Homology classes with compact support for mapping class groups of infinite type surfaces

(Joint work with Xiaolei Wu)

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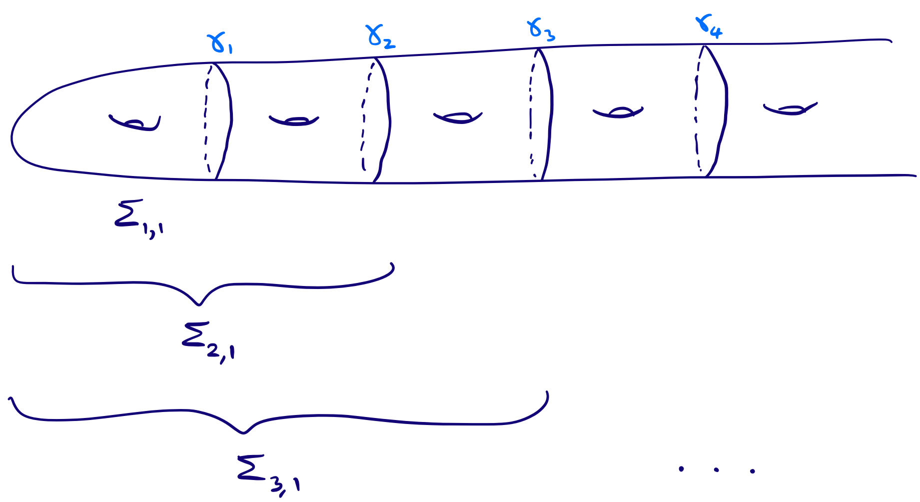
Consider the surface:

$\Sigma_\infty =$

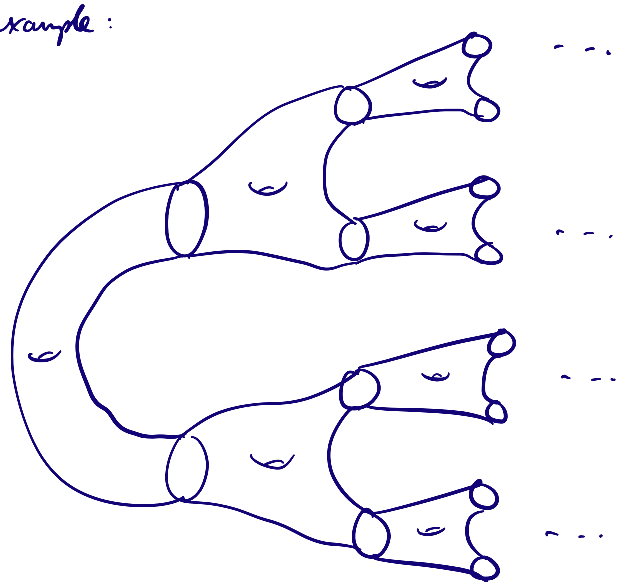


... and the homomorphisms:

$$\begin{array}{ccccccc} \mathrm{Mod}(\Sigma_{1,1}) & \longrightarrow & \mathrm{Mod}(\Sigma_{2,1}) & \longrightarrow & \mathrm{Mod}(\Sigma_{3,1}) & \longrightarrow & \cdots \\ \| & & & & & & \\ \pi_0 \mathrm{Homeo}_\partial(\Sigma_{1,1}) & & & & & & \end{array}$$

extend homeomorphisms by the identity

Another example:

$\Sigma_e =$ 

$$\mathrm{Mod}(\Sigma_{1,2}) \to \mathrm{Mod}(\Sigma_{3,4}) \to \cdots \to \mathrm{Mod}(\Sigma_{2^n-1,\,2^n}) \to \cdots$$

<u>Thm</u> (Harer '85)

$H_i(\mathrm{Mod}(\Sigma_{g,b}))$ is independent of g, b when $g \gg i$

In particular the maps above induce iso. on H_i when $g \gg i$.

$\rightsquigarrow$ "stable homology"

$$\lim_{g\to\infty} H_*(\mathrm{Mod}(\Sigma_{g,b})) \cong H_*(\mathrm{Mod}_c(\Sigma_\infty))$$
$$\cong H_*(\underbrace{\mathrm{Mod}_c(\Sigma_e)}_{})$$

group of mapping classes with compact support

Thm (Madsen–Weiss '02)

The stable homology is isomorphic to $H_*(\Omega_0^\infty MTSO(2))$.

