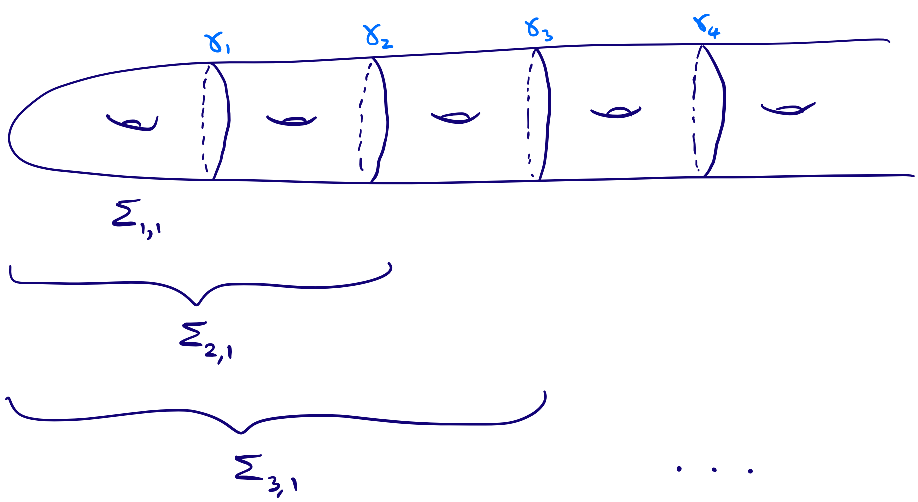
Are there non-zero compactly-supported homology classes on MCGs of infinite-type surfaces? Part II

Joint work with Xiaolei Wu
Based on arXiv: 2405.03512

GeMAT seminar
IMAR
21 May 2024

Consider the surface:

$\Sigma_\infty =$



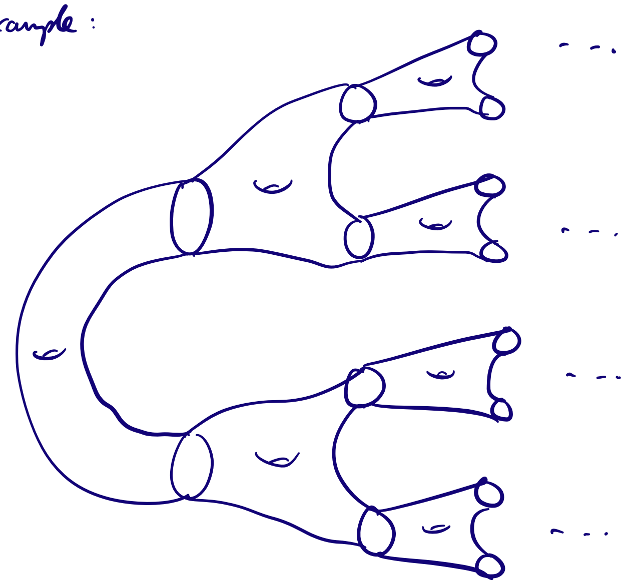
. . .

...and the homomorphisms:

$$\begin{tikzcd} \mathrm{Mod}(\Sigma_{1,1}) \arrow[r] \arrow[d, equal] & \mathrm{Mod}(\Sigma_{2,1}) \arrow[r] & \mathrm{Mod}(\Sigma_{3,1}) \arrow[r] & \cdots \\ \pi_0 \mathrm{Homeo}_\partial(\Sigma_{1,1}) & & & \end{tikzcd}$$

extend homeomorphisms
by the identity

Another example:

$\Sigma_e =$ 

$$\mathrm{Mod}(\Sigma_{1,2}) \longrightarrow \mathrm{Mod}(\Sigma_{3,4}) \longrightarrow \cdots \longrightarrow \mathrm{Mod}(\Sigma_{2^n-1,\,2^n}) \longrightarrow \cdots$$

<u>Thm</u> (Harer '85)

$H_i(\mathrm{Mod}(\Sigma_{g,b}))$ is independent of g, b when $g \gg i$.

