Homology of big MCGs supported on compact subsurfaces

Joint work with Xiaolei Wu
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Topology of moduli spaces
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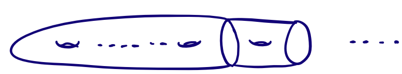
S = surface of infinite type (connected, $\partial S = \emptyset$, orientable)

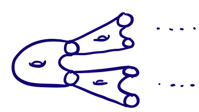
i.e. $\pi_1(S)$ not fin. gen. (but S 2nd-countable)

Typical examples

$S_\infty = \operatorname{colim}(\Sigma_{n,1})$



$S_{\mathcal{C}} = \operatorname{colim}(\Sigma_{2^n-1,1})$



Classified by (Kérekjártó, Richards)

- genus(S) — $\mathbb{N} \cup \{\infty\}$
- $\mathcal{E}(S) = \{\text{ends of } S\}$ — space homeo to closed $\subseteq$ Cantor
- $\mathcal{E}_{np}(S) = \{\text{non-planar ends of } S\}$ — closed subspace ($\emptyset$ iff genus $< \infty$), with $\mathcal{E}_{np}(S) \subseteq \mathcal{E}(S)$

Thm (Freudenthal)

$\forall$ "nice" X, $\exists$ (unique) maximal "compact" of X : $\bar{X} - X$ is 0-dim.

connected
second-countable
rim-compact ——— $\exists$ basis $\{U_i\}$ s.t. ∂U_i is compact

has a basis of clopen sets

Def $\mathcal{E}(S) := \bar{S} - S$

$\mathcal{E}_{np}(S) := \{ e \in \mathcal{E}(S) \mid e \text{ does not have a nbhd in } \bar{S} \text{ homeo to } \mathbb{R}^2 \}$

$P(S) := \{\text{punctures of } S\} = \{ e \in \mathcal{E}(S) \mid e \text{ is planar \& isolated in } \mathcal{E}(S) \}$

Obs S has finite type $\iff$ genus$(S) < \infty$ and $|\mathcal{E}(S)| < \infty$.

Def $\mathrm{MCG}(S) := \pi_0 \mathrm{Homeo}^+_\partial(S)$ or-preserving
id on ∂ when we allow S to have ∂

Rmk This is
- uncountable
- Polish group (separable + completely metrisable)
- totally disconnected

Homology

$$H_* \mathrm{MCG}(S_g) = \begin{cases} 0 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases} \qquad \text{(Calegari - Chen)}$$

$$H_* \mathrm{MCG}(S_g - \text{point}) = \begin{cases} 0 & * \text{ odd} \\ \mathbb{Z} & * \text{ even} \end{cases} \qquad \text{(P. - Wu)}$$

$$H_* \mathrm{MCG}(S_\infty) \supseteq \Lambda^*\Big(\bigoplus_{\mathbb{R}} \mathbb{Z}\Big) \qquad \begin{matrix}\text{(Domat in degree } *=1) \\ \text{(P. - Wu)}\end{matrix}$$

Homology supported on compact subsurfaces

Whenever genus$(S) = \infty$,

Thm [Madsen-Weiss] (+ [Harer] stability)

$$\operatorname*{colim}_{\substack{\Sigma \subset S \\ \text{cpt}}} H_* \mathrm{MCG}(\Sigma) \cong H_* \Omega^\infty_0 \mathrm{MTSO}(2)$$

Q: Are there non-0 classes in $\widetilde{H}_* \mathrm{MCG}(S)$ supported on $\mathrm{MCG}(\Sigma)$ for a compact $\Sigma \subset S$?

More precisely

$$MCG_{cpt}(S) = \operatorname*{colim}_{\substack{\Sigma \subset S \\ \text{compact}}} MCG(\Sigma) \xrightarrow{\varphi_c(S)} MCG(S)$$

$$MCG_{fin}(S) = \operatorname*{colim}_{\substack{\Sigma \subset S \\ \text{f. type} \\ \text{+ pr. emb.}}} PMCG(\Sigma) \xrightarrow{\varphi_f(S)} MCG(S)$$

Lemma

$$\begin{tikzcd} MCG_{cpt}(S) \arrow[r, two heads] \arrow[rr, bend left, "\varphi_c(S)"] & MCG_{fin}(S) \arrow[r, hook, "\varphi_f(S)"] & MCG(S) \end{tikzcd}$$

central extension by $\mathbb{Z}^p$ $\quad p = |P(S)|$

injective

Q: How do $\varphi_c(S)$ and $\varphi_f(S)$ act on $\tilde{H}_*$?

