

## Homology of **big** MCGs

## Setting

$S =$  *infinite-type* surface  
 $\text{Mod}(S) = \text{Homeo}(S)/\text{iso}$

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### Examples

$$S_\infty = \text{[Diagram of a surface with three handles and an ellipsis]} \dots$$

$$S_C = \text{[Diagram of a diamond-shaped region with a dashed orange line and an ellipsis]} \dots$$

$$= \mathbb{R}^2 \setminus \text{Cantor}$$

$$F_1 = \text{[Diagram of a surface with four punctures and an ellipsis]} \dots$$

$$F_n = \text{[Diagram of a surface with five punctures and an ellipsis]} \dots$$

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## Questions

(1)  $H_*(\text{Mod}(S)) \cong ?$

(2) What is the map

$$\text{colim}_{\substack{\Sigma \subset S \\ \text{compact}}} (H_*(\text{Mod}(\Sigma)))$$



(\*)

$$H_*(\text{Mod}(S)) \quad ?$$

(What is its image?)

(Is it zero?)

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## Remark

Domain of (\*) is understood by [Madsen-Weiss, 2007] in the case  $\text{genus}(S) = \infty$ .

Rationally  $\cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]^*$

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$$S_\infty = \text{[Diagram of a surface with three handles and a boundary, with an ellipsis indicating infinite handles]}$$

$$S_c = \text{[Diagram of a diamond-shaped region with a dashed orange line representing a Cantor set]} \\ = \mathbb{R}^2 \setminus \text{Cantor}$$

$$F_1 = \text{[Diagram of a surface with one boundary component and a Cantor set]} \\ \dots$$

$$F_n = \text{[Diagram of a surface with n boundary components and a Cantor set]} \\ \dots$$

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## Answers

Theorem [P.-Wu, 2022-24]

	(1)	(2)
$S_\infty$	$\supseteq \mathbb{Z}^{\oplus \mathfrak{c}} \vee *$	0
$S_c$	$\left\{ \begin{array}{l} \mathbb{Z} \text{ * even} \\ 0 \text{ * odd} \end{array} \right\}$	0
$F_1$	$\supseteq \mathbb{Z}^{\oplus \mathfrak{c}} \vee *$	0
$F_n$	??	$\neq 0$

( $n \geq 4$ )

$\mathfrak{c} = |\text{continuum}| = 2^{\aleph_0}$

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$$F_n = \text{[Diagram of a surface with four arms and a central point]} \dots$$

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## Proofs – key ideas

*Homological stability*

*Pseudo-mitotic/binate grps*

*Top. distinguished ends*

*Escaping sequences of SCCs*

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## Open questions

(A)  $\exists$  surface  $S$ :

$H_*(\text{Mod}(S)) \supseteq A$ ,

$A$  torsion & uncountable?

(B) Is there a dichotomy

•  $\forall i, H_i(\text{Mod}(S))$  fin. gen.

•  $\forall i, |H_i(\text{Mod}(S))| > \aleph_0$  ?