Untwisting Heisenberg homological representations of mapping class groups, II

IMAR
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Outline

- Motivation
- Constructing twisted rep's of $MCG(\Sigma_{g,1})$

  depending on a choice of representation V of the Heisenberg group $\mathcal{H}_g$
- Untwisting for special choices of V
  - $\mathcal{H}_g \oplus \mathbb{Z}$
  - Schrödinger

Motivation — linearity

M : smooth manifold

$MCG(M) := \pi_0(\mathrm{Diff}_\partial(M))$

Q: Is $MCG(M)$ linear?

↳ Does it embed into $GL_n(\mathbb{F})$?

(2000) Yes if $M = D^2 \setminus$ punctures
Bigelow, Krammer

$MCG(M) \cong$ braid groups

(??) Unknown if $M = \Sigma_{g,1} = (S^1 \times S^1)^{\# g} \setminus$ disc

proof via faithfulness of certain "Lawrence-Bigelow" representations

Idea: Construct analogues for $MCG(\Sigma_{g,1})$

Twisted homological rep's of $MCG(M)$

Assume $\partial M \neq \emptyset$.

Choose integer $n \geq 1$.

$$C_n(M) = \{ (x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ if } i \neq j \} / \mathfrak{S}_n$$

basepoint configuration $\subseteq \partial M$

Choose a quotient

$$B_n(M) = \pi_1(C_n(M)) \xrightarrow{\phi} Q$$

and a Q-representation V (over R).

$$\rightsquigarrow H_*(C_n(M); V)$$

possibly Borel-Moore

R-module

$MCG(M)$ acts on $C_n(M)$ fixing its basepoint

$\rightsquigarrow$ it acts on $B_n(M)$

Lemma:

Suppose $\forall f \in MCG(M)$, we have $f_*(\ker(\phi)) = \ker(\phi)$.

Then $\exists$ well-defined action $\Phi : MCG(M) \longrightarrow \mathrm{Aut}(Q)$

s.t.

$$\begin{tikzcd} B_n(M) \arrow[r, "f_*"] \arrow[d, two heads, "\phi"'] & B_n(M) \arrow[d, two heads, "\phi"] \\ Q \arrow[r, "\Phi(f)"] & Q \end{tikzcd}$$

& $\exists$ well-defined twisted representation of $MCG(M)$ on $H_*(C_n(M); V)$.

Def: For a Q-representation V with action $Q \xrightarrow{\varrho} \mathrm{Aut}_R(V)$, and an automorphism $\tau \in \mathrm{Aut}(Q)$, we denote by ${}_\tau V$ the Q-representation $Q \xrightarrow{\tau} Q \xrightarrow{\varrho} \mathrm{Aut}_R(V)$.

The twisted representation of $MCG(M)$ consists of

- R-modules $H_*(C_n(M); {}_\tau V)$ for $\tau \in \mathrm{Aut}(Q)$
- R-linear isomorphisms

$$H_*(C_n(M); {}_{\tau \circ \Phi(f)} V) \longrightarrow H_*(C_n(M); {}_\tau V)$$

for $f \in MCG(M)$ and $\tau \in \mathrm{Aut}(Q)$.

Note:

Each ${}_\tau V$ is isomorphic as a Q-representation to V, but not canonically.

Heisenberg twisted representations

$M = \Sigma_{g,1} = \Sigma \qquad (g \geq 2)$

Def: $\mathcal{H}_g := \mathbb{Z} \times H_1(\Sigma)$

with $(k, x)(\ell, y) = (k + \ell + x \cdot y,\ x + y)$

Discrete Heisenberg group of genus g.

Fact: $\exists$ central extension

$$1 \to \mathbb{Z} \to \mathcal{H}_g \to H_1(\Sigma) \to 1$$

Proposition [Blanchet - P. - Shaukat]

- $\exists$ quotient $B_n(\Sigma) \xrightarrow{\phi} \mathcal{H}_g$
- $\ker(\phi)$ is preserved by the action of $MCG(\Sigma)$

Idea:

- Quotient by $\langle\langle [\sigma_1, b] : b \in B_n(\Sigma) \rangle\rangle$ $\; = \ker(\phi)$
- $\cong \mathcal{H}_g$ (check by presentations)
- Enough to check that σ_1 is fixed by $MCG(\Sigma)$ action.
- ✓ because $MCG(\Sigma)$ action fixes a nbhd of $\partial\Sigma$.

