Are there non-zero compactly-supported homology classes on mapping class groups of infinite-type surfaces?

Martin Palmer-Anghel // GeMAT seminar, IMAR // 23 January 2024

Abstract.

The Mumford conjecture (proven by Madsen and Weiss) describes the (rational) homology of the colimit of the mapping class groups $\mathrm{Mod}(\Sigma_{g,1})$ as $g \to \infty$. One may alternatively take the colimit of the surfaces $\Sigma_{g,1}$ themselves, to obtain an infinite-type surface Σ_∞, and then consider the homology of the mapping class group $\mathrm{Mod}(\Sigma_\infty)$. This is known to be uncountably generated in every degree, but its precise structure is very mysterious. A natural question is whether any non-zero homology classes survive under the natural homomorphism

$$H_*(\Omega_0^\infty \mathrm{MT}SO(2)) \cong H_*(\mathrm{colim}_{g\to\infty}(\mathrm{Mod}(\Sigma_{g,1}))) \longrightarrow H_*(\mathrm{Mod}(\Sigma_\infty)), \tag{1}$$

where the left-hand isomorphism is the Madsen-Weiss theorem. Indeed, more generally, for any infinite-type surface S, one may ask whether the natural homomorphism

$$H_*(\mathrm{colim}_{\Sigma\subset S}(\mathrm{Mod}(\Sigma))) \longrightarrow H_*(\mathrm{Mod}(S)) \tag{2}$$

is trivial or non-trivial, where the colimit is taken over all *compact* subsurfaces $\Sigma \subset S$. Elements in the image of (2) are *compactly-supported homology classes* on $\mathrm{Mod}(S)$. We give a complete answer to this question when $\mathrm{genus}(S) > 0$ and a partial answer when $\mathrm{genus}(S) = 0$. In particular, in the case $S = \Sigma_\infty$, we prove that (1) is the zero homomorphism.

This represents joint work with Xiaolei Wu.