⤳ "stable homology"

$$\lim_{g\to\infty} H_*(\mathrm{Mod}(\Sigma_{g,b})) \cong H_*(\mathrm{Mod}_c(\Sigma_\infty))$$

$$\cong H_*(\underbrace{\mathrm{Mod}_c(\Sigma_e)})$$

group of mapping classes with compact support

Thm (Madsen–Weiss '02)

- The stable homology is isomorphic to $H_*(\Omega_0^\infty MTSO(2))$.
- In particular, with $\mathbb{Q}$ coeffs, the stable cohomology is

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots] \qquad |\kappa_i| = 4i.$$

Miller - Morita - Mumford classes

Question:

Do the duals of the MMM classes vanish in $H_*(\mathrm{Mod}(\Sigma_\infty))$ or $H_*(\mathrm{Mod}(\Sigma_c))$?

<u>full</u> mapping class group

$\Updownarrow$

Does the inclusion $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induce a non-0 map on H_* for $S = \Sigma_\infty$ or Σ_c ?

More generally, we may ask the same question for <u>any</u> infinite-type surface S.

[<u>Note</u>: classified by von Kerékjártó / Richards.]

Q_1: Does $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induce 0 on $\widetilde{H}_*$?

Q_2: Does $\mathrm{Mod}_f(S) \hookrightarrow \mathrm{Mod}(S)$ induce 0 on $\widetilde{H}_*$?

mapping class with support in a finite-type subsurface.

(**Note:** $1 \to \mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}_f(S) \twoheadrightarrow \mathrm{Bij}_f(P(S)) \to 1$

punctures of S (isolated, planar ends))

What is known about $H_*(\mathrm{Mod}(S))$:

Thm [Calegari-Chen '21 '22]

$$H_1(\mathrm{Mod}(\Sigma_e)) = 0$$

$$H_2(\mathrm{Mod}(\Sigma_e)) \cong \mathbb{Z}/2$$

Thm [P.-Wu '22]

$$H_i(\mathrm{Mod}(\Sigma_e \setminus pt)) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Thm [P.-Wu '24]

$H_i(\mathrm{Mod}(\Sigma_\omega))$ is not countably generated $\forall i \geq 1$.

Answers

(Focus on $\mathrm{genus}(S) = \infty$)

$$\begin{tikzcd} \mathrm{Mod}_c(S) \arrow[r, hook] \arrow[rr, bend right, "\iota_c"'] & \mathrm{Mod}_f(S) \arrow[r, hook, "\iota_f"] & \mathrm{Mod}(S) \end{tikzcd}$$

Theorem [P.-Wu '24]

Assume that $\mathrm{genus}(S) = \infty$.

Let $p = \#$ of punctures of S $\in \{0, 1, 2, \dots, \infty\}$.

isolated ends that are not accumulated by genus

① $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces the 0 map on $\widetilde{H}_*$ with field coefficients.

Rmk It does not follow for $\mathbb{Z}$ coefficients! ($S^1 \longrightarrow B(\mathbb{Q}/\mathbb{Z})$)

Rmk Hence the dual MMM classes all vanish in $H_*(\mathrm{Mod}(S))$.

② The same is true for $\mathrm{Mod}_f(S) \hookrightarrow \mathrm{Mod}(S)$ if either

- $p = 0$
- $p = \infty$ and S has a "mixed end".

end accumulated by genus and by punctures.

③ If $p \in [1, \infty)$ then the image of $H_* \mathrm{Mod}_f(S) \longrightarrow H_* \mathrm{Mod}(S)$ contains a $\mathbb{Z}$ summand in every even degree.

Answers (detailed version)

O / O — $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ / $\mathrm{Mod}_f(S)$ induces 0 on $\tilde{H}_*$ with any field coeffs.