In particular, with $\mathbb{Q}$ coeffs, the stable cohomology is

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots] \qquad |\kappa_i| = 4i.$$

Miller – Morita – Mumford classes

Q: Are the (duals of the) MMM classes non-0 in $H_*(\mathrm{Mod}(\Sigma_\infty))$
or $H_*(\mathrm{Mod}(\Sigma_e))$?

full mapping class group

I.e. Does the inclusion $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induce
a non-0 map on $\tilde{H}_*$ for $S = \Sigma_\infty$ or Σ_e ?

More generally — the same question for any infinite type surface S.

About $H_*(\mathrm{Mod}(S))$:

- **Thm** [P.–Wu '24]
  $H_i(\mathrm{Mod}(\Sigma_\infty))$ is not countably generated $\forall i \geq 1$.

- $H_i(\mathrm{Mod}(\Sigma_e))$ is unknown, but:

  Thm [P.–Wu '22] $H_i(\mathrm{Mod}(\Sigma_e \setminus pt)) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

From now on — take coeffs in a field $\mathbb{F}$.

(The case of $\mathbb{Z}$ coeffs is work in progress)

Thm [P.-Wu]

Ⓐ If genus$(S) = \infty$ then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces the zero map on $\widetilde{H}_*$.

In particular, the MMM classes are sent to 0 in $H_*(\mathrm{Mod}(S))$.

Ⓑ If genus$(S) \in [1, \infty)$ then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces a non-zero map on $\widetilde{H}_*$.

Ⓒ If genus$(S) = 0$, it is complicated.

For example:

Cantor set

- If $S = \mathbb{R}^2 \setminus \mathcal{C}$, the induced map on $\widetilde{H}_*$ is zero.
- If $S = \underbrace{(\mathbb{R}^2 \setminus \mathbb{N}) \# \cdots \# (\mathbb{R}^2 \setminus \mathbb{N})}_{k}$, — " — $\begin{cases} \text{zero} & k=1 \\ \text{non-zero} & k \geq 4 \end{cases}$

Plan: Give ideas of proof of Thm Ⓐ (2 steps)

First observation:

Thm Ⓐ is equivalent to: if genus$(S) = \infty$ and $\Sigma \subset S$ is a compact subsurface then $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ induces 0 on $\widetilde{H}_*$. (*)

By classification, $\Sigma \cong \Sigma_{g,b}$ for $g \geq 0$, $b \geq 1$

Step $\boxed{1}$ Reduction to $b = 1$.

Step $\boxed{2}$ Proof when $b = 1$.

Step $\boxed{1}$.

Assume that $(*)$ is true for $b=1$.

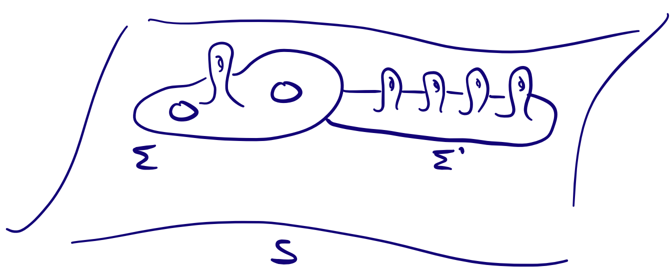
Let $\Sigma_{g,b} \cong \Sigma \hookrightarrow S$ be any compact subsurface.

Since $\mathrm{genus}(S) = \infty$ we may find another compact subsurface $\Sigma_{h,1} \cong \Sigma' \hookrightarrow S$ such that

(i) h is arbitrarily large

(ii) $\Sigma' \cap \Sigma$ = interval in $\partial\Sigma' \cap \partial\Sigma$ $\left(\Rightarrow \Sigma' \cup \Sigma \cong \Sigma' \natural \Sigma \right)$

boundary connected sum



Let $i \geq 1$ and choose $h \geq \frac{3}{2} i$.

$$
\begin{tikzcd}
H_i(\mathrm{Mod}(\Sigma)) \arrow[dr] \arrow[drr, bend left, "(*)"] & & \\
& H_i(\mathrm{Mod}(\Sigma' \cup \Sigma)) \arrow[r] & H_i(\mathrm{Mod}(S)) \\
H_i(\mathrm{Mod}(\Sigma')) \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

by assumption

surjective by [Harer '85].

