Do the dual Miller-Morita-Mumford classes vanish in the homology of the big mapping class group?

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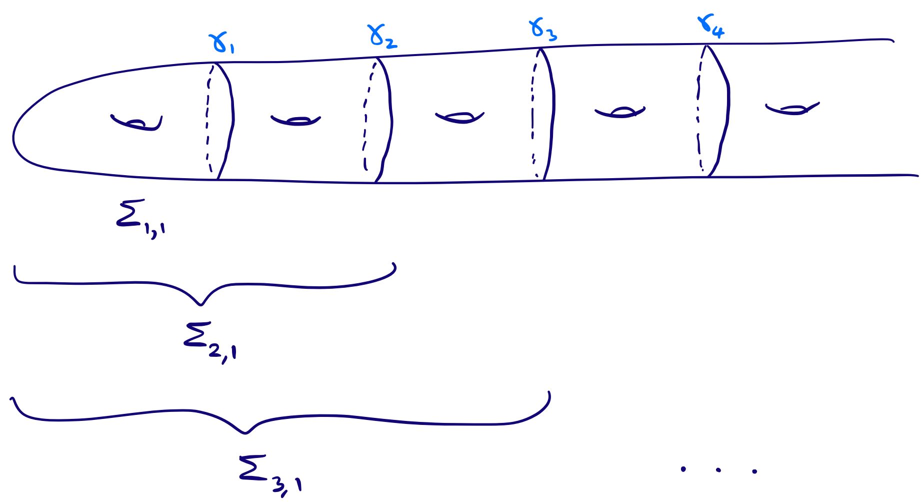
Consider the surface:

$\Sigma_\infty =$

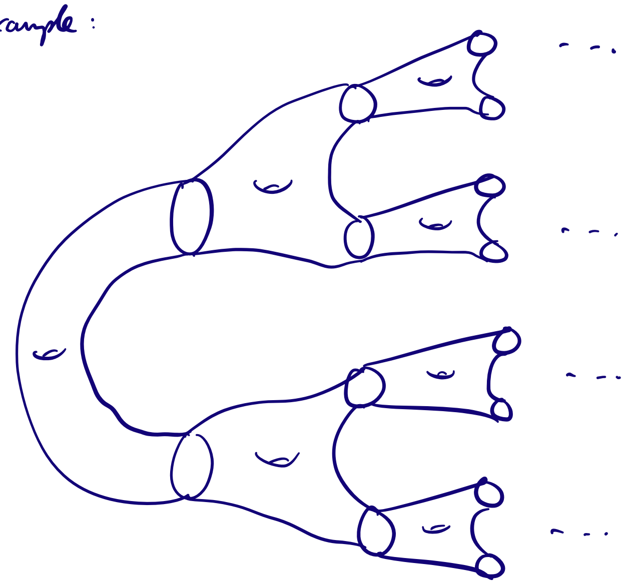


... and the homomorphisms:

$$\begin{array}{ccccccc} \mathrm{Mod}(\Sigma_{1,1}) & \longrightarrow & \mathrm{Mod}(\Sigma_{2,1}) & \longrightarrow & \mathrm{Mod}(\Sigma_{3,1}) & \longrightarrow & \cdots \\ \| & & & & & & \\ \pi_0 \mathrm{Homeo}_\partial(\Sigma_{1,1}) & & & & & & \end{array}$$

extend homeomorphisms by the identity

Another example:

$\Sigma_e =$ 

$$\mathrm{Mod}(\Sigma_{1,2}) \to \mathrm{Mod}(\Sigma_{3,4}) \to \cdots \to \mathrm{Mod}(\Sigma_{2^n-1,\,2^n}) \to \cdots$$

<u>Thm</u> (Harer '85)

$H_i(\mathrm{Mod}(\Sigma_{g,b}))$ is independent of g, b when $g \gg i$.

