Compactly-supported H_* classes for big MCGs

Joint work with Xiaolei Wu
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Low-dimensional topology days
Nantes
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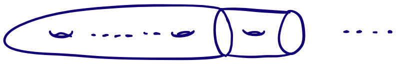
S = surface of infinite type (connected, $\partial S = \emptyset$, orientable)

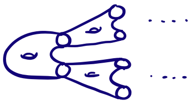
i.e. $\pi_1(S)$ not fin. gen. (but S 2nd-countable)

Typical examples

$S_\infty = \mathrm{colim}(\Sigma_{n,1})$



$S_e = \mathrm{colim}(\Sigma_{2^n-1,2^n})$



Classified by (Kérékjártó, Richards)

- genus(S) $\qquad \mathbb{N} \cup \{\infty\}$
- $\mathcal{E}(S) = \{\text{ends of } S\}$ $\qquad$ space homeo to closed $\subseteq$ Cantor
  $\cup$
- $\mathcal{E}_{np}(S) = \{\text{non-planar ends of } S\}$ $\qquad$ closed subspace ($\emptyset$ iff genus $< \infty$)

Thm (Freudenthal)

$\forall$ "nice" X, $\exists$ (unique) maximal compact" of X : $\bar{X} - X$ is 0-dim.

connected
second-countable
rim-compact ——— $\exists$ basis $\{U_i\}$ s.t. ∂U_i is compact

has a basis of clopen sets

Def $\mathcal{E}(S) := \bar{S} - S$

$\mathcal{E}_{np}(S) := \{e \in \mathcal{E}(S) \mid e \text{ does } \underline{\text{not}} \text{ have a nbhd in } \bar{S} \text{ homeo to } \mathbb{R}^2\}$

$P(S) := \{\text{punctures of } S\} = \{e \in \mathcal{E}(S) \mid e \text{ is planar \& isolated in } \mathcal{E}(S)\}$

Obs S has finite type $\iff$ genus$(S) < \infty$ and $|\mathcal{E}(S)| < \infty$.

Def $\mathrm{MCG}(S) := \pi_0 \mathrm{Homeo}^+_\partial(S)$ or-preserving
id on ∂ when we allow S to have ∂

Homology

$$H_* \mathrm{MCG}(S_e) = \begin{cases} 0 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases}$$ (Calegari-Chen)

$$H_* \mathrm{MCG}(S_e - \text{point}) = \begin{cases} 0 & * \text{ odd} \\ \mathbb{Z} & * \text{ even} \end{cases}$$ (P.-Wu)

$$H_* \mathrm{MCG}(S_\infty) \supseteq \Lambda^*\Big(\bigoplus_{\mathbb{R}} \mathbb{Z}\Big)$$ (Domat in degree $*=1$)
(P.-Wu)

Q: How well is $H_* \mathrm{MCG}(S)$ approximated by:

(1) $H_* \mathrm{MCG}(\Sigma)$ for Σ compact $\subset S$? "compactly-supported classes"

(2) $H_* \mathrm{PMCG}(\Sigma)$ for Σ fin. type $\subset S$? "finitely-supported classes"

fixing punctures

properly embedded

Rmk When $\text{genus}(S) = \infty$, the colimits of (1) + (2) are "fully understood" by

(1) Madsen-Weiss '02 (incl. Mumford Conj.)
(2) Madsen-Weiss '02 & Bödigheimer-Tillmann '01

& Harer '85

Theorem [P.-Wu '24] (also partial results when $\text{genus}(S) < \infty$)

Let $\text{genus}(S) = \infty$. Then the image of

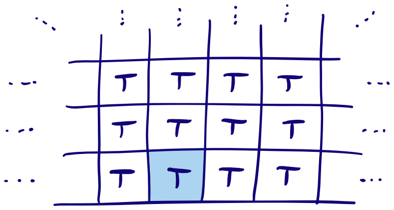
(1) $$\operatorname*{colim}_{\substack{\Sigma \subset S \\ \text{cpt}}} \widetilde{H}_* \mathrm{MCG}(\Sigma) \longrightarrow \widetilde{H}_* \mathrm{MCG}(S)$$ is zero (with field coeffs)

(2) $$\operatorname*{colim}_{\substack{\Sigma \subset S \\ \text{f. type}}} \widetilde{H}_* \mathrm{PMCG}(\Sigma) \longrightarrow \widetilde{H}_* \mathrm{MCG}(S)$$ contains a $\mathbb{Z}$ summand in all even degrees <u>if</u> $\# P(S) \in [1, \infty)$

Compact support

Def T surface with $\partial T \cong S^1$

$$T \underset{\text{proper embedding}}{\hookrightarrow} \mathrm{Grid}(T) =$$



$$MCG(T) \longrightarrow MCG(\mathrm{Grid}(T))$$

Thm (P.-Wu) This induces 0 on $\tilde{H}_*$ with field coeffs.

Proof is via a 2-dim. "infinite iteration trick". Here we require field coeffs — need natural splitting in the Univ. Coeff. SES.