Hence $(*) = 0$. $\square$

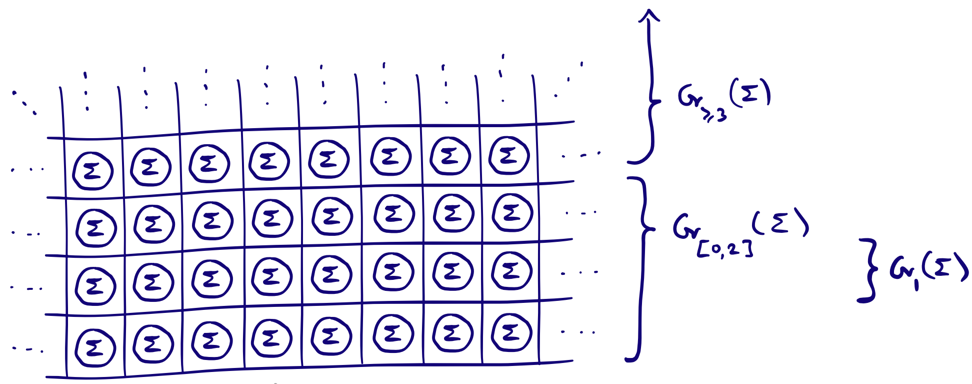
Step $\boxed{2}$.

$\mathrm{genus}(S) = \infty$

$\Sigma_{g,1} \cong \Sigma \hookrightarrow S$ compact subsurface.

Definition Grid surface:

$Gr(\Sigma) =$



denote the copies of Σ by $\Sigma_{i,j}$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$

Since $\mathrm{genus}(S) = \infty$, the inclusion $\Sigma \hookrightarrow S$ extends to a proper embedding $Gr(\Sigma) \hookrightarrow S$.

Hence we have a factorisation of $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ into

$$\mathrm{Mod}(\Sigma) \xrightarrow{(*)} \mathrm{Mod}(Gr(\Sigma)) \longrightarrow \mathrm{Mod}(S)$$

The proof reduces to:

Prop. The map (∗) induces the zero map on $\widetilde{H}_j$ for all $j \geq 0$.

Proof — by induction on j.

$\underline{\mathcal{H}(n,j)}$: The map $\mathrm{Mod}(\Sigma) \xrightarrow{i_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$
$(n \geq 0,\ j \geq 0)$ $\qquad [\varphi] \longmapsto \begin{bmatrix} \varphi & \text{on } \Sigma_{0,n} \\ id & \text{elsewhere} \end{bmatrix}$

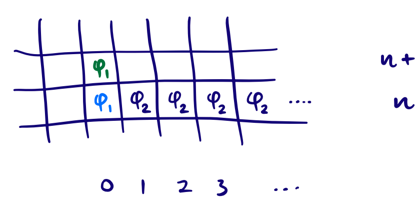
induces the zero map on $\widetilde{H}_j$.

$\underline{j=0}$ ✓

$\underline{j \geq 1}$:

$$\mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \xrightarrow{\eta_n \quad \nu_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$$

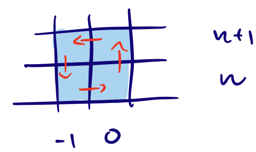
$$(\varphi_1, \varphi_2) \longmapsto$$



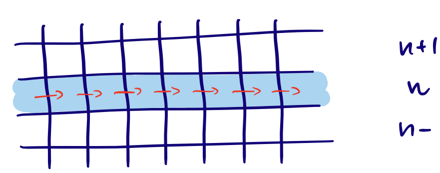
$$\Psi_{0,n}(\varphi) := \eta_n(\varphi, \varphi)$$

$$\Psi_{1,n}(\varphi) := \eta_n(id, \varphi)$$

$\underline{\text{Obs}}$: $\quad \eta_n \sim \nu_n$



$$\Psi_{0,n} \sim \Psi_{1,n}$$



$$
\begin{tikzcd}
 & & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
\mathrm{Mod}(\Sigma) \arrow[urr, bend left, "\psi_{1,n}"] \arrow[r, "\Delta"] & \mathrm{Mod}(\Sigma)\times\mathrm{Mod}(\Sigma) \arrow[r, "\eta_n"] \arrow[d, "i_{n+1}\times\psi_{1,n}"] \arrow[dr, "\nu_n"] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
 & \mathrm{Mod}(\mathrm{Gr}_{\geq n+1}(\Sigma))\times\mathrm{Mod}(\mathrm{Gr}_n(\Sigma)) \arrow[r, "\text{glue}"'] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma))
\end{tikzcd}
$$

$\alpha \in H_j(\mathrm{Mod}(\Sigma))$

[Künneth]

$$\Delta_*(\alpha) \in H_j(\mathrm{Mod}(\Sigma)) \cong \bigoplus_{k+l=j} H_k(\mathrm{Mod}(\Sigma)) \otimes H_l(\mathrm{Mod}(\Sigma))$$

$$\| $$

$$\alpha \otimes 1 + \underbrace{\cdots\cdots}_{} + 1 \otimes \alpha$$

vanish under the map $(i_{n+1})_* \otimes (\psi_{1,n})_*$ by induction.

Image of α in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$ is $(i_{n+1})_*(\alpha) + (\psi_{1,n})_*(\alpha)$

But its image is also $(\psi_{1,n})_*(\alpha)$.