$\rightsquigarrow$ "stable homology"

$$\lim_{g\to\infty} H_*(\mathrm{Mod}(\Sigma_{g,b})) \cong H_*(\mathrm{Mod}_c(\Sigma_\infty))$$

$$\cong H_*(\underbrace{\mathrm{Mod}_c(\Sigma_e)})$$

group of mapping classes with compact support

Thm (Madsen–Weiss '02)

- The stable homology is isomorphic to $H_*(\Omega^\infty_0 MTSO(2))$.
- In particular, with $\mathbb{Q}$ coeffs, the stable cohomology is

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots] \qquad |\kappa_i| = 4i.$$

Miller - Morita - Mumford classes

Question:

Do the duals of the MMM classes vanish in $H_*(\mathrm{Mod}(\Sigma_\infty))$
or $H_*(\mathrm{Mod}(\Sigma_\varepsilon))$?

full mapping class group

Does the inclusion $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induce a non-0 map on $\tilde{H}_*$ for $S = \Sigma_\infty$ or Σ_ε ?

More generally, we may ask the same question for any infinite-type surface S.

[Note: classified by von Kerékjártó / Richards.]

Q_1 : Does $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induce 0 on $\tilde{H}_*$?

Q_2 : Does $\mathrm{Mod}_f(S) \hookrightarrow \mathrm{Mod}(S)$ induce 0 on $\tilde{H}_*$?

mapping class with support in a finite-type subsurface.

Note: $1 \to \mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}_f(S) \twoheadrightarrow \mathrm{Bij}_f(P(S)) \to 1$

punctures of S (isolated, planar ends)

What is known about $H_*(\mathrm{Mod}(S))$:

Thm [P.-Wu '24]

$H_i(\mathrm{Mod}(\Sigma_\infty))$ is not countably generated $\forall i \geq 1$.

Thm [Calegari-Chen '21 '22]

$$H_1(\mathrm{Mod}(\Sigma_e)) = 0$$

$$H_2(\mathrm{Mod}(\Sigma_e)) \cong \mathbb{Z}/2$$

Thm [P.-Wu '22]

$$H_i(\mathrm{Mod}(\Sigma_e \setminus pt)) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Answers

0 — $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces 0 on $\tilde{H}_*$ with any field coeffs.

$\times$ — $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces a non-0 map on $\tilde{H}_*$ with $\mathbb{Z}$ coeffs

Theorem [P.-Wu '24]

Ⓐ $\mathrm{genus}(S) = \infty \longrightarrow 0$

Ⓑ $\mathrm{genus}(S) \in [1, \infty) \longrightarrow \times$

Ⓒ $\mathrm{genus}(S) = 0 \longrightarrow$ complicated!

Cantor set

$$\left.\begin{array}{l} S = \mathbb{R}^2 \setminus \mathcal{C} \\ S = \mathbb{R}^2 \setminus \mathbb{Z}^2 \end{array}\right\} \longrightarrow 0$$

$$S = \#_{n} (\mathbb{R}^2 \setminus \mathbb{Z}^2) \longrightarrow \times$$

$n \geq 4$

In particular, the MMM classes do vanish in $H_*(\mathrm{Mod}(S))$ when $\mathrm{genus}(S) = \infty$.

Plan: Give ideas of proof of Thm Ⓐ (2 steps)

Obs: Thm Ⓐ is equivalent to:

(*) if $\mathrm{genus}(S) = \infty$ and $\Sigma \subset S$ is a compact subsurface then $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ induces 0 on $\tilde{H}_*$.

By classification, $\Sigma \cong \Sigma_{g,b}$ for $g \geq 0$, $b \geq 1$

Step $\boxed{1}$ Reduction to $b = 1$.

Step $\boxed{2}$ Proof when $b = 1$.

Step $\boxed{1}$.

Assume that $(\ast)$ is true for $b=1$.

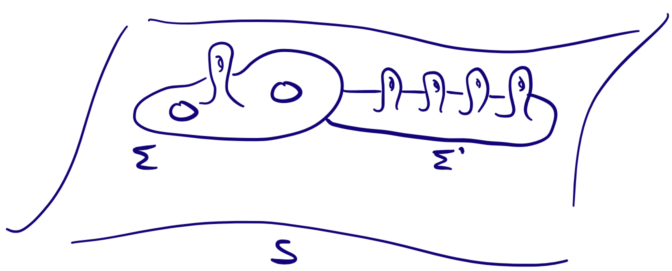
Let $\Sigma_{g,b} \cong \Sigma \hookrightarrow S$ be any compact subsurface.