Rmks

- [Varadarajan, Berrick] prove that pseudo-mitotic (binate) groups are acyclic via a 1-dim. "infinite iteration trick".
- $X \to Y$ inducing 0 on H_* with field coeffs $\not\Rightarrow$ same with $\mathbb{Z}$ coeffs $\left(B\mathbb{Z} \to B(\mathbb{Q}/\mathbb{Z}) \right)$

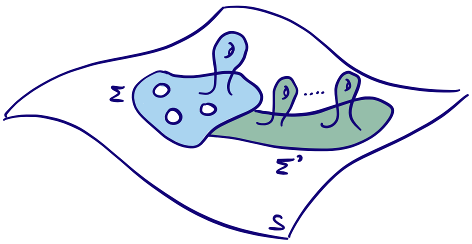
Proof of (1)

Let $\underset{\text{compact}}{\Sigma} \subset \underset{\infty \text{ genus}}{S}$

Aim: $\tilde{H}_* MCG(\Sigma) \longrightarrow \tilde{H}_* MCG(S)$ is the zero map

① find $\Sigma' \hookrightarrow S$ with $\Sigma' \cap \Sigma$ = interval for any $h \geq 0$

$\Sigma' \cong \Sigma_{h,1}$



② extend $\Sigma' \hookrightarrow S$

to $\mathrm{Grid}(\Sigma') \longhookrightarrow S$ (proper embedding)

For each $i \geq 1$ this induces:

$$\begin{tikzcd}
H_i MCG(\Sigma) \arrow[d] \arrow[rrd] & & \\
H_i MCG(\Sigma \natural \Sigma') \arrow[rr] & & H_i MCG(S) \\
H_i MCG(\Sigma') \arrow[u, "\cong \text{ for } h \gg i \text{ [Harer]}"] \arrow[rru] \arrow[rr, "0"'] & & H_i MCG(\mathrm{Grid}(\Sigma')) \arrow[u]
\end{tikzcd}$$

□

Finite-type support

Thm if $\mathrm{genus}(S) = \infty$

$|P(S)| = p \in [1,\infty)$

then $\mathrm{image}\left(H_* MCG_{fin}(S) \to H_* MCG(S)\right) \underset{\oplus}{\supseteq} H_*(\mathbb{CP}^\infty)$

Idea

Def [Bödigheimer–Tillmann in fin. type case]

$$MCG(S) \cong MCG(\bar{S}, P(S)) \qquad \bar{S} = S \cup P(S)$$

$$= \pi_0 \mathrm{Homeo}^+(\bar{S}, P(S))$$

$$\simeq \mathrm{Homeo}^+(\bar{S}, P(S))$$

- [Hamstrom '66] fin. type
  [Yagasaki '00] in general
  $\hookrightarrow$ contractible components
- $MCG(S)$ is totally disconnected

$$\simeq \mathrm{Diff}^+(\bar{S}, P(S)) \qquad \text{[Kirby–Siebenmann '77]}$$

$$\begin{tikzcd} \mathrm{Diff}^+(\bar{S}, P(S)) \arrow[d] \\ S^1 \wr \mathfrak{S}_p \end{tikzcd}$$

permutation of $P(S)$
rotations of tangent planes to $\bar{S}$ at $P(S)$
(fix framing on $P(S)$)

$$B\,MCG(S) \longrightarrow B(S^1 \wr \mathfrak{S}_p)$$

Obs Restricting to $B\,MCG_{fin}(S)$:

$$\begin{tikzcd} B\,MCG_{fin}(S) \arrow[r, "(*)"'] \arrow[d] & (BS^1)^p \arrow[d] \\ B\,MCG(S) \arrow[r] & B(S^1 \wr \mathfrak{S}_p) \end{tikzcd}$$

Corollary of [Bödigheimer - Tillmann '01]

(*) admits a section on H_*

Hence we have:

$$\begin{tikzcd}
H_*(BS^1)^p \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & H_* MCG_{fin}(S) \arrow[r, "(*)"'] \arrow[d, "(*)"] & H_*(BS^1)^p \arrow[d] \\
& H_* MCG(S) \arrow[r] & H_* B(S^1 \wr \mathfrak{S}_p) \arrow[d, "{\begin{array}{c} S^1 \wr \mathfrak{S}_p \longrightarrow S^1 \\ (t_1,\dots,t_p,\sigma) \mapsto \sum_i t_i \end{array}}"] \\
H_*(BS^1) \arrow[uu, "{\begin{array}{c} S^1 \to (S^1)^p \\ t \mapsto (t,1,\dots,1) \end{array}}"] \arrow[rr, "\mathrm{id}"'] & & H_*(BS^1)
\end{tikzcd}$$

$\Rightarrow \quad \text{image } (*) \underset{\oplus}{\supseteq} H_*(BS^1).$

$\square$

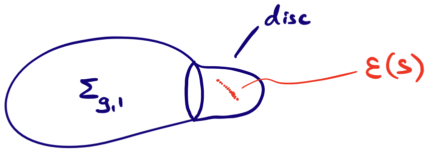
(BONUS 1)

Non-vanishing when $g \in [1, \infty)$

Thm $\exists$ cpt $\Sigma \subset S$: $H_* MCG(\Sigma) \longrightarrow H_* MCG(S)$ is non-0.