Hence $(i_{n+1})_*(\alpha) = 0$ in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$.

$$\|$$

$(i_n)_*(\alpha)$

since $i_n \sim i_{n+1}$

$\square$

BONUS : Ⓑ $\operatorname{genus}(S) \in [1,\infty)$

Ⓒ $\operatorname{genus}(S) = 0$

Ⓑ

Suppose $\operatorname{genus}(S) = g \in [1,\infty)$.

- $\Sigma_{g,1} \hookrightarrow S$ $\longrightarrow$ $\operatorname{Mod}(\Sigma_{g,1}) \to \operatorname{Mod}(S)$
- [filling in all "ends" of S] $\cong \Sigma_g$ $\longrightarrow$ $\operatorname{Mod}(S) \to \operatorname{Mod}(\Sigma_g)$

(Freudenthal compactification)

$$\begin{tikzcd} \operatorname{Mod}(\Sigma_{g,1}) \arrow[r, "(*)"] \arrow[rr, bend right, "\text{surjective on } H_2 \text{ when } g \geq 2 \text{ [Harer '85] (+ Ivanov, Boldsen)}"'] & \operatorname{Mod}(S) \arrow[r] & \operatorname{Mod}(\Sigma_g) \end{tikzcd}$$

$H_2(\operatorname{Mod}(\Sigma_g)) \neq 0$ when $g \geq 2$ $\left(\begin{array}{ll} \mathbb{Z}/2 & g=2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & g=3 \\ \mathbb{Z} & g \geq 4 \end{array} \right)$

Hence $(*)$ is non-0 on H_2 for $g \geq 2$.

For $g=1$, $H_1(\operatorname{Mod}(\Sigma_{1,1})) \twoheadrightarrow H_1(\operatorname{Mod}(\Sigma_1))$

$\qquad H_1(\operatorname{Mod}(\Sigma_{1,1})) \cong \mathbb{Z}$, $\quad H_1(\operatorname{Mod}(\Sigma_1)) \cong \mathbb{Z}/12$

is surjective, so $(*)$ is non-0 on H_1 for $g=1$.

(c)

If $S = \underbrace{(\mathbb{R}^2 \setminus \mathbb{N}) \# \cdots \# (\mathbb{R}^2 \setminus \mathbb{N})}_{k}$, then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$

induces $\begin{cases} \text{the zero map on } \widetilde{H}_* \text{ if } k=1 \\ \text{a non-zero map on } \widetilde{H}_* \text{ if } k \geq 4. \end{cases}$

<u>$k=1$</u>

$\mathbb{R}^2 \setminus \mathbb{N} \cong \mathbb{R}^2 \setminus \mathbb{Z}^2$

One can use a "grid surface argument" like in the proof for genus$(S) = \infty$ above.

<u>$k \geq 4$</u>

More generally, if genus$(S) = 0$
and Ends(S) has a $\overbrace{\text{topologically-distinguished}}^{\text{i.e. fixed setwise by all homeomorphisms}}$ subset A
of cardinality $|A| = k \in [4, \infty)$

then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces a non-0 map on H_1.

<u>Proof</u>: Embed $\Sigma_{0,k} \hookrightarrow S$ so that each component of $S \setminus \Sigma_{0,k}$ contains one point of $A \subseteq \mathrm{Ends}(S)$.

$\longrightarrow \mathrm{Mod}(\Sigma_{0,k}) \to \mathrm{Mod}(S)$

Since $A \subseteq \mathrm{Ends}(S)$ is top. distinguished, we have

$$\mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S \cup \{\text{all ends except } A\})$$
$$\cong \mathrm{Mod}(S^2 \setminus k \text{ points})$$

For $k \geq 2$:

$$\begin{tikzcd} & & \mathbb{Z}/2 \arrow[d] & & \\ & & B_k(S^2) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2(k-1) \arrow[r, two heads] & \begin{cases} \mathbb{Z}/2(k-1) & k \text{ even} \\ \mathbb{Z}/(k-1) & k \text{ odd} \end{cases} \\ \mathrm{Mod}(\Sigma_{0,k}) \arrow[r, "(*)"] & \mathrm{Mod}(S) \arrow[r] & \mathrm{Mod}(S^2 \setminus k \text{ points}) \arrow[urr, two heads] & & \end{tikzcd}$$

 $\longmapsto 2$

Hence $(*)$ is non-0 on $(-)^{ab} = H_1$ whenever:

(k even): $2 \not\equiv 0 \mod 2(k-1)$ ⟵ iff $k \geq 4$

(k odd): $2 \not\equiv 0 \mod (k-1)$ ⟵ iff $k \geq 5$

$\square$