Moments of families of quadratic L-functions via scanning

(following work of Bergström
Diaconu
Petersen
Westerland)

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Outline

Background
- Dirichlet L-functions
- Analogue over function fields
- Conjectural asymptotic formulas for moments

Thm (BDPW): these asymptotic formulas are correct in the function field setting

Idea of proof
- Reduction to a stable twisted H_* calculation (via twisted homological stability)
- Reduction to a stable untwisted H_* calculation for a different moduli space
- Scanning methods to do this calculation.

Dirichlet characters & series

Dirichlet characters

Fix $d \in \mathbb{Z}$ positive (modulus)

Definition

- A Dirichlet character of modulus d is a group homomorphism

$$\begin{tikzcd}
 & (\mathbb{Z}/d\mathbb{Z})^\times \arrow[r, "\chi"] \arrow[d, hook] & \mathbb{C}^\times \arrow[d, hook] \\
\mathbb{Z} \arrow[r, two heads] \arrow[rr, bend right, "\chi"'] & \mathbb{Z}/d\mathbb{Z} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

extend by 0

- It is imprimitive if it factors through

$$(\mathbb{Z}/d\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/d_0\mathbb{Z})^\times$$

for a proper divisor $d_0 \mid d$. Otherwise it is primitive.

- It is real / quadratic if it takes values in $\mathbb{R}^\times = \{\pm 1\} \subseteq \mathbb{C}^\times$.

Remark χ always takes values in

$$\{ \phi(d)^{th} \text{ roots of } 1 \} \subseteq \mathbb{C}^\times$$

Lemma (consequence of the group structure of $(\mathbb{Z}/d\mathbb{Z})^\times$)

- $d = n$ or $4n$ $\longrightarrow$ $\exists$ unique real primitive χ
- $d = 8n$ $\longrightarrow$ $\exists$ two real primitive χ

  (n odd, square-free)

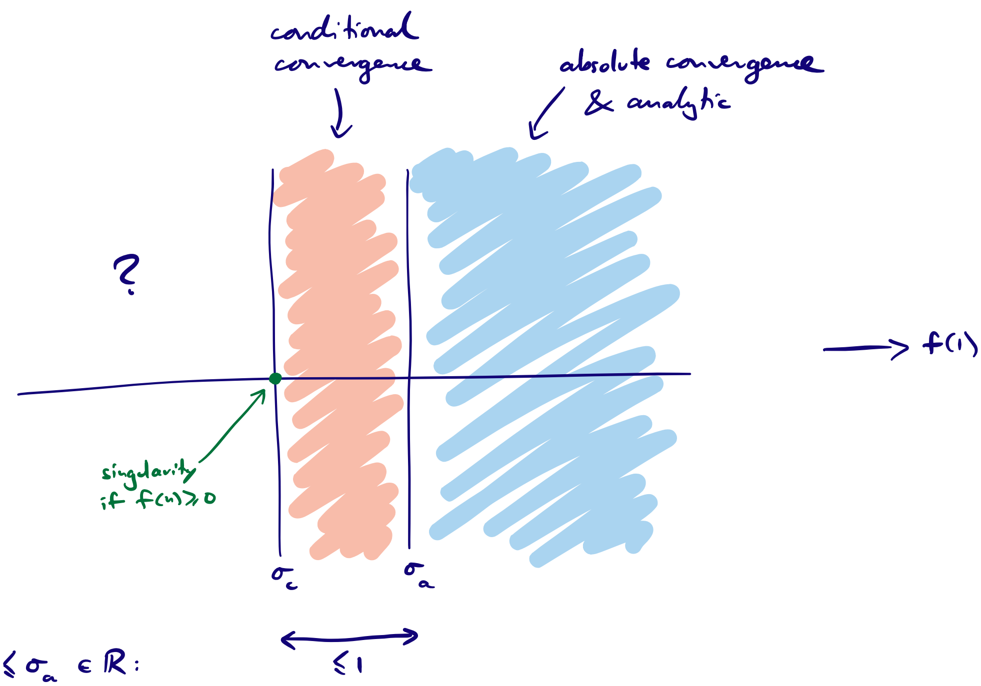
- d not of this form $\longrightarrow$ $\nexists$ real primitive χ

Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad f: \mathbb{N} \to \mathbb{C}, \quad s \in \mathbb{C}$$

Convergence:

If $F(s)$ does not converge everywhere or nowhere, then:



$\exists\, \sigma_c \leqslant \sigma_a \in \mathbb{R}$:

- $F(s)$ converges on $\mathrm{Re}(s) > \sigma_c$ but not on $\mathrm{Re}(s) < \sigma_c$
  - ↳ well-defined on $\mathrm{Re}(s) > \sigma_c$
- $F(s)$ converges absolutely on $\mathrm{Re}(s) > \sigma_a$ but not on $\mathrm{Re}(s) < \sigma_a$
  - ↳ analytic on $\mathrm{Re}(s) > \sigma_a$
- $\sigma_a - \sigma_c \leqslant 1$
- $F(\sigma + it) \to f(1)$ as $\sigma \to \infty$ uniformly in t
- if $f(n) \geqslant 0 \ \forall n$, then $F(s)$ has a singularity at $s = \sigma_c$

Examples

$f(n) = 1 \longrightarrow F(s) = \zeta(s)$ Riemann zeta-function

$\sigma_a = 1$

$\sigma_c = 1$

$f(n) = \mu(n) \longrightarrow F(s) = \dfrac{1}{\zeta(s)}$

Möbius fn

$f(n) = \phi(n) \longrightarrow F(s) = \dfrac{\zeta(s-1)}{\zeta(s)}$

Euler totient fn

$f(n) = \chi(n) \longrightarrow F(s) = L(s, \chi)$

Dirichlet char. modulo d

Dirichlet L-function.

$\sigma_a = 1$

$$\sigma_c = \begin{cases} 0 & \chi \neq \chi_1 \\ 1 & \chi = \chi_1 \end{cases}$$

trivial character mod d

Properties

Euler product formula

If $f : \mathbb{N} \longrightarrow \mathbb{C}$ is a semigroup homomorphism,

then
$$F(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s} f(p)} \qquad \text{for } \operatorname{Re}(s) > \sigma_a$$

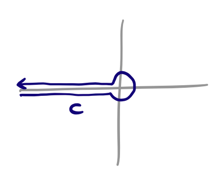
(**Coro** If $\chi = \chi_1$ is the trivial character mod d,

then
$$L(s, \chi_1) = \zeta(s) \prod_{p \mid d} (1 - p^{-s})$$
)

Integral formula & analytic continuation

$\begin{pmatrix} f(n) = 1 \\ F = \zeta \end{pmatrix}$ **Thm** For $\operatorname{Re}(s) > 1$,
$$\zeta(s) = \underbrace{\Gamma(1-s)}_{\text{poles at } s = 1, 2, 3, \ldots} \underbrace{I(s)}_{\text{analytic on } \mathbb{C}}$$

where
$$I(s) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^z}{1 - e^z} \, dz$$



Coro $\zeta(s)$ may be analytically continued to $\mathbb{C} \setminus \{1\}$ with a pole at $s = 1$.

$\begin{pmatrix} f = \chi \\ F = L(-,\chi) \end{pmatrix}$ For $\mathrm{Re}(s) > 1$, $L(s,\chi) = (\text{"}\Gamma\text{-factor"})(\text{"integral factor"})$

This gives an analytic continuation to
$$\begin{cases} \mathbb{C} & \chi \neq \chi_1 \\ \mathbb{C} \setminus \{1\} & \chi = \chi_1 \end{cases}$$

Functional equations relating $F(s)$ and $F(1-s)$

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)\zeta(1-s)$$

$\left(\Rightarrow \quad \zeta(-2n) = 0 \quad \text{for} \quad n \in \mathbb{N} \right.$
"trivial zeros" $\left.\right)$

$$L(1-s,\chi) = \frac{d^{s-1}}{(2\pi)^s}\Gamma(s)\left(e^{-\pi i s/2} + \chi(-1)e^{\pi i s/2}\right)\left(\sum_{r=1}^{d}\chi(r)e^{2\pi i r/d}\right)L(s,\bar{\chi})$$