Cor: • There is an action $\Phi : MCG(\Sigma) \longrightarrow Aut(\mathcal{H}_g)$.

• For any $\mathcal{H}_g$-representation V, there is a twisted $MCG(\Sigma)$-representation on $H_n^{BM}(C_n(\Sigma); V)$.

Notes:

- We actually work with homology relative to a certain subspace of $C_n(\Sigma)$ — namely all configurations that intersect a fixed interval in $\partial\Sigma$.
- Prop [BPS]: As a module, $H_n^{BM}(C_n(\Sigma); V)$ is just a finite $\oplus$ of copies of V.
- Prop [BPS]: The kernel of the $MCG(\Sigma)$-action on $H_n^{BM}(C_n(\Sigma); \mathbb{Z}[\mathcal{H}_g])$ is contained in $J(n) \cap$ "Magnus kernel"

General strategy for untwisting

Idea: Try to find coefficient isomorphisms

$$V \cong {}_{\Phi(f)}V \qquad (*)$$

depending "naturally" (i.e. homomorphically) on $f \in MCG(\Sigma)$.

Then:

$$\begin{tikzcd} H_n^{BM}(C_n(\Sigma); {}_{\Phi(f)}V) \arrow[r, "\text{from the twisted representation}"] & H_n^{BM}(C_n(\Sigma); V) \\ H_n^{BM}(C_n(\Sigma); V) \arrow[u, "\text{induced by } (*)"] \arrow[ur, "\text{untwisted representation}"'] & \end{tikzcd}$$

Thm [BPS]:

This strategy may be carried out, if:

(1) We restrict to the Torelli group

(2) $V = \mathcal{H}_g \oplus \mathbb{Z}$

(3) $V =$ Schrödinger and:

(a) we pass to the (stably) universal central extension of $MCG(\Sigma_{g,1})$

(b) we restrict to an "Earle-Morita subgroup".

Last time: (1)

Today: (2) and (3).

The linearised translation action of $\mathcal{H}_g$

Affine space (over R)

- R-module M
- set A with a free, transitive action of $(M, +)$

Affine automorphism

bijection $f : A \longrightarrow A$

s.t. $\exists\ \varphi : M \longrightarrow M$ R-linear automorphism (necessarily unique)

s.t. $\forall a \in A, \forall m \in M, \quad f(a+m) = f(a) + \varphi(m)$.

Linearisation

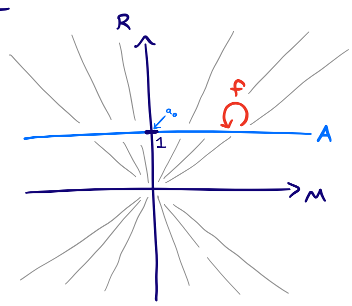
$Aff(A) \hookrightarrow Aut_R(M \oplus R)$ depending on $a_0 \in A$

Formally:

$$f \longmapsto \begin{tikzcd} M \arrow[r, "\varphi"] & M \\ \oplus & \oplus \\ R \arrow[r, "id"'] \arrow[uur, "m_0"] & R \end{tikzcd}$$

$m_0 \in M$ unique : $f(a_0) = a_0 + m_0$

Pictorial idea:



Affine structure on $\mathcal{H}_g$

$R = \mathbb{Z}$

$A = \mathcal{H}_g = \mathbb{Z} \times H_1(\Sigma)$ as a set

$M = \mathbb{Z} \oplus H_1(\Sigma)$ as a $\mathbb{Z}$-module

Observation

Fix $(k, x) \in \mathcal{H}_g$.

Then $(\ell, y) \longmapsto (k + \ell + x.y,\ x + y) \ : \ A \longrightarrow A$ is affine.

Cor: $\mathcal{H}_g \hookrightarrow Aff(A)$

Linearise this:

take $a_0 = 0$

$$\mathcal{H}_g \hookrightarrow \mathrm{Aff}(A) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(\mathbb{Z}\oplus H_1(\Sigma)\oplus\mathbb{Z}\right)$$

$$(k,x) \longmapsto \begin{pmatrix} 1 & x\cdot - & k \\ 0 & I & x \\ 0 & 0 & 1 \end{pmatrix}$$

Def $L := \overbrace{\mathbb{Z}\oplus H_1(\Sigma)}^{\mathcal{H}_g}\oplus\mathbb{Z}$ with this action of $\mathcal{H}_g$.