Idea — explicit construction + detection

- $\Sigma_{g,1} \hookrightarrow S$
- $\bar{S} \cong \Sigma_g$



homeo's extend (uniquely) from S to $\bar{S}$

$$\begin{tikzcd}
MCG(\Sigma_{g,1}) \arrow[r, "(*)"] \arrow[rr, bend right] & MCG(S) \arrow[r] & MCG(\Sigma_g)
\end{tikzcd}$$

surjective on H_2 when $g \geq 2$ [Harer '85]
(+ Ivanov, Boldsen, Randal-Williams)

$H_2(MCG(\Sigma_g)) \neq 0$ when $g \geq 2$ $\quad \left(\begin{array}{ll} \mathbb{Z}/2 & g=2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & g=3 \\ \mathbb{Z} & g \geq 4 \end{array} \right)$

Hence (*) is non-0 on H_2 for $g \geq 2$.

For $g=1$,
$$\begin{array}{ccc} H_1(MCG(\Sigma_{1,1})) & \twoheadrightarrow & H_1(MCG(\Sigma_1)) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z}/12 \end{array}$$

is surjective, so (*) is non-0 on H_1 for $g=1$.

(BONUS 2)

Vanishing for $S^2 \setminus \mathcal{C}$ and $\mathbb{R}^2 \setminus \mathcal{C}$

Thm If $\underset{\text{fin. type}}{\Sigma} \subset S = \begin{cases} S^2 \setminus \mathcal{C} \\ \mathbb{R}^2 \setminus \mathcal{C} \end{cases}$

then $\tilde{H}_* MCG(\Sigma) \longrightarrow \tilde{H}_* MCG(S)$ is the zero map

Idea

- If Σ compact
  $\Sigma \cong \Sigma_{0,b}$

  then $\Sigma \hookrightarrow S$ factors through D^2 or $D^2 \setminus \mathcal{C}$

  Thm [P.-Wu'22] $MCG(D^2 \setminus \mathcal{C})$ is acyclic.

- Otherwise :

$$\begin{tikzcd}
\Sigma_{0,b+1} & D^2 \setminus \mathcal{C} \\
\Sigma' \arrow[u, equal] \arrow[r, hook] \arrow[d, hook] & S' \arrow[u, equal] \arrow[d, hook] \\
\Sigma \arrow[r, hook] \arrow[d, equal] & S \arrow[d, equal] \\
\Sigma_{0,b} & \mathbb{R}^2 \setminus \mathcal{C}
\end{tikzcd}$$

$\rightsquigarrow$ map of central extensions:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] & \text{acyclic} \arrow[dl, squiggly] \\
MCG(\Sigma') \arrow[r] \arrow[d, shift right] & MCG(S') \arrow[d] & \\
\underbrace{MCG(\Sigma)}_{\text{trivial extension}} \arrow[r, "(*)"] \arrow[u, shift right] & MCG(S) &
\end{tikzcd}$$

($MCG(\Sigma')$ = pure ribbon braid group, whose centre splits off as a direct factor)

$\square$

(BONUS 3)

Non-vanishing for $S = (S^2 \setminus \mathcal{C}) \setminus \{1, \dots, p\}$ if $p \geqslant 4$

Thm $\exists$ cpt $\Sigma \subset S$: $H_1 \mathrm{MCG}(\Sigma) \longrightarrow H_1 \mathrm{MCG}(S)$ is non-0.

Idea For $p \geqslant 2$:

$$\begin{tikzcd}
& & & \mathbb{Z}/2 \arrow[d] \arrow[dr, "p(p-1)"] & & \\
& & & B_p(S^2) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2(p-1) \arrow[r, two heads] & \left\{ \begin{array}{ll} \mathbb{Z}/2(p-1) & p \text{ even} \\ \mathbb{Z}/(p-1) & p \text{ odd} \end{array} \right. \\
& \mathrm{MCG}(\Sigma_{0,p}) \arrow[r, "(*)"] & \mathrm{MCG}(S) \arrow[r] & \mathrm{MCG}(S^2 \setminus p \text{ points}) \arrow[urr, two heads] & & \rotatebox{90}{$\in$} \\
& \rotatebox{90}{$\in$} & & & & 2 \\
& {} \arrow[urrrr, mapsto] & & & &
\end{tikzcd}$$

Hence $(*)$ is non-0 on $(-)^{ab} = H_1$ whenever :

(p even): $2 \not\equiv 0 \mod 2(p-1)$ $\longleftarrow$ iff $p \geqslant 4$

(p odd): $2 \not\equiv 0 \mod (p-1)$ $\longleftarrow$ iff $p \geqslant 5$ $\square$

Rmk This works whenever genus$(S) = 0$ and $\mathcal{E}(S)$ has a finite, topologically-distinguished subset of size $\geqslant 4$.