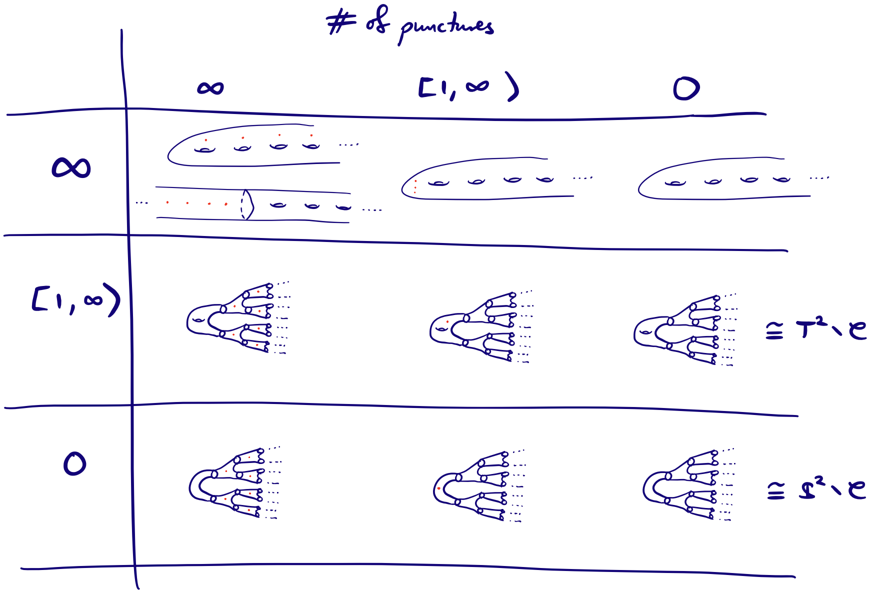
X / X — $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ / $\mathrm{Mod}_f(S)$ induces a non-0 map on $\tilde{H}_*$ with $\mathbb{Z}$ coeffs

Theorem [P.-Wu '24]

(O ⇒ O, X ⇒ X)

genus \ # of punctures	∞	$[1,\infty)$	0
∞	O; mixed end : O; no mixed end : ?	O X	O O
$[1,\infty)$	X X	X X	X X
0	$\mathrm{Ends}(S) \cong [0,\omega^\alpha]$: O O; $\mathrm{Ends}(S) \supseteq A$ topologically distinguished subset with $4 \leq \lvert A\rvert < \infty$: X X; Otherwise : ? ?	$p=1$: O O; $p=2,3$: X ?; $p \geqslant 4$: X X	O O

Examples



Plan:

- Proof of Theorem 1 (2 is very similar)
- Proof of Theorem 3

Proof of Theorem 1

Observation

Theorem 1 $\iff$ if $\text{genus}(S) = \infty$ and $\Sigma \subset S$ is a compact subsurface then $\text{Mod}(\Sigma) \to \text{Mod}(S)$ induces 0 on $\tilde{H}_*$. $(*)$

By classification, $\Sigma \cong \Sigma_{g,b}$ for $g \geq 0$, $b \geq 1$

Step $\boxed{1}$: proof for $b=1$

Step $\boxed{2}$: ($b=1$ case) $\Rightarrow$ ($b \geq 1$ case)

Step $\boxed{2}$.

Assume that (*) is true for $b=1$.

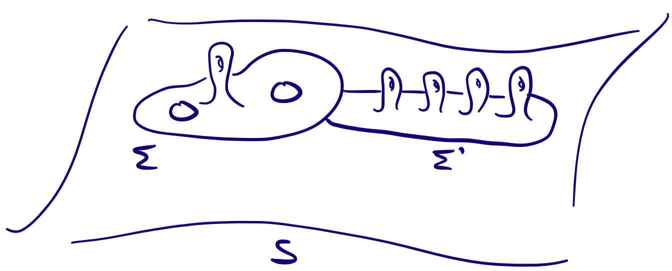
Let $\Sigma_{g,b} \cong \Sigma \hookrightarrow S$ be any compact subsurface.

Since $\mathrm{genus}(S) = \infty$ we may find another compact subsurface $\Sigma_{h,1} \cong \Sigma' \hookrightarrow S$ such that

(i) h is arbitrarily large

(ii) $\Sigma' \cap \Sigma$ = interval in $\partial\Sigma' \cap \partial\Sigma$ $\quad \left(\Rightarrow \Sigma' \cup \Sigma \cong \Sigma' \natural \Sigma \right)$

boundary connected sum



Let $i \geq 1$ and choose $h \geq \frac{3}{2} i$.

$$\begin{tikzcd}
H_i(\mathrm{Mod}(\Sigma)) \arrow[dr] \arrow[drr, bend left, "(*)"] & & \\
& H_i(\mathrm{Mod}(\Sigma' \cup \Sigma)) \arrow[r] & H_i(\mathrm{Mod}(S)) \\
H_i(\mathrm{Mod}(\Sigma')) \arrow[ur, two heads] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

by assumption

surjective by [Harer '85].