Since $\mathrm{genus}(S)=\infty$ we may find another compact subsurface $\Sigma_{h,1} \cong \Sigma' \hookrightarrow S$ such that

(i) h is arbitrarily large

(ii) $\Sigma' \cap \Sigma$ = interval in $\partial\Sigma' \cap \partial\Sigma$ $\left(\Rightarrow \Sigma' \cup \Sigma \cong \Sigma' \natural \Sigma \right)$

boundary connected sum



Let $i \geq 1$ and choose $h \geq \frac{3}{2} i$.

$$\begin{tikzcd}
H_i(\mathrm{Mod}(\Sigma)) \arrow[dr] \arrow[drr, bend left, "(\ast)"] & & \\
& H_i(\mathrm{Mod}(\Sigma' \cup \Sigma)) \arrow[r] & H_i(\mathrm{Mod}(S)) \\
H_i(\mathrm{Mod}(\Sigma')) \arrow[ur, two heads] \arrow[urr, "0"'] & &
\end{tikzcd}$$

by assumption

surjective by [Harer '85].

Hence $(\ast) = 0$. $\square$

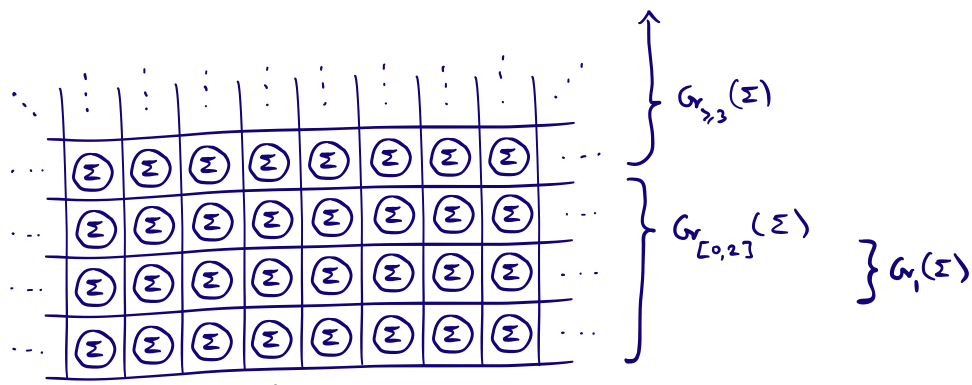
Step $\boxed{2}$.

$\mathrm{genus}(S) = \infty$

$\Sigma_{g,1} \cong \Sigma \hookrightarrow S$ compact subsurface.

Definition Grid surface:

$\mathrm{Gr}(\Sigma) =$



denote the copies of Σ by $\Sigma_{i,j}$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$

Since $\mathrm{genus}(S) = \infty$, the inclusion $\Sigma \hookrightarrow S$ extends to a proper embedding $\mathrm{Gr}(\Sigma) \hookrightarrow S$.

Hence we have a factorisation of $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ into

$$\mathrm{Mod}(\Sigma) \xrightarrow{(*)} \mathrm{Mod}(\mathrm{Gr}(\Sigma)) \longrightarrow \mathrm{Mod}(S)$$

The proof reduces to:

Prop. The map (*) induces the zero map on $\tilde{H}_j$ for all $j \geq 0$.

Proof — by induction on j.

$\mathcal{H}(n,j)$: The map

$$\begin{aligned} Mod(\Sigma) &\xrightarrow{\;i_n\;} Mod(Gr_{\geq n}(\Sigma)) \\ [\varphi] &\longmapsto \begin{bmatrix} \varphi & \text{on } \Sigma_{0,n} \\ id & \text{elsewhere} \end{bmatrix} \end{aligned}$$

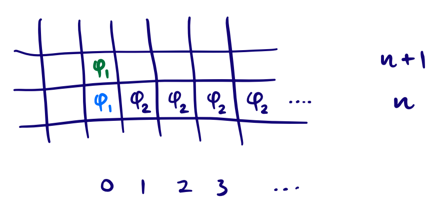
$(n \geq 0,\ j \geq 0)$

induces the zero map on $\tilde{H}_j$.