Selberg class

Definition

A Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ with $f(1)=1$ is in the Selberg class if

- It converges absolutely on $\mathrm{Re}(s) > 1$
- It can be analytically continued to $\mathbb{C} \setminus \{1\}$ with a pole at $s=1$.
- $\exists$ functional equation (involving exponentials and Γ) relating $\overline{F(s)}$ to $F(1-\bar{s})$
- $\exists$ Euler product formula
- $|f(n)| \ll n^{\varepsilon}$ $\forall \varepsilon > 0$

Examples

- $\zeta(s)$
- $L(s, \chi)$ χ Dirichlet character
- $K/\mathbb{Q}$ finite extension $\longrightarrow$ $\zeta_K(s)$ Dedekind zeta-function
- ......

Function field analogues

$\mathbb{F}$ = finite field of order q

Dirichlet characters

Fix $d \in \mathbb{F}[t]$ monic (modulus)

Definition

- A Dirichlet character of modulus d is a group homomorphism

$$\begin{tikzcd}
 & \left(\mathbb{F}[t]/d\mathbb{F}[t]\right)^{\times} \arrow[r, "\chi"] \arrow[d, hook] & \mathbb{C}^{\times} \arrow[d, hook] \\
\mathbb{F}[t] \arrow[r, two heads] \arrow[rr, bend right, "\chi"'] & \mathbb{F}[t]/d\mathbb{F}[t] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

extend by 0

- "primitive" & "real"/"quadratic" defined as before.

Lemma (for q odd)

- d square-free $\longrightarrow$ $\exists$ unique real primitive χ
- otherwise $\longrightarrow$ $\nexists$ real primitive χ

Dirichlet series

$$\zeta_q(s) = \sum_{\substack{f \in \mathbb{F}[t] \\ \text{monic}}} \frac{1}{|f|^s} \qquad |f| := q^{\deg(f)}$$

$$= \frac{1}{1 - q^{1-s}} \quad \text{on } \operatorname{Re}(s) > 1$$

Analytic continuation to $\mathbb{C} \setminus \{1\}$ with pole at $s = 1$.

$$L(s, \chi) = \sum_{\substack{f \in \mathbb{F}[t] \\ \text{monic}}} \frac{\chi(f)}{|f|^s} \qquad \chi \text{ Dirichlet character}$$

Converges absolutely on $\operatorname{Re}(s) > 1$.

Analytic continuation to $\mathbb{C} \setminus \{1\}$.

In fact, if $\chi \neq \chi_1$ then $L(s, \chi) =$ polynomial in q^{-s}

so it is analytic on $\mathbb{C}$

These have analogous Euler product formulas, etc.

	Number field setting	Function field setting $\lvert \mathbb{F} \rvert = q$
Riemann zeta	$\zeta(s)$	$\zeta_q(s)$
Dedekind zeta	$\zeta_K(s)$ $K/\mathbb{Q}$ finite ext. $\left(\zeta_{\mathbb{Q}}(s) = \zeta(s)\right)$	$\zeta_K(s)$ $K/\mathbb{F}(t)$ finite ext. $\left(\zeta_{\mathbb{F}(t)}(s) = \zeta_q(s)(1-q^{-s})^{-1}\right)$
Dirichlet L-functions	$L(s,\chi)$ $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $d \in \mathbb{Z}$ positive	$L(s,\chi)$ $\chi : (\mathbb{F}[t]/d\mathbb{F}[t])^\times \longrightarrow \mathbb{C}^\times$ $d \in \mathbb{F}[t]$ monic

Questions:

Outside of the central line $\mathrm{Re}(s) = 1/2$

$\hookrightarrow$ Generalised Riemann Hypothesis

" $\zeta_K(s)$ has no $\underbrace{\text{(non-trivial)}}$ zeros outside of the central line $\mathrm{Re}(s) = 1/2$ "

in the number field setting, there are "trivial" zeros at non-positive integers

Function field setting : Theorem (Weil '48)