Hence (*) $= 0$. $\square$

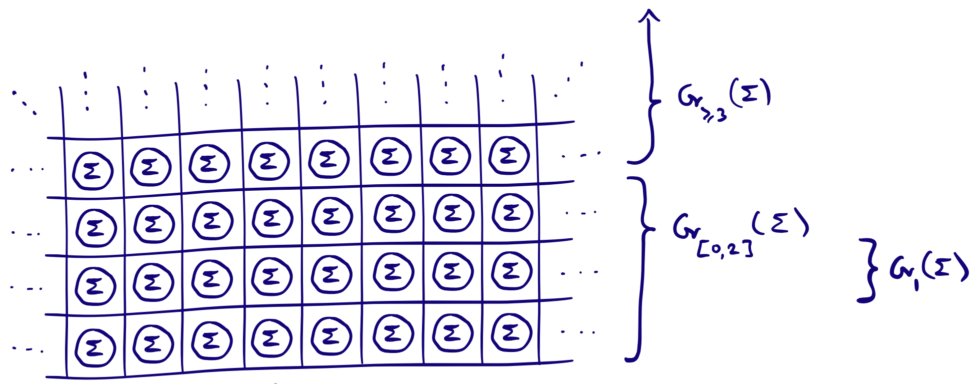
Step $\boxed{1}$.

$\text{genus}(S) = \infty$

$\Sigma_{g,1} \cong \Sigma \hookrightarrow S$ compact subsurface.

Definition Grid surface:

$Gr(\Sigma) =$



denote the copies of Σ by $\Sigma_{i,j}$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$

Since $\text{genus}(S) = \infty$, the inclusion $\Sigma \hookrightarrow S$ extends to a proper embedding $Gr(\Sigma) \hookrightarrow S$.

Hence we have a factorisation of $Mod(\Sigma) \to Mod(S)$ into

$$Mod(\Sigma) \xrightarrow{(*)} Mod(Gr(\Sigma)) \longrightarrow Mod(S)$$

The proof reduces to:

Prop. The map (*) induces the zero map on $\tilde{H}_j$ for all $j \geq 0$.

Proof — by induction on j.

$\mathcal{H}(n,j)$: The map $\mathrm{Mod}(\Sigma) \xrightarrow{i_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$

$(n \geq 0,\ j \geq 0)$ $\qquad [\varphi] \longmapsto \begin{bmatrix} \varphi & \text{on } \Sigma_{0,n} \\ id & \text{elsewhere} \end{bmatrix}$

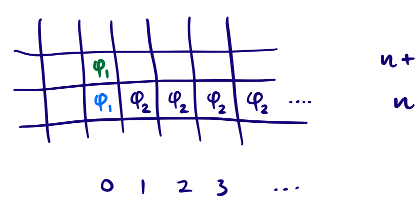
induces the zero map on $\tilde{H}_j$.

$j=0$ ✓

$j \geq 1$:

$$\mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \xrightarrow{\eta_n \quad \nu_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$$

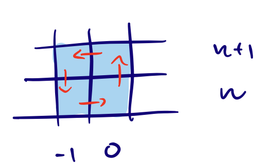
$(\varphi_1, \varphi_2) \longmapsto$



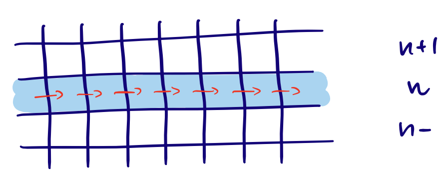
$$\psi_{0,n}(\varphi) := \eta_n(\varphi, \varphi)$$

$$\psi_{1,n}(\varphi) := \eta_n(id, \varphi)$$

Obs: $\qquad \eta_n \sim \nu_n$



$$\psi_{0,n} \sim \psi_{1,n}$$



$$\begin{tikzcd}
 & & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
\mathrm{Mod}(\Sigma) \arrow[r, "\Delta"] \arrow[urr, bend left, "\psi_{1,n}"] & \mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \arrow[r, "\eta_n"] \arrow[d, "i_{n+1} \times \psi_{1,n}"] \arrow[dr, "\nu_n"] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
 & \mathrm{Mod}(\mathrm{Gr}_{\geq n+1}(\Sigma)) \times \mathrm{Mod}(\mathrm{Gr}_{n}(\Sigma)) \arrow[r, "\text{glue}"'] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma))
\end{tikzcd}$$

