,1})$ — depending on a choice of representation V of the Heisenberg group $\mathcal{H}_g$
- Untwisting them on $Tor(\Sigma_{g,1})$ — for any V
- Part II (September): untwisting on $MCG(\Sigma_{g,1})$ for special choices of V

Motivation — linearity

M : smooth manifold

$MCG(M) := \pi_0(\mathrm{Diff}_\partial(M))$

Q: Is $MCG(M)$ linear?

↳ Does it embed into $GL_n(\mathbb{F})$?

Eg (1992) No if $M = (S^1 \times S^2)^{\# g} \setminus \text{disc}$
Formanek-Procesi

(2000) Yes if $M = D^2 \setminus \text{punctures}$
Bigelow, Krammer
$MCG(M) \cong$ braid groups

(??) Unknown if $M = \Sigma_{g,1} = (S^1 \times S^1)^{\# g} \setminus \text{disc}$

- Non-linearity of $MCG((S^1 \times S^2)^{\# g} \setminus \text{disc})$ is proven by:
  - inventing a class of "FP-groups"
  - proving that all FP-groups are not linear
  - embedding an FP-group into $MCG(\cdots)$.

Thm [Brendle - Hamidi-Tehrani '01]
No FP-groups embed into $MCG(\Sigma_{g,1})$.

- Linearity of B_n is proven by showing that the Lawrence - Bigelow representations are faithful.

Aim: Define analogues of these for $MCG(\Sigma_{g,1})$.

Twisted homological rep's of MCG(M)

Assume $\partial M \neq \emptyset$.

Choose integer $n \geq 1$.

$$C_n(M) = \{ (x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ if } i \neq j \} / \mathfrak{S}_n$$

basepoint configuration $\subseteq \partial M$

Choose a quotient

$$B_n(M) = \pi_1(C_n(M)) \overset{\phi}{\twoheadrightarrow} Q$$

and a Q-representation V (over R).

$\rightsquigarrow H_*(C_n(M); V)$

possibly Borel-Moore — R-module

$MCG(M)$ acts on $C_n(M)$ fixing its basepoint

$\rightsquigarrow$ it acts on $B_n(M)$

Lemma: Suppose $\forall f \in MCG(M)$,

$$\begin{tikzcd} B_n(M) \arrow[rr, "f_*"] \arrow[dr, two heads, "\phi"'] & & B_n(M) \arrow[dl, two heads, "\phi"] \\ & Q & \end{tikzcd}$$

commutes.

Then the local system V on $C_n(M)$ is preserved by the $MCG(M)$-action and $\exists$ well-defined representation of $MCG(M)$ on $H_*(C_n(M); V)$.

Lemma:

Suppose $\forall f \in MCG(M)$, we have $f_*(\ker(\phi)) = \ker(\phi)$.

Then $\exists$ well-defined action $\Phi : MCG(M) \longrightarrow \mathrm{Aut}(Q)$

s.t.

$$\begin{tikzcd} B_n(M) \arrow[r, "f_*"] \arrow[d, two heads, "\phi"'] & B_n(M) \arrow[d, two heads, "\phi"] \\ Q \arrow[r, "\Phi(f)"] & Q \end{tikzcd}$$

& $\exists$ well-defined twisted representation of $MCG(M)$ on $H_*(C_n(M); V)$.

... more precisely...

Def: For a Q-representation V with action $Q \xrightarrow{\rho} \mathrm{Aut}_R(V)$, and an automorphism $\tau \in \mathrm{Aut}(Q)$, we denote by ${}_\tau V$ the Q-representation $Q \xrightarrow{\tau} Q \xrightarrow{\rho} \mathrm{Aut}_R(V)$.

The twisted representation of $MCG(M)$ consists of

- R-modules $H_*(C_n(M); {}_\tau V)$ for $\tau \in \mathrm{Aut}(Q)$
- R-linear isomorphisms

$$H_*(C_n(M); {}_{\tau \circ \Phi(f)} V) \longrightarrow H_*(C_n(M); {}_\tau V)$$

for $f \in MCG(M)$ and $\tau \in \mathrm{Aut}(Q)$.