When are they non-zero?

integral coeffs unless otherwise specified

Theorem [P.-Wu '24]

$g = \infty$ $\quad$ · $\varphi_c(S)_* = 0$ $\quad$ with field coeffs

· if $p \in [1, \infty)$ then image$(\varphi_f(S)_*) \supseteq H_*(\mathbb{CP}^\infty)$

$g \in [1, \infty)$ $\quad$ · $\varphi_c(S)_* \neq 0$

$g = 0$ $\quad$ · Assume $p \in [1, \infty)$, so $S = (S^2 \setminus e) \setminus \{1, \dots, p\}$

if $p = 0, 1$ then $\varphi_f(S)_* = 0$

if $p \geq 4$ then $\varphi_c(S)_* \neq 0$

Vanishing in ∞ genus

(for this section take field coeffs)

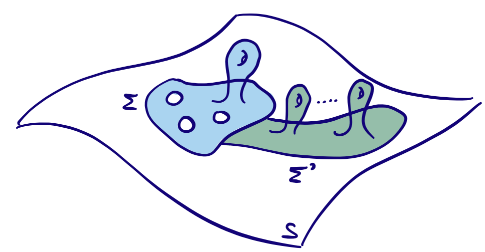
Thm If $\underset{\text{cpt}}{\Sigma} \subset \underset{\infty \text{ genus}}{S}$

then $\widetilde{H}_* MCG(\Sigma) \longrightarrow \widetilde{H}_* MCG(S)$ is the zero map

Idea

① find $\Sigma' \hookrightarrow S$ with $\Sigma' \cap \Sigma = \text{interval}$ for any $h \geq 0$

$\Sigma' \cong \Sigma_{h,1}$

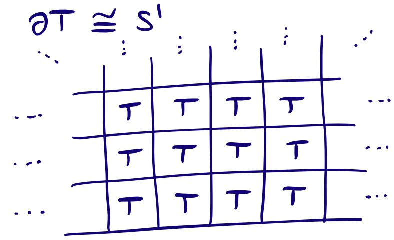


② extend $\Sigma' \hookrightarrow S$

to $\mathrm{Grid}(\Sigma') \hookrightarrow S$ (proper embedding)

Def T surface with $\partial T \cong S^1$

$\mathrm{Grid}(T) :=$



This induces $(i \geq 1)$:

$$
\begin{tikzcd}
H_i MCG(\Sigma) \arrow[d] \arrow[dr] & \\
H_i MCG(\Sigma \natural \Sigma') \arrow[r] & H_i MCG(S) \\
H_i MCG(\Sigma') \arrow[u, two heads] \arrow[ur] \arrow[r, "(*)"'] & H_i MCG(\mathrm{Grid}(\Sigma')) \arrow[u]
\end{tikzcd}
$$

surjective for $h \gg i$

[Harer]

$\Rightarrow$ enough to show that $(*)$ is 0.

③ prove this via a 2-dim. "infinite iteration trick"

<u>Rmks</u>

- [Varadarajan, Berrick] prove that pseudo-mitotic (binate) groups are acyclic via a 1-dim. "infinite iteration trick".

- Here we require field coeffs — need <u>natural</u> splitting in the Univ. Coeff. SES.

- $X \to Y$ inducing 0 on H_* with field coeffs

  $\not\Rightarrow$ same with $\mathbb{Z}$ coeffs $\left(B\mathbb{Z} \to B(\mathbb{Q}/\mathbb{Z}) \right)$

Non-vanishing in ∞ genus

(now go back to $\mathbb{Z}$ coeffs)

Thm if $\text{genus}(S) = \infty$

$|P(S)| = p \in [1, \infty)$

then $\text{image}\left(H_* MCG_{fin}(S) \to H_* MCG(S) \right) \supseteq H_*(\mathbb{C}P^\infty)$

$\hookrightarrow \oplus$

Idea

Def [Bödigheimer–Tillmann in fin. type case]

$$\begin{aligned} MCG(S) &\cong MCG(\bar{S}, P(S)) \\ &= \pi_0 \operatorname{Homeo}^+(\bar{S}, P(S)) \\ &\simeq \operatorname{Homeo}^+(\bar{S}, P(S)) \end{aligned}$$