$\alpha \in H_j(\mathrm{Mod}(\Sigma))$

[Künneth] (field coeffs!)

$$\Delta_*(\alpha) \in H_j(\mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma)) \cong \bigoplus_{k+l=j} H_k(\mathrm{Mod}(\Sigma)) \otimes H_l(\mathrm{Mod}(\Sigma))$$

$$\parallel$$

$$\alpha \otimes 1 + \underbrace{\cdots\cdots}_{} + 1 \otimes \alpha$$

vanish under the map $(i_{n+1})_* \otimes (\psi_{1,n})_*$ by induction.

Image of α in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$ is $(i_{n+1})_*(\alpha) + (\psi_{1,n})_*(\alpha)$

But its image is also $(\psi_{1,n})_*(\alpha)$.

Hence $(i_{n+1})_*(\alpha) = 0$ in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$.

$$\parallel$$

$(i_n)_*(\alpha)$ since $i_n \sim i_{n+1}$

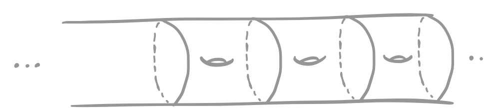
$\square$

Proof of Theorem ③

S surface of genus $= \infty$
and <u>no</u> punctures

$P \subseteq S$ finite, non-empty

Example:



Claim: $\mathrm{image}\Big(H_* \mathrm{Map}_f(S \setminus P) \longrightarrow H_* \mathrm{Map}(S \setminus P) \Big) \supseteq H_*(\mathbb{C}P^\infty)$

Proof:

Def [Bödigheimer–Tillmann '01]

$$\mathrm{Diff}(S, P) \xrightarrow{\quad \tau \quad} (S^1)^p \rtimes \mathfrak{S}_p$$

set $P = \{x_1 \ldots x_p\}$

choose tangent vector v_i to S at x_i

$$\tau(\varphi) = \begin{cases} \text{permutation } \sigma = \tau|_P \\ \forall i = 1 \ldots p, \text{ angle between } D\varphi(v_i) \text{ and } v_{\sigma(i)} \end{cases}$$

Now let $\Sigma \subseteq S$ be any compact subsurface with $P \subset \mathring{\Sigma}$.

$$
\begin{tikzcd}
\operatorname{colim}_\Sigma \mathrm{Diff}(\Sigma, P) \arrow[r] \arrow[d, "\simeq"', "{[\text{K-S '77}]}"] & \mathrm{Diff}(S, P) \arrow[r, "\tau"] \arrow[d, "\simeq", "{[\text{Kirby-Siebenmann '77}]}"'] & (\mathbb{S}^1)^P \rtimes \mathfrak{S}_P \\
\operatorname{colim}_\Sigma \mathrm{Homeo}(\Sigma, P) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Homeo}(S, P) \arrow[d, phantom, "\cong" rotate=-90] & \\
\operatorname{colim}_\Sigma \mathrm{Homeo}(\Sigma \setminus P) \arrow[r] \arrow[d, "\simeq"', "{[\text{Hamstrom '66}]}"] & \mathrm{Homeo}(S \setminus P) \arrow[d, "\simeq", "{[\text{Yagasaki '00}]}"'] & \\
\operatorname{colim}_\Sigma \pi_0 \mathrm{Homeo}(\Sigma \setminus P) \arrow[r] \arrow[d, equal] & \pi_0 \mathrm{Homeo}(S \setminus P) \arrow[d, equal] & \\
\operatorname{colim}_\Sigma \mathrm{Mod}(\Sigma \setminus P) \arrow[r] \arrow[d, equal] & \mathrm{Mod}(S \setminus P) & \\
\mathrm{Mod}_f(S \setminus P) \arrow[ur] \arrow[uuuuurr, bend right, "(*)"'] & &
\end{tikzcd}
$$