$j=0$ ✓

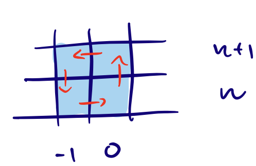
$j \geq 1$:

$$\begin{aligned} Mod(\Sigma) \times Mod(\Sigma) &\xrightarrow{\;\eta_n \quad \nu_n\;} Mod(Gr_{\geq n}(\Sigma)) \\ (\varphi_1, \varphi_2) &\longmapsto \end{aligned}$$

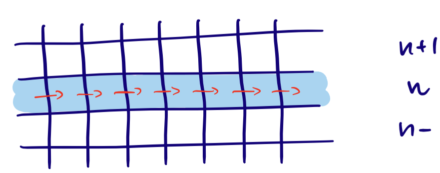


$$\Psi_{0,n}(\varphi) := \eta_n(\varphi, \varphi)$$
$$\Psi_{1,n}(\varphi) := \eta_n(id, \varphi)$$

Obs : $\eta_n \sim \nu_n$



$\Psi_{0,n} \sim \Psi_{1,n}$



$$
\begin{tikzcd}
 & & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \\
\mathrm{Mod}(\Sigma) \arrow[r, "\Delta"] \arrow[urr, bend left, "\psi_{1,n}"] & \mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \arrow[r, "\eta_n"] \arrow[d, "i_{n+1} \times \psi_{1,n}"] \arrow[dr, "\nu_n"] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[u, leftrightarrow, "\mathrm{conj.}"'] \arrow[d, leftrightarrow, "\mathrm{conj.}"] \\
 & \mathrm{Mod}(\mathrm{Gr}_{\geq n+1}(\Sigma)) \times \mathrm{Mod}(\mathrm{Gr}_{n}(\Sigma)) \arrow[r, "\mathrm{glue}"'] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma))
\end{tikzcd}
$$

$\alpha \in H_j(\mathrm{Mod}(\Sigma))$

[Künneth] (field coeffs!)

$$\Delta_*(\alpha) \in H_j(\mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma)) \cong \bigoplus_{k+l=j} H_k(\mathrm{Mod}(\Sigma)) \otimes H_l(\mathrm{Mod}(\Sigma))$$

$$\| $$

$$\alpha \otimes 1 + \underbrace{\cdots\cdots}_{} + 1 \otimes \alpha$$

vanish under the map $(i_{n+1})_* \otimes (\psi_{1,n})_*$ by induction.

Image of α in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$ is $(i_{n+1})_*(\alpha) + (\psi_{1,n})_*(\alpha)$

But its image is also $(\psi_{1,n})_*(\alpha)$.

Hence $(i_{n+1})_*(\alpha) = 0$ in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$.

$$\|$$

$(i_n)_*(\alpha)$ since $i_n \sim i_{n+1}$

$\square$

BONUS : Ⓑ $\mathrm{genus}(S) \in [1,\infty)$
Ⓒ $\mathrm{genus}(S) = 0$

Ⓑ

Suppose $\mathrm{genus}(S) = g \in [1,\infty)$.

- $\Sigma_{g,1} \hookrightarrow S \quad \longrightarrow \quad \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(S)$
- $[\text{filling in all "ends" of } S] \cong \Sigma_g \quad \longrightarrow \quad \mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma_g)$
  (Freudenthal compactification)

$$\begin{tikzcd} \mathrm{Mod}(\Sigma_{g,1}) \arrow[r, "(*)"] \arrow[rr, bend right] & \mathrm{Mod}(S) \arrow[r] & \mathrm{Mod}(\Sigma_g) \end{tikzcd}$$

surjective on H_2 when $g \geq 2$ [Harer '85]
(+ Ivanov, Boldsen, Randal-Williams)

$H_2(\mathrm{Mod}(\Sigma_g)) \neq 0$ when $g \geq 2$ $\quad \begin{pmatrix} \mathbb{Z}/2 & g=2 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & g=3 \\ \mathbb{Z} & g\geq 4 \end{pmatrix}$

Hence $(*)$ is non-0 on H_2 for $g \geq 2$.

