Moduli spaces and their fundamental groups

homology, representations and lower central series

18 June 2024

IMAR

Outline

1. Examples of moduli spaces & motion groups
2. Homological stability — examples & applications
3. Non-local configuration spaces
4. } Big mapping class groups
5. }
6. Linearity — background
7. } Heisenberg homology & rep's of MCGs
8. }
9. The stopping problem for LCS

①

Moduli space $\overset{\pi_1}{\rightsquigarrow}$ Motion group

Space whose points parametrise a fixed kind of geometric / topological object

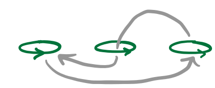
All ways of travelling through the moduli space & returning to start point

- Configurations of n points in the plane
- Riemann surfaces of genus $g \geq 0$

- Trivial links in $\mathbb{R}^3$ with n components 

Braid groups B_n 

$MCG(\Sigma_g)$ (topological symmetries)

Loop braid groups LB_n 

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②

Homological stability

Def $X_1 \to X_2 \to X_3 \to \cdots$ is hom. stable if $H_*(X_n) \to H_*(X_{n+1})$ is an isom. for $* \leq f(n)$ ← diverging function

Ex
- $C_n(M)$ ← connected open mfld. [McDuff, Segal '70s]
- $MCG(\Sigma_g)$ [Harer '85] ⟶ used by [Madsen-Weiss '02] to prove the Mumford Conj. ('83) (Alg. Geom.)
- Hurwitz spaces [Ellenberg-Venkatesh-Westerland '16] ⟶ asymptotic Cohen-Lenstra Conj. (N. Theory)
- B_n with coeffs twisted by irred. symplectic rep's [Bergström-Diaconu-Petersen-Westerland '23] [Miller-Patzt-Petersen-Randal-Williams '24] ⟶ asymptotic formula for moments of families of L-functions

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③

Non-local configuration spaces

① Configuration-mapping spaces

$$CMap_n(M, X) = \left\{ \begin{array}{c|c} S \subseteq M & |S| = n \\ M \setminus S \xrightarrow{f} X & \text{condition on behavior of } f \text{ near } S \end{array} \right\}$$

singularities, field, charge

Ex: Hurwitz spaces

② Mod. spaces of asymptotic magnetic monopoles

Idea: built out of torus bundles over $C_n(\mathbb{R}^3)$; each S^1 factor encodes pairwise interactions

$$\widetilde{C}_n(\mathbb{R}^3) \to \widetilde{C}_2(\mathbb{R}^3) \simeq S^2 \xleftarrow{\text{Hopf}} S^3$$

Thm (P.-Tillmann, Res. Math. Sci '21, R.S. Proc A. '23) Homological stability for ① and ②.

"Big" mapping class groups

S surface

$\mathcal{C} \subset S$ embedded Cantor set

$\pi_1(S \setminus \mathcal{C})$ is not fin. gen.
$MCG(S \setminus \mathcal{C})$ is uncountable.

$MCG(S \setminus \mathcal{C})$ — example of a "big" MCG.

arises when studying dynamical systems on S with attractor $= \mathcal{C}$

Thm (P.-Wu, J.Top '24)

(*) $H_*(MCG(\mathbb{R}^2 \setminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$

Proof — via hom. stability!

Contrast:

Thm (P.-Wu, Doc. Math '24)

$H_*(MCG(\mathbb{R}^2 \setminus N)) \supseteq \Lambda^*(\underbrace{\bigoplus_{\mathfrak{c}}}_{\text{continuum}} \mathbb{Z})$

Idea: $1 \longrightarrow \langle T_\partial \rangle \longrightarrow MCG(D^2 \setminus \mathcal{C}) \longrightarrow MCG(\mathbb{R}^2 \setminus \mathcal{C}) \longrightarrow 1$

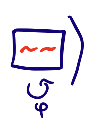
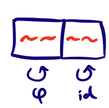
$\langle T_\partial \rangle \cong \mathbb{Z}$

(*)

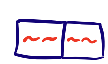
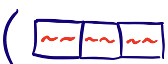
$\Uparrow$ spectral sequence

$\widetilde{H}_*(MCG(D^2 \setminus \mathcal{C})) = 0$

$\Uparrow$ ∞ iteration trick

$MCG($  $) \longrightarrow MCG($  $)$ induces $\cong$ on H_* in all degrees.

$\Uparrow$

Hom. stability for $MCG($  $) \longrightarrow MCG($  $) \longrightarrow MCG($  $) \longrightarrow \cdots$

Linearity

General Q: for a manifold M, when is $MCG(M)$ linear?

embeds into $GL_n(\mathbb{F})$

$\longleftrightarrow$ has a faithful fin. dim. representation

Ex $MCG(D^2 \setminus n \text{ points}) \cong B_n$ is linear [Bigelow, Krammer, 2000]

LKB representations [Lawrence '90]

$MCG($  $)$ ..... ????

$\Sigma_{g,1}$

$MCG(\#_g(S^1 \times S^2) \setminus \text{disc}) \cong Aut(F_g)$ is <u>not</u> linear [Formanek-Procesi '92]

Heisenberg homology (7)

Thm [Blanchet - P. Shaukat '21 + '23]

(1) V repr. of "discrete Heisenberg group" $\mathcal{H}_g$

$\rightsquigarrow W_k(V)$ twisted repr. of $MCG(\Sigma_{g,1})$

action by isomorphisms on a collection of objects

formally: repr. of a certain groupoid

Idea: $\underset{\displaystyle \pi_1 C_k(\Sigma_{g,1})}{\overset{B_k(\Sigma_{g,1})}{\|}} \overset{\phi}{\twoheadrightarrow} \mathcal{H}_g$

$\ker(\phi)$ is $MCG(\Sigma_{g,1})$ - invariant

$W_k(V) = \{ H_k(C_k(\Sigma_{g,1}); V_\tau) \mid \tau \in Aut(\mathcal{H}_g) \}$

$\pi_1 C_k(\Sigma_{g,1}) \overset{\phi}{\twoheadrightarrow} \mathcal{H}_g \xrightarrow{\tau} \mathcal{H}_g \longrightarrow Aut_R(V)$

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(2) $W_k(V)$ may be untwisted to give a genuine representation on: (8)

- Torelli — for any V
- MCG — for V = Schrödinger

(3) $\ker(W_k(V)) \subsetneq J_k$

strict inclusion (calc.)

Johnson filtration, $\longrightarrow 0$ as $k \to \infty$

Rmk When $k \geq 3$, $\mathcal{H}_g$ is the quotient of $B_k(\Sigma_{g,1})$ by $\Gamma_3(B_k(\Sigma_{g,1}))$.

Def. $\Gamma_1(G) = G$

$\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$ Lower central series

Problem: Study $\Gamma_*(B_k(S))$ for any surface S... (9)

Q: Does it stop?

$\exists i \quad \Gamma_i(\cdots) = \Gamma_{i+1}(\cdots)$?

Thm [Darné - P. - Soulié, Memo. AMS, '23]

(We also answer the question for virtual braid groups, welded braid groups and partitioned versions of all of the above.)

The lower central series $\Gamma_*(B_k(S))$

$(k \geq 3)$ stops at $i=2$ if $S \subseteq S^2$ or non-or.

stops at $i=3$ if $S \not\subseteq S^2$ & or.

$(k=2)$ does not stop if $S \neq D^2, S^2, P^2$

$(k=1)$ does not stop if $S \neq$ (6 exceptions)