Notes

- $MCG(M)$ acts by isomorphisms between different modules, not by automorphisms of a single module.
- Each ${}_\tau V$ is isomorphic as a Q-representation to V, but not canonically.

Eg — Lawrence–Bigelow representations

$M = D^2 \smallsetminus k$ points $\qquad MCG(M) \cong B_k$

$n \geq 2$

$$B_n(D^2 \smallsetminus k \text{ points}) \xrightarrow{\ \phi\ } \!\!\!\!\rightarrow Q = \{ q^r t^s \mid r,s \in \mathbb{Z} \} \cong \mathbb{Z}^2$$

$$b \longmapsto q^{\text{writhe}(b)}\, t^{\text{winding}\,\#(b)}$$

V = regular representation of Q, over $R = \mathbb{Z}[Q]$

$= \mathbb{Z}[Q] = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$

Lemma: ϕ is invariant under the B_k-action on $B_n(D^2 \smallsetminus k \text{ points})$

⤷ representation of B_k on

$$H_n^{BM}(C_n(D^2 \smallsetminus k \text{ points}); \mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$$

over the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.

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Eg - Moriyama representations

$M = \Sigma_{g,1}$

$B_n(\Sigma_{g,1}) \xrightarrow{\ \phi\ } \!\!\!\!\rightarrow \mathfrak{S}_n$ induced permutation of base config.

$V = \mathbb{Z}[\mathfrak{S}_n]$ regular representation (over $\mathbb{Z}$)

⤷ representation (over $\mathbb{Z}$) of $MCG(\Sigma_{g,1})$ on

$$H_n^{BM}(C_n(\Sigma_{g,1}); \mathbb{Z}[\mathfrak{S}_n])$$
$$\cong \quad \text{Shapiro's Lemma}$$
$$H_n^{BM}(F_n(\Sigma_{g,1}); \mathbb{Z})$$

ordered config. space

Thm [Moriyama '07]

The kernel of this representation is $J(n)$, the n^{th} term of the Johnson filtration of $MCG(\Sigma_{g,1})$.

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Heisenberg representations

$M = \Sigma_{g,1} \qquad n \geq 2$

Def / Lemma:
- The genus-g discrete Heisenberg group $\mathcal{H}_g$ is the central extension of $H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ with kernel $\mathbb{Z}$ and cocycle given by the intersection form.
- It may be written as $\mathbb{Z} \times H_1(\Sigma_{g,1}; \mathbb{Z})$ with $(k,x)(l,y) = (k + l + x.y\,,\ x + y)$.
- It has presentation

$$\mathcal{H}_g = \Big\langle \underbrace{\begin{matrix} a_1 \ldots a_g \\ b_1 \ldots b_g \end{matrix}}_{\text{symplectic basis for } H_1},\ u \ \Big|\ \begin{matrix} \text{all pairwise commute,} \\ \text{except } a_i b_i = u^2 b_i a_i \end{matrix} \Big\rangle$$

$(1,0)$

Lemma [Blanchet - P. - Shaukat]
- $\exists$ quotient $B_n(\Sigma_{g,1}) \xrightarrow{\ \phi\ } \!\!\!\!\rightarrow \mathcal{H}_g$
- $\ker(\phi)$ is preserved by the action of $MCG(\Sigma_{g,1})$.

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Cor: For any $\mathcal{H}_g$-representation V, there is a twisted $MCG(\Sigma_{g,1})$-representation on $H_n^{BM}(C_n(\Sigma_{g,1}); V)$.