Suitably reinterpreted Thm of [Bödigheimer–Tillmann '01]:

The map $(*)$ admits a section on H_*.

$$
\begin{tikzcd}
(\mathbb{S}^1)^P \rtimes \mathfrak{S}_P \arrow[r, dashed] & \mathrm{Mod}_f(S \setminus P) \arrow[r] & \mathrm{Mod}(S \setminus P) \arrow[r] & (\mathbb{S}^1)^P \rtimes \mathfrak{S}_P \arrow[d] \\
\mathbb{S}^1 \arrow[u] \arrow[rrr, "\mathrm{id}"] & & & \mathbb{S}^1
\end{tikzcd}
$$

$$
\mathrm{image}\Big(H_* \mathrm{Map}_f(S \setminus P) \longrightarrow H_* \mathrm{Map}(S \setminus P) \Big) \supseteq H_*(\mathbb{S}^1) = H_*(\mathbb{CP}^\infty)
$$

as a topological group

∎

$$
\begin{tikzcd}
& \mathrm{Diff}(\Sigma, P) \arrow[rr] \arrow[dl, swap, "\text{[EE]69, [ES]70}"] \arrow[dd, swap, "\simeq"] \arrow[dd, "\text{[KS]77}"] \arrow[rrr, bend left] & & \mathrm{Diff}(S, P) \arrow[r] \arrow[dl] \arrow[dd, swap, "\simeq"] \arrow[dd, "\text{[KS]77}"] & \mathbb{S}^1 \wr \mathfrak{S}_p \\
\pi_0(\mathrm{Diff}(\Sigma, P)) \arrow[rr] \arrow[dd, swap, "\cong"] & & \pi_0(\mathrm{Diff}(S, P)) \arrow[dd, swap, "\cong"] & & \\
& \mathrm{Homeo}(\Sigma, P) \arrow[rr] \arrow[dl] \arrow[dd, swap, "\cong"] & & \mathrm{Homeo}(S, P) \arrow[dl] \arrow[dd, swap, "\cong"] & \\
\pi_0(\mathrm{Homeo}(\Sigma, P)) \arrow[rr] \arrow[dd, swap, "\cong"] & & \pi_0(\mathrm{Homeo}(S, P)) \arrow[dd, swap, "\cong"] & & \\
& \mathrm{Homeo}(\Sigma \setminus P) \arrow[rr] \arrow[dl, swap, "\text{66[H]}"] & & \mathrm{Homeo}(S \setminus P) \arrow[dl, "\text{[Y]00}"] & \\
\pi_0(\mathrm{Homeo}(\Sigma \setminus P)) \arrow[rr] \arrow[d, equal] & & \pi_0(\mathrm{Homeo}(S \setminus P)) \arrow[d, equal] & & \\
\mathrm{Map}(\Sigma \setminus P) \arrow[rr] \arrow[dr] & & \mathrm{Map}(S \setminus P) & & \\
& \mathrm{Map}(S \setminus P)^\delta \arrow[ur, swap, "\simeq"] & & &
\end{tikzcd}
$$

BONUS : (A) $\text{genus}(S) \in [1,\infty)$
(B) $\text{genus}(S) = 0$

(A)

Suppose $\text{genus}(S) = g \in [1,\infty)$.

- $\Sigma_{g,1} \hookrightarrow S \quad \Longrightarrow \quad \text{Mod}(\Sigma_{g,1}) \to \text{Mod}(S)$
- [filling in all "ends" of S] $\cong \Sigma_g \quad \Longrightarrow \quad \text{Mod}(S) \to \text{Mod}(\Sigma_g)$

(Freudenthal compactification)

$$\begin{tikzcd} \text{Mod}(\Sigma_{g,1}) \arrow[r, "(*)"] \arrow[rr, bend right] & \text{Mod}(S) \arrow[r] & \text{Mod}(\Sigma_g) \end{tikzcd}$$

surjective on H_2 when $g \geq 2$ [Harer '85]
(+ Ivanov, Boldsen, Randal-Williams)

$H_2(\text{Mod}(\Sigma_g)) \neq 0$ when $g \geq 2$

$$\left(\begin{array}{ll} \mathbb{Z}/2 & g=2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & g=3 \\ \mathbb{Z} & g \geq 4 \end{array} \right)$$

Hence $(*)$ is non-0 on H_2 for $g \geq 2$.