Proposition [Blanchet - P. - Shaukat]

For any $\tau\in\mathrm{Aut}^+(\mathcal{H})$, there is a canonical isomorphism of $\mathcal{H}_g$-representations $L \cong {}_\tau L$ given by $\tau\oplus\mathrm{id}_{\mathbb{Z}}$.

Corollary: $\exists$ untwisted representation of $MCG(\Sigma)$ on $H^{BM}_n(C_n(\Sigma);L)$.

The Schrödinger representation of $\mathcal{H}_g$

Real Heisenberg group

$$\begin{array}{ccccc} \mathcal{H}_g & = & \mathbb{Z} & \tilde{\times} & H_1(\Sigma) \\ \downarrow & & & & \downarrow \\ \mathcal{H}_g^{\mathbb{R}} & = & \mathbb{R} & \tilde{\times} & H_1(\Sigma;\mathbb{R}) \end{array}$$

product twisted via $(k,x)(l,y) = (k+l+x\cdot y,\ x+y)$

The Schrödinger representation

- $W = L^2(\mathbb{R}^g)$ complex Hilbert space
- $\rho : \mathcal{H}_g^{\mathbb{R}} \longrightarrow U(W)$ unitary representation (depending on $\hbar>0$)

Facts: (1) ρ is irreducible

(2) $\mathbb{R}\subset\mathcal{H}_g^{\mathbb{R}} \xrightarrow{\rho} U(W)$

$t \longmapsto e^{i\hbar/2\, t}\cdot\mathrm{id}_W$.

Theorem (corollary of Stone - von Neumann)

If $\rho' : \mathcal{H}_g^{\mathbb{R}} \longrightarrow U(W)$ is another unitary representation satisfying (1) and (2), then $\exists\, u\in U(W)$, unique up to rescaling by S^1, s.t. $\rho' = u.\rho.u^{-1}$.

Def

$$T:\ \mathrm{Aut}(\mathcal{H}_g^{\mathbb{R}}) \longrightarrow PU(W) = U(W)/S^1$$

$\varphi \longmapsto$ consider $\rho\circ\varphi$ $\quad ({}_\varphi W)$

$T(\varphi) :=$ the unique $u\in PU(W)$ s.t. $\rho\circ\varphi = u\rho u^{-1}$.

(Segal - Shale - Weil representation)

$$\begin{tikzcd} S^1 \arrow[d] & & & & S^1 \arrow[d] \\ \overline{MCG}(\Sigma) \arrow[rrrr, "Q"] \arrow[d, "\pi"'] & & & & U(W) \arrow[d] \\ MCG(\Sigma) \arrow[r, "\Phi"] & \mathrm{Aut}(\mathcal{H}_g) \arrow[r, phantom, "\subset"] & \mathrm{Aut}(\mathcal{H}_g^{\mathbb{R}}) \arrow[rr, "T"] & & PU(W) \end{tikzcd}$$

$f\in\overline{MCG}(\Sigma)$, $Q(f)\in U(W)$, $\pi(f)\in MCG(\Sigma)$, $\Phi(\pi(f))\in\mathrm{Aut}(\mathcal{H}_g)$

Proposition [Blanchet - P. - Shaukat]

As representations of $\mathcal{H}_g$,

$$Q(f):\ W \cong {}_{\Phi(\pi(f))}W$$

Corollary: $\exists$ untwisted unitary representation of $\overline{MCG(\Sigma)}$ on $H^{BM}_n(C_n(\Sigma);W)$.

Proof:

$f\in\overline{MCG(\Sigma)}$

twisted representation applied to $\pi(f)$

induced by $Q(f)$

$$H^{BM}_n(C_n(\Sigma);W) \longrightarrow H^{BM}_n(C_n(\Sigma);{}_{\Phi(\pi(f))}W) \longrightarrow H^{BM}_n(C_n(\Sigma);W)$$

Universal central extensions

A central extension $\tilde{G}\to G$ is universal if it admits a unique morphism to any other central ext. of G.

Lemma: G has a universal central extension (UCE) iff $H_1(G)=0$.
In this case, the kernel of the UCE is $H_2(G)$.

Fact: When $g\geq 4$, $H_1(MCG(\Sigma))=0$
$H_2(MCG(\Sigma))\cong\mathbb{Z}$.

Hence:

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & & \\ \widetilde{MCG}(\Sigma) \arrow[r] \arrow[d] & \overline{MCG}(\Sigma) \arrow[r] & U\left(H^{BM}_n(C_n(\Sigma);W)\right) \\ MCG(\Sigma) & & \end{tikzcd}$$