Notes:
- We actually work with homology relative to a certain subspace of $C_n(\Sigma_{g,1})$ — namely all configurations that intersect a fixed interval in $\partial \Sigma_{g,1}$.
- **Prop** [BPS]: As a module, $H_n^{BM}(C_n(\Sigma_{g,1}); V)$ is just a finite $\oplus$ of copies of V.
- **Prop** [BPS]: The kernel of the $MCG(\Sigma_{g,1})$-action on $H_n^{BM}(C_n(\Sigma_{g,1}); \mathbb{Z}[\mathcal{H}_g])$ is contained in $J(n) \cap$ "Magnus kernel"

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General strategy for untwisting

Idea: Try to find coefficient isomorphisms

$$V \cong {}_{\Phi(f)}V \qquad (*)$$

depending "naturally" (i.e. homomorphically) on $f \in MCG(\Sigma_{g,1})$.

If we can do this, then we have:

$$\begin{array}{ccc} H_n^{BM}(C_n(\Sigma_{g,1}); {}_{\Phi(f)}V) & \longrightarrow & H_n^{BM}(C_n(\Sigma_{g,1}); V) \\ \uparrow & & \\ H_n^{BM}(C_n(\Sigma_{g,1}); V) & & \end{array}$$

induced by $(*)$

from the twisted representation

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This will be an untwisted representation on the module $H_n^{BM}(C_n(\Sigma_{g,1}); V)$.

Thm [BPS]:

This strategy may be carried out, if:

(1) We restrict to the Torelli group

(2) $V = \mathcal{H}_g \oplus \mathbb{Z}$

(3) V = Schrödinger and:

(a) we pass to the (stably) universal central extension of $MCG(\Sigma_{g,1})$

(b) we restrict to an "Earle - Morita subgroup".

Today: (1)

Part II (September): (2) and (3).

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Untwisting on Torelli

Strategy:

$MCG(\Sigma_{g,1}) \xrightarrow{\Phi} Aut(\mathcal{H}_g)$

Prop: $\Phi^{-1}(Inn) = Torelli$.

(0)

(1) Consider a central ext. $\widetilde{Tor} \to Tor$

(2) Untwist on $\widetilde{Tor}$

(3) Show that $\widetilde{Tor} \to Tor$ is a trivial extension.

$\Phi^{-1}(Inn) = Torelli$

(i) Structure of $Aut(\mathcal{H}_g)$

Recall that $\mathcal{H}_g = \mathbb{Z} \times H$ $\qquad H = H_1(\Sigma_{g,1};\mathbb{Z})$

with $(k,x)(l,y) = (k+l+x.y,\ x+y)$

Its centre is $\mathbb{Z}\times\{0\}$ generated by $(1,0) = u$.

Any aut. must send $(1,0) \mapsto (\pm 1, 0)$.

Def: $Aut^+(\mathcal{H}_g) :=$ subgroup of aut. fixing $(1,0)$.

Obs: The action of $MCG(\Sigma_{g,1})$ fixes $(1,0)$.

Prop: $Aut^+(\mathcal{H}_g) \cong Hom(H,\mathbb{Z}) \rtimes Sp(H)$

$$\varphi \longleftrightarrow (\varphi^\circ, \varphi_*)$$

$$\varphi(k,x) = (k + \varphi^\circ(x),\ \varphi_*(x))$$

(ii) A crossed homomorphism

$$\begin{tikzcd}
 & & H^1(\Sigma_{g,1};\mathbb{Z}) = H^1 \arrow[d, phantom, sloped, "\cong"] \\
 & & Hom(H,\mathbb{Z}) \\
MCG(\Sigma_{g,1}) \arrow[rru, bend left, "\partial"] \arrow[r, "\Phi"] \arrow[rrd, bend right] & Aut^+(\mathcal{H}_g) \arrow[r, phantom, "\cong"] & Hom(H,\mathbb{Z}) \rtimes Sp(H) \arrow[u, two heads] \arrow[d, two heads] \\
 & & Sp(H)
\end{tikzcd}$$

crossed homomorphism

Poincaré duality

usual symplectic action

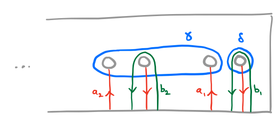
$$\Phi(f)(k,x) = (k + \partial(f)(x),\ f_*(x))$$