For $g=1$,
$$\begin{array}{ccc} H_1(\text{Mod}(\Sigma_{1,1})) & \twoheadrightarrow & H_1(\text{Mod}(\Sigma_1)) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z}/12 \end{array}$$

is surjective, so $(*)$ is non-0 on H_1 for $g=1$.

(B)

If $S = \underbrace{(\mathbb{R}^2 \setminus \mathbb{Z}^2) \# \cdots \# (\mathbb{R}^2 \setminus \mathbb{Z}^2)}_{n}$, then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$

induces $\begin{cases} \text{the zero map on } \widetilde{H}_* \text{ if } n=1 & \text{(field coeffs)} \\ \text{a non-zero map on } \widetilde{H}_* \text{ if } n \geq 4 & (\mathbb{Z} \text{ coeffs}) \end{cases}$

$n=1$

One can use a "grid surface argument" like in the proof for genus$(S) = \infty$ above.

$n \geq 4$

More generally, if genus$(S) = 0$ and Ends(S) has a $\overbrace{\text{topologically-distinguished subset}}^{\text{i.e. fixed setwise by all homeomorphisms}}$ A of cardinality $|A| = n \in [4, \infty)$

then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces a non-0 map on H_1.

Proof: Embed $\Sigma_{0,n} \hookrightarrow S$ so that each component of $S \setminus \Sigma_{0,n}$ contains one point of $A \subseteq \mathrm{Ends}(S)$.

$\longrightarrow \mathrm{Mod}(\Sigma_{0,n}) \longrightarrow \mathrm{Mod}(S)$

Since $A \subseteq \mathrm{Ends}(S)$ is top. distinguished, we have

$$\mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S \cup \{\text{all ends except } A\}) \cong \mathrm{Mod}(\mathbb{S}^2 \setminus n \text{ points})$$

For $n \geqslant 2$:

$$\begin{tikzcd} & & \mathbb{Z}/2 \arrow[d] & & \\ & & B_n(S^2) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2(n-1) \arrow[r, two heads] & \begin{cases} \mathbb{Z}/2(n-1) & n \text{ even} \\ \mathbb{Z}/(n-1) & n \text{ odd}\end{cases} \\ \mathrm{Mod}(\Sigma_{0,n}) \arrow[r, "(*)"] & \mathrm{Mod}(S) \arrow[r] & \mathrm{Mod}(S^2 \setminus n \text{ points}) \arrow[urr, two heads] & & \end{tikzcd}$$

$\mathrm{Mod}(\Sigma_{0,n}) \ni$  $\longmapsto 2 \in \begin{cases} \mathbb{Z}/2(n-1) \\ \mathbb{Z}/(n-1) \end{cases}$

Hence $(*)$ is non-0 on $(-)^{ab} = H_1$ whenever:

$(n \text{ even})$: $2 \not\equiv 0 \mod 2(n-1)$ ⟵ iff $n \geqslant 4$

$(n \text{ odd})$: $2 \not\equiv 0 \mod (n-1)$ ⟵ iff $n \geqslant 5$

$\square$

If $S = \mathbb{R}^2 \setminus \mathcal{C}$ then $\mathrm{Mod}_f(S) \hookrightarrow \mathrm{Mod}(S)$ induces the zero map on $\widetilde{H}_*$ with $\mathbb{Z}$ (hence any) coefficients.

Proof: Any finite type $\Sigma \subset \mathbb{R}^2 \setminus \mathcal{C}$ must be contained in:

$$\begin{tikzcd} \Sigma \arrow[r, hook] & \Sigma' \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{R}^2 \setminus \mathcal{C} \\ & \mathbb{D}^2 \setminus \mathcal{C} & \end{tikzcd}$$

But $\widetilde{H}_*(\mathrm{Mod}(\mathbb{D}^2 \setminus \mathcal{C})) = 0$ by [P.-Wu '22].

$\square$.