$\bar{S} = S \cup P(S)$

- [Hamstrom '66] fin. type
  [Yagasaki '00] in general
  $\hookrightarrow$ contractible components
- $MCG(S)$ is totally disconnected

$$\simeq \operatorname{Diff}^+(\bar{S}, P(S))$$

[Kirby–Siebenmann '77]

$$\begin{array}{c} \operatorname{Diff}^+(\bar{S}, P(S)) \\ \downarrow \\ S^1 \wr \mathfrak{S}_p \end{array}$$

permutation of $P(S)$
rotations of tangent planes to $\bar{S}$ at $P(S)$
(fix framing on $P(S)$)

$$B\,MCG(S) \longrightarrow B(S^1 \wr \mathfrak{S}_p)$$

Obs Restricting to $B\,MCG_{fin}(S)$:

$$\begin{array}{ccc} B\,MCG_{fin}(S) & \xrightarrow{(*)} & (BS^1)^p \\ \downarrow & & \downarrow \\ B\,MCG(S) & \longrightarrow & B(S^1 \wr \mathfrak{S}_p) \end{array}$$

Thm [Bödigheimer–Tillmann '01 (+ Madsen–Weiss '07)]

$B\,MCG_{fin}(S)$ splits as $\Omega_0^\infty MTSO(2) \times (BS^1)^p$ on H_* / + construction

$(*)$ is the projection onto the second factor

Coro $(*)$ has a section on H_*

$$\begin{tikzcd}
H_*(BS^1)^p \arrow[r] \arrow[rr, bend left, "id"] & H_*\,MCG_{fin}(S) \arrow[r, "(*)"'] \arrow[d, "(*)"] & H_*(BS^1)^p \arrow[d] \\
 & H_*\,MCG(S) \arrow[r] & H_*B(S^1 \wr \mathfrak{S}_p) \arrow[d] \\
H_*(BS^1) \arrow[uu] \arrow[rr, "id"'] & & H_*(BS^1)
\end{tikzcd}$$

$S^1 \to (S^1)^p$

$t \mapsto (t, 1, \dots, 1)$

$S^1 \wr \mathfrak{S}_p \longrightarrow S^1$

$(t_1, \dots, t_p, \sigma) \mapsto \sum_i t_i$

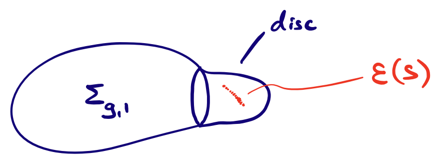
$\Rightarrow$ image $(*)$ $\supseteq_{\oplus}$ $H_*(BS^1)$.

Non-vanishing when $g \in [1, \infty)$

Thm $\exists$ cpt $\Sigma \subset S$: $H_* MCG(\Sigma) \longrightarrow H_* MCG(S)$ is non-0.

Idea — explicit construction + detection

- $\Sigma_{g,1} \hookrightarrow S$
- $\overline{S} \cong \Sigma_g$



homeo's extend (uniquely) from S to $\overline{S}$

$$\begin{tikzcd} MCG(\Sigma_{g,1}) \arrow[r, "(*)"] \arrow[rr, bend right] & MCG(S) \arrow[r] & MCG(\Sigma_g) \end{tikzcd}$$

surjective on H_2 when $g \geqslant 2$ [Harer '85]
(+ Ivanov, Boldsen, Randal-Williams)

$H_2(MCG(\Sigma_g)) \neq 0$ when $g \geqslant 2$ $\left(\begin{array}{ll} \mathbb{Z}/2 & g=2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & g=3 \\ \mathbb{Z} & g \geqslant 4 \end{array} \right)$

Hence $(*)$ is non-0 on H_2 for $g \geqslant 2$.

For $g=1$, $\underset{\mathbb{Z}}{\overset{\cong}{H_1(MCG(\Sigma_{1,1}))}} \twoheadrightarrow \underset{\mathbb{Z}/12}{\overset{\cong}{H_1(MCG(\Sigma_1))}}$

is surjective, so $(*)$ is non-0 on H_1 for $g=1$.

Vanishing for $S^2 \setminus \mathcal{C}$ and $\mathbb{R}^2 \setminus \mathcal{C}$

Thm If $\underset{\text{fin. type}}{\Sigma} \subset S = \begin{cases} S^2 \setminus \mathcal{C} \\ \mathbb{R}^2 \setminus \mathcal{C} \end{cases}$

then $\widetilde{H}_* MCG(\Sigma) \longrightarrow \widetilde{H}_* MCG(S)$ is the zero map

Idea

- If Σ compact
  $\Sigma \cong \Sigma_{0,b}$

  then $\Sigma \hookrightarrow S$ factors through D^2 or $D^2 \setminus \mathcal{C}$

  Thm [P.-Wu'22] $MCG(D^2 \setminus \mathcal{C})$ is acyclic.