(iii) Proposition [BPS]:

$$Image(\partial|_{Torelli}) \subseteq 2.H^1 \qquad (\text{in fact, } =)$$

Proof

[Johnson]: Torelli is normally generated by the single element $T_\gamma T_\delta^{-1}$:



interior bdy circles are identified in pairs by reflection

$\Rightarrow$ it suffices to check that

$$\partial(T_\gamma T_\delta^{-1}) \in 2.H^1$$

Direct calculation $\rightsquigarrow$ $\partial(T_\gamma T_\delta^{-1})$: $a_1 \mapsto 2$, $b_1 \mapsto 0$, $a_i, b_i \mapsto 0$ $\quad i \geq 2$

(iv) Proof that $\Phi^{-1}(Inn) = Torelli$

($\subseteq$) If $\Phi(f) \in Aut(\mathcal{H}_g)$ is inner, then $f_* \in Aut(H)$ must also be inner. But H is abelian, so $f_* = id$, i.e. $f \in Torelli$.

($\supseteq$) If $f \in Torelli$, then $f_* = id$ and $\partial(f) \in 2.H^1$ so $\exists y \in H : \partial(f)(x) = 2x.y$

Thus
$$\begin{aligned}\Phi(f)(k,x) &= (k + \partial(f)(x),\ f_*(x)) \\ &= (k + 2x.y,\ x) \\ &= (0,-y)(k,x)(0,y)\end{aligned}$$

So $\Phi(f) =$ conjugation by $(0,-y)$.

A central extension of the Torelli group

Step (1)

In general, $Inn(G) \cong G/Z(G)$.

For $G = \mathcal{H}_g$, $Z(\mathcal{H}_g) \cong \mathbb{Z}$ (gen. by $(1,0)$)

$Inn(\mathcal{H}_g) \cong H$

pullback

$$\begin{tikzcd}
 & Z(\mathcal{H}_g) \arrow[r, phantom, "\cong"] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook] \\
\widetilde{Tor} \arrow[r, "\tilde{\partial}"] \arrow[d, two heads, "\pi"'] & \mathcal{H}_g \arrow[r, equal] \arrow[d, two heads, "c"] & \mathcal{H}_g \arrow[d, two heads] \\
Torelli(\Sigma_{g,1}) \arrow[r, "\partial"] \arrow[d, phantom, sloped, "\subseteq"] & Inn(\mathcal{H}_g) \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\subseteq"] & H \arrow[d, phantom, sloped, "\subseteq"] \\
MCG(\Sigma_{g,1}) \arrow[r, "\Phi"] & Aut^+(\mathcal{H}_g) \arrow[r, phantom, "\cong"] & H \rtimes Sp(H)
\end{tikzcd}$$

Step ② Untwisting on $\widetilde{\mathrm{Tor}}$

Recall: To untwist, we need isomorphisms

$$V \xrightarrow{\cong} {}_{\tau}V \qquad (\tau \in \mathrm{Aut}(\mathcal{H}_g))$$

of $\mathcal{H}_g$-representations over R.

($\mathcal{H}_g$ acts on the left, R acts on the right)

Obs: When $\tau \in \mathrm{Inn}(\mathcal{H}_g)$, one has:

$$v \mapsto h \cdot v \; : \; V \xrightarrow{\cong} {}_{c_h}V$$

where

$$\begin{array}{c} h \in \mathcal{H}_g \\ \downarrow \\ \tau = c_h \in \mathrm{Inn}(\mathcal{H}_g) \\ \| \\ h - h^{-1} \end{array}$$

Note: This requires a choice of lift of $\tau = c_h \in \mathrm{Inn}(\mathcal{H}_g)$ to $\mathcal{H}_g$!