For $g=1$, $\quad \underset{\mathbb{Z}}{\underset{\cong}{H_1(\mathrm{Mod}(\Sigma_{1,1}))}} \twoheadrightarrow \underset{\mathbb{Z}/12}{\underset{\cong}{H_1(\mathrm{Mod}(\Sigma_1))}}$

is surjective, so $(*)$ is non-0 on H_1 for $g=1$.

(c)

If $S = \underbrace{(\mathbb{R}^2 \setminus \mathbb{Z}^2) \# \cdots \# (\mathbb{R}^2 \setminus \mathbb{Z}^2)}_{n}$, then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$

induces $\begin{cases} \text{the zero map on } \widetilde{H}_* \text{ if } n=1 & (\text{field coeffs}) \\ \text{a non-zero map on } \widetilde{H}_* \text{ if } n \geq 4 & (\mathbb{Z} \text{ coeffs}) \end{cases}$

<u>$n=1$</u>

One can use a "grid surface argument" like in the proof for genus$(S) = \infty$ above.

<u>$n \geq 4$</u>

More generally, if genus$(S) = 0$ and $\mathrm{Ends}(S)$ has a $\overbrace{\text{topologically-distinguished}}^{\text{i.e. fixed setwise by all homeomorphisms}}$ subset A of cardinality $|A| = n \in [4, \infty)$

then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces a non-0 map on H_1.

<u>Proof</u>: Embed $\Sigma_{0,n} \hookrightarrow S$ so that each component of $S \setminus \Sigma_{0,n}$ contains one point of $A \subseteq \mathrm{Ends}(S)$.

$$\longrightarrow \mathrm{Mod}(\Sigma_{0,n}) \longrightarrow \mathrm{Mod}(S)$$

Since $A \subseteq \mathrm{Ends}(S)$ is top. distinguished, we have

$$\mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S \cup \{\text{all ends except } A\})$$
$$\cong$$
$$\mathrm{Mod}(\mathbb{S}^2 \setminus n \text{ points})$$

For $n \geq 2$:

$$\begin{tikzcd}
 & & \mathbb{Z}/2 \arrow[d] & & \\
 & & B_n(S^2) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2(n-1) \arrow[r, two heads] & \left\{ \begin{array}{ll} \mathbb{Z}/2(n-1) & n \text{ even} \\ \mathbb{Z}/(n-1) & n \text{ odd} \end{array} \right. \\
\mathrm{Mod}(\Sigma_{0,n}) \arrow[r, "(*)"] & \mathrm{Mod}(S) \arrow[r] & \mathrm{Mod}(S^2 \setminus n \text{ points}) \arrow[urr] & &
\end{tikzcd}$$

$\mathrm{Mod}(\Sigma_{0,n}) \ni$  $\longmapsto 2 \in$ (the group above)

Hence $(*)$ is non-0 on $(-)^{ab} = H_1$ whenever:

(n even): $2 \not\equiv 0 \mod 2(n-1)$ ← iff $n \geq 4$

(n odd): $2 \not\equiv 0 \mod (n-1)$ ← iff $n \geq 5$

$\square$

If $S = \mathbb{R}^2 \setminus \mathcal{C}$ then $\mathrm{Mod}_c(S) \hookrightarrow \mathrm{Mod}(S)$ induces the zero map on $\tilde{H}_*$ with $\mathbb{Z}$ (hence any) coefficients.

Proof: Any finite type $\Sigma \subset \mathbb{R}^2 \setminus \mathcal{C}$ must be contained in:

$$\begin{tikzcd}
\Sigma \arrow[r, hook] & \Sigma' \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{R}^2 \setminus \mathcal{C} \\
 & \mathbb{D}^2 \setminus \mathcal{C} &
\end{tikzcd}$$

But $\tilde{H}_*(\mathrm{Mod}(\mathbb{D}^2 \setminus \mathcal{C})) = 0$ by [P.-Wu '22].

$\square$.