Number field setting : Conjecture (\$1,000,000)

On the central line $\mathrm{Re}(s) = 1/2$

Conjecture (Chowla '65)

if χ is a real, non-trivial Dirichlet character mod d

then $\underbrace{L(1/2, \chi)}$ $\neq 0$

the "central value" of the L-function associated to χ

Number field setting: open

Function field setting: FALSE (Li 2018)

More refined question:

$\mathcal{D} = \{$real, primitive Dirichlet characters mod $d\}$

$\mathcal{D}_{<N} = \{$real, primitive Dirichlet characters mod d with $|d| < N\}$

$\mathcal{D}_{=N} = \{$real, primitive Dirichlet characters mod d with $|d| = N\}$

Number field setting: $\exists$ one if $d = n$ or $4n$ (n odd square-free)
$\exists$ two if $d = 8n$
$\nexists$ otherwise

Function field setting (odd q): $\exists$ one if d is square-free
$\nexists$ otherwise

What is the distribution of the central values $L(1/2, \chi)$ for $\chi \in \mathcal{D}$?

More precisely: fix $r \in \mathbb{N}$.

What is an asymptotic formula for $\displaystyle\sum_{\chi \in \mathcal{D}_{<N}} L(\tfrac{1}{2}, \chi)^r$ as $N \to \infty$?

$r=1 \longleftrightarrow$ average central value

$r=2 \longleftrightarrow$ variance of the central values

$r\geq 3 \longleftrightarrow$ higher moments of the distribution of central values

Asymptotic formulas known / conjectured:

In the number field setting:

$r=1$, $r=2$ } Jutila '81

$r=3$ Soundararajan '00

$r=4$ Shen '21 and Li '22

$r\geq 5$???

$\forall r$: CONJECTURE by Conrey - Farmer - Keating - Rubinstein - Snaith '05

$$\sum_{\chi\in\mathcal{D}_{<N}} L(\tfrac{1}{2},\chi)^r \sim N\,P_r(\log N)$$

explicit polynomial of degree $\frac{1}{2}r(r+1)$

In the function field setting:

$r=1$ Hoffstein - Rosen '92

$r=2$, $r=3$, $r=4$ } Florea '17

$r\geq 5$???

$\forall r\geq 4$: CONJECTURE by Andrade - Keating '14

following the heuristics of CFKRS

(*)
$$\sum_{\chi\in\mathcal{D}_{=N}} L(\tfrac{1}{2},\chi)^r \sim N\,Q_r(\log_q N)$$

explicit polynomial of degree $\frac{1}{2}r(r+1)$

Theorem (Bergström - Diaconu - Petersen - Westerland '23)

The conjectured asymptotic formula (*) is true for odd q ......

...... modulo another conjecture about twisted homological stability for the braid groups.

This has recently been proven by Miller - Patzt - Petersen - Randal-Williams using work-in-progress by Sierra - Wahl.

Strategy of proof

via homological stability

$\boxed{1}$ Reduction to a stable twisted H_* calculation

$\boxed{2}$ Reduction to a stable untwisted H_* calculation of a different moduli space

$\boxed{3}$ Do this calculation via "scanning".

Step $\boxed{1}$

Set $N = q^{2g+1}$.

asymptotically as $N \to \infty$ $(g \to \infty)$

Aim: $$\frac{1}{N} \sum_{\chi \in D_{=N}} L(\tfrac{1}{2}, \chi)^r \sim \text{poly of degree } \tfrac{1}{2}r(r+1) \text{ in } \underset{2g+1}{\underset{\parallel}{\log_q N}}$$

$$M_r(g) \overset{?}{=} \quad \sim \text{poly of degree } \tfrac{1}{2}r(r+1) \text{ in } g$$

Grothendieck-Lefschetz trace formula:

λ reflected in $g \times r$ box

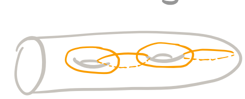
$$M_r(g) = \sum_{\substack{\lambda \\ \cap\!\shortparallel \\ g \times r \text{ box}}} \sum_{k=0}^{\infty} (-1)^k \operatorname{trace}\left(\mathrm{Frob}_q \circlearrowright H_k(\mathcal{H}_g^{1,0}; V_\lambda)\right) \cdot \dim(V_{\lambda^t})$$

moduli space of hyperelliptic curves of genus g with a marked point fixed under the hyperelliptic involution

irred. repr. of $Sp_{2g}(\mathbb{Z})$ assoc. to $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0)$

$$\mathcal{H}_g^{1,0} \longrightarrow \mathcal{M}_g^1$$

$$\pi_1: \; B_{2g+1} \hookrightarrow \mathrm{Mod}_g^1 \twoheadrightarrow Sp_{2g}(\mathbb{Z})$$



Why $\mathcal{H}_g^{1,0}$?

Because $\mathcal{H}_g^{1,0}(\mathbb{F}_q) = \{$ monic square-free polynomials over $\mathbb{F}_q$ of degree $2g+1$ $\}$

$= \{$ real, primitive Dirichlet char's with modulus d where $|d| = q^{2g+1}$ $\}$

$= \mathcal{D}_{=N}$

$$M_r(g) = \sum_{\substack{\lambda \\ \cap| \\ g\,\square}} \sum_{k=0}^{\infty} (-1)^k \operatorname{trace}\left(\mathrm{Frob}_q \circlearrowright H_k\left(\mathcal{H}_g^{1,0}; V_\lambda\right)\right) . \underbrace{\dim\left(V_{\lambda^+}\right)}$$

Obs: This is polynomial in g of degree $\frac{1}{2}r(r+1)$

Define:

$$Q_r(g) := \underbrace{\sum_{\substack{\lambda \\ \cap| \\ \infty\,\square}} \sum_{k=0}^{\infty} (-1)^k \operatorname{trace}\left(\mathrm{Frob}_q \circlearrowright H_k\left(\mathcal{H}_\infty^{1,0}; V_\lambda\right)\right) . \dim\left(V_{\lambda^+}\right)}_{\text{polynomial in } g \text{ of degree } \frac{1}{2}r(r+1)}$$

Aim: Show that $M_r(g) - Q_r(g) \sim 0$

Idea: Low degree terms dominate, so show that

- terms for small k cancel
- terms for large k can be asymptotically bounded for $M_r(g)$ and $Q_r(g)$ separately.

① Twisted homological stability for $H_k(\mathcal{H}^{1,0}_g; V_\lambda) = H_k(B_{2g+1}; V_\lambda)$

Conj BDPW

Proved by Miller, Patzt, Petersen, Randal-Williams

with $\theta(n) = \frac{1}{4}n - c$

in a range $k \leq \theta(2g+1)$ independent of λ

$(\theta : \mathbb{R} \to \mathbb{R}$ divergent$)$

$\Rightarrow$ all terms in $M_r(g) - Q_r(g)$ with $k \leq \theta(2g+1)$ cancel

② Asymptotic bound on terms in $M_r(g)$ with $k > \theta(2g+1)$

Easy by counting cells in a classifying CW-complex for B_{2g+1}

③ Asymptotic bound on terms in $Q_r(g)$ with $k > \theta(2g+1)$

Harder: Need a calculation of $H_k(\mathcal{H}^{1,0}_\infty; V_\lambda)$

or (equivalently) of $\lim\limits_{g \to \infty} H_k(\mathcal{H}^{1,0}_g; S^\lambda(V))$

$V = H^1(\Sigma^1_g; \mathbb{Q})$

Step $\boxed{2}$ (Reduction to a stable untwisted H_* calculation of a different moduli space)

$\mathcal{H}_g^{1,0}(X)$ = moduli space of hyperelliptic surfaces equipped with a continuous map to X taking ∂ to $*$.