Untwisting on $\widetilde{\mathrm{Tor}}$:

$$\begin{tikzcd}
 & h := \tilde{\partial}(f) \in \mathcal{H}_g \arrow[r] & \begin{array}{c} V \\ \downarrow \\ {}_{c_h}V \end{array} \arrow[r, "\Phi(\pi(f)) = c_h"] & H_n^{BM}(C_n(\Sigma_{g,1}); V) \arrow[d] \\
f \in \widetilde{\mathrm{Tor}} \arrow[ur] \arrow[dr] & & & H_n^{BM}(C_n(\Sigma_{g,1}); {}_{\Phi(\pi(f))}V) \arrow[d] \\
 & \pi(f) \in \mathrm{Torelli} \arrow[rr] & & H_n^{BM}(C_n(\Sigma_{g,1}); V)
\end{tikzcd}$$

Step ③ $\widetilde{\mathrm{Tor}}$ is a trivial extension

Lemma

Every $\mathbb{Z}$-central ext. of $MCG(\Sigma_{g,1})$ becomes trivial when restricted to $\mathrm{Torelli}(\Sigma_{g,1})$.

Proof (for $g \geq 3$)

Central extensions of G by $\mathbb{Z}$ are classified by $H^2(G;\mathbb{Z})$, so it is enough to show that the restriction map

$$H^2(\mathrm{Torelli};\mathbb{Z}) \longleftarrow H^2(MCG;\mathbb{Z})$$

is zero. This follows from:

$$\begin{tikzcd}
H^2(\mathrm{Torelli};\mathbb{Z}) & H^2(MCG;\mathbb{Z}) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] \arrow[ll, bend right, "0"'] \\
 & \mathbb{Z} & \mathbb{Z} \arrow[l, "\cong"']
\end{tikzcd}$$

$\widetilde{\mathrm{Tor}} \downarrow \mathrm{Torelli}$ is pulled back from $\mathcal{H}_g \downarrow H$ along $\mathrm{Torelli} \xrightarrow{\partial} H$

So a cocycle for $\widetilde{\mathrm{Tor}}$ is

$$\begin{aligned} \mathrm{Torelli} \times \mathrm{Torelli} &\longrightarrow \mathbb{Z} \\ (f,g) &\longmapsto \partial(f).\partial(g) \\ & \quad \| \\ & \; \delta(f).\delta(g) \\ &\overset{f_* = id}{} \quad \| \\ & \; f_*^{-1}(\delta(f)).\delta(g) \\ & \quad \| \\ & \; -c(f,g) \end{aligned}$$

Prop [BPS]:

The crossed homomorphism

$$\partial : MCG(\Sigma_{g,1}) \longrightarrow H$$

is equal to δ.

another crossed hom. defined by Morita.

Thm [Morita]:

$\exists$ cocycle $MCG(\Sigma_{g,1}) \times MCG(\Sigma_{g,1}) \longrightarrow \mathbb{Z}$

$$(f,g) \longmapsto \underset{\substack{\| \\ c(f,g)}}{\delta(f^{-1}).\delta(g)} = -f_*^{-1}(\delta(f)).\delta(g)$$

general properties of crossed homom.'s

Corollary:

- The cocycle for $\widetilde{\mathrm{Tor}}$ extends to the cocycle $-c$ on the full MCG.
- Hence by the lemma, $\widetilde{\mathrm{Tor}}$ is the trivial $\mathbb{Z}$-central extension of Torelli, ie.

$$\begin{tikzcd}
\mathrm{Torelli}(\Sigma_{g,1}) \times \mathbb{Z} \arrow[r, phantom, "\cong"] & \widetilde{\mathrm{Tor}} \arrow[r] \arrow[d] & \mathrm{Aut}_R\big(H_n^{BM}(C_n(\Sigma_{g,1}); V)\big) \\
 & \mathrm{Torelli}(\Sigma_{g,1}) \arrow[ul, dashed] &
\end{tikzcd}$$