Do the dual MMM classes vanish in the big MCG ?

Joint work with Xiaolei Wu
Based on arXiv: 2405.03512

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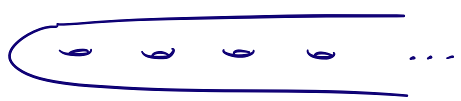
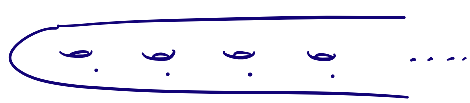
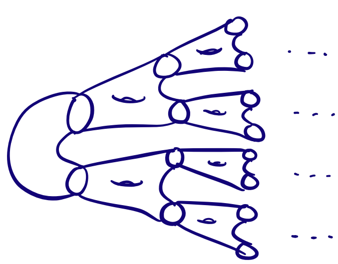
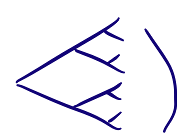
S — connected, orientable surface $(\partial = \emptyset)$ of ∞ genus.

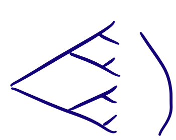
Examples

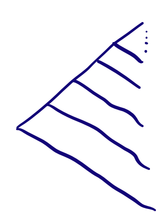
S_∞ 

S'_∞ 

S_e  $= \partial\big($reg. nbhd. of  $\big)$

take $\# T^2$ at each vertex

S'_e $\partial\big($reg. nbhd. of  $\big)$

take $\# T^2$ at each vertex of subtree 

Classified by $\quad \mathrm{Ends}_{np}(S) \subseteq \mathrm{Ends}(S)$

non-$\emptyset$ closed subspace

compact tot. discon. 2^{nd}-countable space

<u>Ex</u>

S	$\mathrm{Ends}(S)$	$\mathrm{Ends}_{np}(S)$
S_∞	$*$	$*$
S_∞'	$\mathbb{N}^+$	$*$
S_e	Cantor	Cantor
S_e'	Cantor	$\mathbb{N}^+$

<u>Def</u> $\quad$ <u>puncture</u> $=$ isolated point of $\mathrm{Ends}(S) \setminus \mathrm{Ends}_{np}(S)$

$p(S) = \#$ of punctures $\in \mathbb{N} \cup \{\infty\}$

<u>mixed end</u> $=$ point of $\mathrm{Ends}_{np}(S)$ that is a limit of punctures

Big mapping class groups

$\mathrm{Mod}(S) := \pi_0(\mathrm{Homeo}^+(S))$

Q(1): What is $H_*(\mathrm{Mod}(S))$?

Σ — connected, orientable, finite-type surface

$\cong$

$\Sigma_{g,b}^{n}$ $\quad g, b, n \geq 0$

f. type

$\mathrm{Mod}_f(S) := \{ [\varphi] \in \mathrm{Mod}(S) \mid \mathrm{supp}(\varphi) \subseteq \Sigma \subset S \}$

Q(2): What is $H_*(\mathrm{Mod}_f(S))$?

Q(3): How does the induced map $H_*(\mathrm{Mod}_f(S)) \to H_*(\mathrm{Mod}(S))$ act? Is it zero?

Thm (combining Harer '85 / Bödigheimer–Tillmann '01 / Madsen–Weiss '07 / Hatcher–Wahl '10)

$$H_*(\mathrm{Mod}_f(S)) \cong \begin{cases} H_*\left(\Omega_0^\infty MTSO(2) \times B(S^1 \wr \mathfrak{S}_{p(S)})\right) & p(S) < \infty \\ H_*\left(\Omega_0^\infty MTSO(2) \times \Omega_0^\infty \Sigma^\infty(\mathbb{CP}^\infty_+)\right) & p(S) = \infty \end{cases}$$

<u>Cor</u> (Mumford conjecture)

$$H_*(\mathrm{Mod}_f(S); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots]^* \otimes (\cdots\cdots)$$

Miller - Morita - Mumford classes

$H_*(\mathrm{Mod}(S))$ is less well-understood

<u>Thm</u> [Calegari - Chen '21] $S = S_\infty$ $H_1 = 0$, $H_2 \cong \mathbb{Z}/2$

<u>Thm</u> [Domat '20] $S = S_\infty$ $H_1 \supseteq \bigoplus_c \mathbb{Q}$ $c = |\mathbb{R}|$

<u>Rmk</u> Depends on more than $p(S)$.

<u>Thm</u> [P. - Wu '22]

① $S = S_\infty$ $H_* \supseteq \Lambda^*\left(\bigoplus_c \mathbb{Z}\right)$

② $S = S_e - \text{point}$ $H^* \cong \mathbb{Z}[e]$ $|e| = 2$

Euler class of the central ext.

$$1 \to \mathbb{Z} \to \mathrm{Mod}(S_e - \mathring{D}^2) \to \mathrm{Mod}(S_e - pt) \to 1$$

$Q(3)$: image $\left(H_*(\mathrm{Mod}_f(S)) \longrightarrow H_*(\mathrm{Mod}(S)) \right) = ?$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; I(S)$

__Thm__ [P.-Wu '24]

Take coeffs in a __field__.

- Any class with support in __compact__ $\Sigma \subset S$ vanishes in $H_*(\mathrm{Mod}(S))$

  $\hookrightarrow$ the MMM classes all vanish.

- The image $I(S)$ is precisely :

$p(S)$	$I(S)$
0	0
$[1,\infty)$	$H_*(B(S^1 \wr \mathfrak{S}_{p(S)}))$
∞	0 if $\exists$ mixed end
	$?$ if $\nexists$ mixed end

__Proof outline__

① 2-dim version of an "infinite iteration argument" (Mather, Baumslag-Dyer-Heller, Varadarajan, Berrick)

(Künneth SES $\rightsquigarrow$ field coeffs)

$\Rightarrow$ no non-0 H_* classes supported on $\Sigma_{g,1} \hookrightarrow S$

② if $\exists$ non-0 H_* class supported on $\Sigma_{g,b} \hookrightarrow S$

$\Big\downarrow$ hom. stab^y trick

then $\exists$ non-0 H_* class supported on $\Sigma_{h,1} \hookrightarrow S$

for $h \geqslant g + \frac{3}{2}i$

Question

$\exists$ spectrum $\mathbb{X}$: $\quad H_*(\mathrm{Mod}(S)) \cong H_*(\Omega^\infty_\circ \mathbb{X})$?

Rmks

- Madsen-Weiss '07 prove this for $\mathrm{Mod}_f(S)$ when $p(S)=0$ with $\mathbb{X} = MTSO(2)$

- Szymik-Wahl '17 prove this for Higman-Thompson groups $V_{n,r}$ with $\mathbb{X}$ = Moore spectrum of $\mathbb{Z}/(n-1)$.

- $\mathbb{X}$ would have to depend strongly on S.