Fibration sequence: $\mathrm{Map}_*(\Sigma_g^1/_\partial, X) \longrightarrow \mathcal{H}_g^{1,0}(X) \longrightarrow \mathcal{H}_g^{1,0}$

Set $X = K(A,n)$ Eilenberg-Mac Lane space for A = $\mathbb{Q}$-vector space

Serre spectral sequence collapses $\implies$ for any A:

$$\underbrace{H_*\left(\mathcal{H}_g^{1,0}(K(A,n));\mathbb{Q}\right)}_{\substack{\text{functor of } A \\ (\text{"analytic"})}} \cong \bigoplus_\lambda S^\lambda(A) \otimes \underbrace{H_{*-|\lambda|(n-1)}\left(\mathcal{H}_g^{1,0}; S^\lambda(V)\right)}_{\text{coefficients}}$$

This is what we want to compute when $g \to \infty$.

$\implies$ it suffices to compute $H_*\left(\mathcal{H}_\infty^{1,0}(K(A,n));\mathbb{Q}\right)$ as an analytic functor of A.

Step $\boxed{3}$ (Do this calculation via "scanning".)

Note: Computation of $H_*(\mathcal{M}^1_\infty(K(A,n));\mathbb{Q})$ follows from

via the "group-completion theorem"

$$\Omega B\left(\coprod_g \mathcal{M}^1_g(X)\right) \simeq \Omega^\infty(MTSO(2)\wedge X_+)$$

Galatius Madsen Tillmann Weiss (2009)

So we need to calculate $\Omega B\left(\coprod_g \mathcal{H}^{1,0}_g(X)\right) \simeq\ ?$

(at least for $X = K(A,n)$)

When X = point:

Theorem (Segal '73) $\quad \Omega B\left(\coprod_g \mathcal{H}^{1,0}_g\right) \simeq \Omega^2 S^2$

Proof via "scanning".

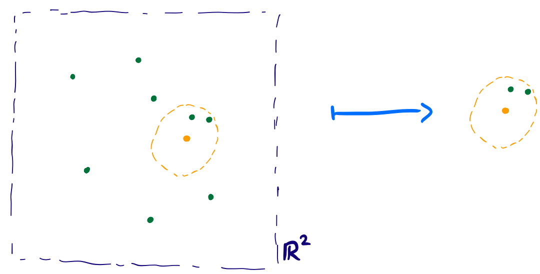
Theorem (BDPW '23) $\quad \Omega B\left(\coprod_g \mathcal{H}^{1,0}_g(K(A,n))\right) \simeq \Omega^2\left(S^2 \vee \frac{K(A,n)}{(-1)}\right)$

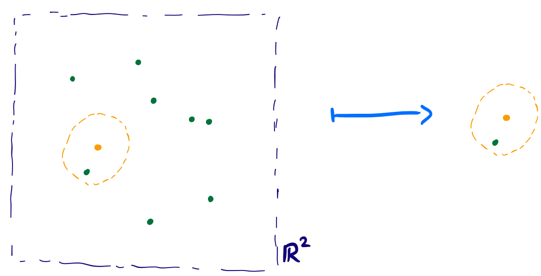
Proof via "scanning".

Idea of proof in the classical setting of Segal:

(a) $\mathcal{H}_g^{1,0} \simeq \underset{\substack{\rotatebox{90}{$\in$} \\ \{p_1,\dots,p_{2g+1}\}}}{\mathrm{Conf}_{2g+1}(\mathbb{R}^2)} \xrightarrow{\text{scanning map}} \mathrm{Map}_c(\underset{\substack{\rotatebox{90}{$\in$} \\ P}}{\mathbb{R}^2}, C(D^2))$

finite configurations in D^2
points appear/disappear on ∂D^2





(b) $\mathcal{H}_g^{1,0} \longrightarrow \mathrm{Map}_c(\mathbb{R}^2, C(D^2)) \simeq \Omega^2 C(D^2)$ for all g

induce $\Omega B\left(\coprod_g \mathcal{H}_g^{1,0}\right) \xrightarrow{\simeq} \Omega^2 C(D^2)$

(c) $C(D^2) \simeq S^2$ (radial expansion) $\square$

In the setting of BDPW:

Steps (a) and (b) are exactly analogous and work for any space X.

Step (c) is more involved:

$$C(D^2; K(A,n)) \simeq S^2 \vee \frac{K(A,n)}{(-1)}$$ (after localising away from 2)