- Otherwise:

$$\begin{tikzcd}
\Sigma_{0,b+1} \arrow[d, equal] & D^2 \setminus \mathcal{C} \arrow[d, equal] \\
\Sigma' \arrow[r, hook] \arrow[d, hook] & S' \arrow[d, hook] \\
\Sigma \arrow[r, hook] \arrow[d, equal] & S \arrow[d, equal] \\
\Sigma'_{0,b} & \mathbb{R}^2 \setminus \mathcal{C}
\end{tikzcd}$$

$\rightsquigarrow$ map of central extensions:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] \\
MCG(\Sigma') \arrow[r] \arrow[d, shift right] & MCG(S') \arrow[d] & \text{acyclic} \arrow[l, squiggly] \\
MCG(\Sigma) \arrow[r, "(*)"] \arrow[u, shift right] & MCG(S)
\end{tikzcd}$$

$\underbrace{MCG(\Sigma)}_{\text{trivial extension}}$

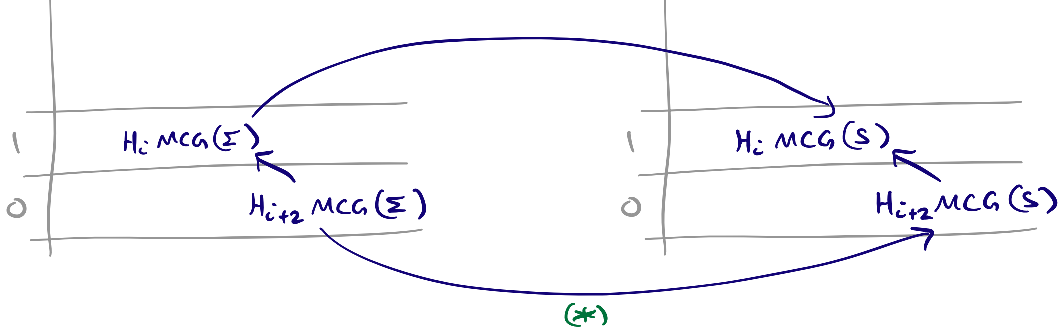
($MCG(\Sigma') =$ pure ribbon braid group, whose centre splits off as a direct factor)

$\rightsquigarrow$ map of Hochschild–Serre spectral sequences:

$$H_* MCG(S) \cong \begin{cases} 0 & * \text{ odd} \\ \mathbb{Z} & * \text{ even} \end{cases}$$

every d^2 is an isom.

$d^2 = 0$



$\square$

(BONUS)

Non-vanishing for $S = (S^2 \setminus e) \setminus \{1, \dots, p\}$ if $p \geq 4$

Thm $\exists$ cpt $\Sigma \subset S$: $H_1 MCG(\Sigma) \longrightarrow H_1 MCG(S)$ is non-0.

Idea For $p \geq 2$:

$$
\begin{tikzcd}
 & & \mathbb{Z}/2 \arrow[d] \arrow[dr, "p(p-1)"] & & \\
 & & B_p(S^2) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2(p-1) \arrow[r, two heads] & \left\{\begin{array}{ll} \mathbb{Z}/2(p-1) & p \text{ even} \\ \mathbb{Z}/(p-1) & p \text{ odd} \end{array}\right. \\
MCG(\Sigma_{0,p}) \arrow[r, "(*)"] & MCG(S) \arrow[r] & MCG(S^2 \setminus p \text{ points}) \arrow[urr, two heads] & &
\end{tikzcd}
$$

$MCG(\Sigma_{0,p}) \ni$  $\longmapsto 2 \in$ (target above)

Hence $(*)$ is non-0 on $(-)^{ab} = H_1$ whenever:

$(p \text{ even})$: $2 \not\equiv 0 \mod 2(p-1)$ $\longleftarrow$ iff $p \geq 4$

$(p \text{ odd})$: $2 \not\equiv 0 \mod (p-1)$ $\longleftarrow$ iff $p \geq 5$ $\square$

Rmk This works whenever genus$(S) = 0$ and $\mathcal{E}(S)$ has a finite, topologically-distinguished subset of size ≥ 4.