The lower central series of partitioned motion groups

Joint work with Jacques Darné
& Arthur Soulié

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Def The lower central series of a group G is

$$G = \Gamma_1(G) \supseteq \Gamma_2 \supseteq \cdots\cdots$$

given by

$$\Gamma_{n+1}(G) = [G, \Gamma_n(G)]$$
$$= \left\{ [g,h] = ghg^{-1}h^{-1} \;\middle|\; g \in G,\ h \in \Gamma_n(G) \right\}$$

$$\Gamma_\infty(G) = \bigcap_{n=1}^{\infty} \Gamma_n(G)$$

Rmk If $\Gamma_i(G) = \Gamma_{i+1}(G)$
then $\Gamma_{i+1}(G) = \Gamma_{i+2}(G)$.

Def The LCS length $\ell(G)$ is the smallest i such that $\Gamma_i(G) = \Gamma_{i+1}(G)$, if it exists. Otherwise $\ell(G) = \infty$.

Ex $A \neq 0$ abelian $\rightsquigarrow$ $\ell(A) = 2$
G perfect $\rightsquigarrow$ $\ell(G) = 1$

We will consider $G =$ surface braid group
loop braid group & generalisations.

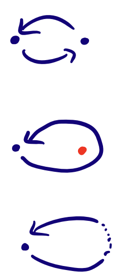
<u>Def</u> S surface

$\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\sum \lambda_i = n$

$B_\lambda(S) = \pi_1 C_\lambda(S)$

where $C_\lambda(S) = \{ (p_1, \ldots, p_n) \in S^n \mid p_i \neq p_j \text{ for } i \neq j \} \big/ \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$

<u>Generators:</u>



<u>Def</u> $a, b, c \in \mathbb{N}$

$wB(a,b,c) = \pi_1 \left(\text{config space of } \begin{array}{l} a \text{ points} \\ b \text{ oriented unknots} \\ c \text{ unoriented unknots} \end{array} \text{ in } \mathbb{R}^3 \right)$

disjoint
unlinked

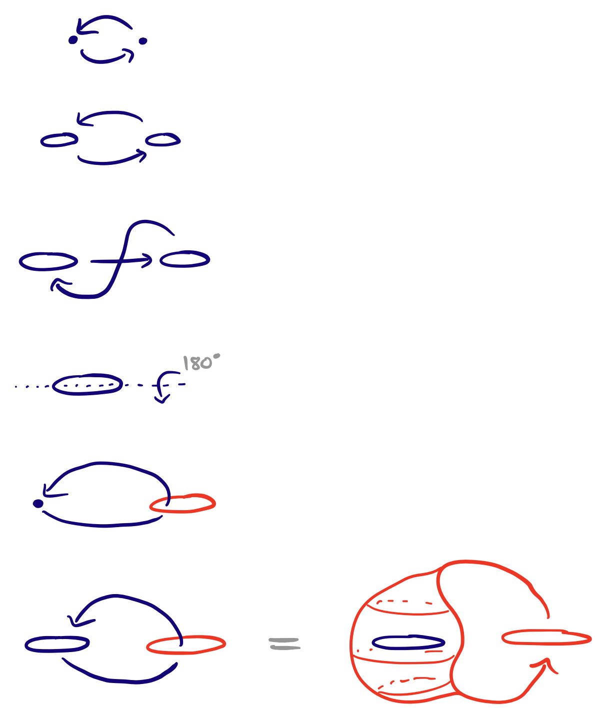
& generalise to <u>partitions</u> of a, b and c

$wB(a,0,0) \cong \Sigma_a$

$wB(0,b,c) = LB_b$ loop braid group

$wB(0,0,c) = \widetilde{LB}_c$ extended loop braid group

Generators:



Theorem (Damé – P. – Sorlié '22)

Complete calculation of $\ell(G)$ for the above groups.

almost! exception is $B_{n,2}(\mathbb{RP}^2)$ for $n \geqslant 3$

Sample:

group	length	
B_n	2	
$B_{n,2}$	∞	
$B_{n,2}(S^2)$	2	$n=1$
	∞	$n=2$
	$\nu_2(m) + (1 \text{ or } 2 \text{ or } 3)$	$n \geqslant 3$
$g \geqslant 2$ $\quad B_n(\Sigma_g)$	∞	$n=1,2$
	3	$n \geqslant 3$
LB_n	∞	$n=2,3$
	2	$n \geqslant 4$
$wB(2,0,n)$	3	$n \geqslant 4$

when $g=1$ the results are the same, except for $n=1$, in which case $B_1(\Sigma_1) \cong \mathbb{Z}^2$ has length 2

see BONUS page 10

= I'll prove in this talk.

Aside — homological representations of motion groups:

Construction scheme (P. - Soulié '19+) (e.g. surface braid groups, but works also for other motion groups) $(\partial S \neq \emptyset)$

$$\begin{tikzcd}
1 \arrow[r] & B_m(S \setminus n) \arrow[r] \arrow[d, two heads, "\phi"] & B_{m,n}(S) \arrow[r] \arrow[d, two heads] & B_n(S) \arrow[r] \arrow[d, two heads] \arrow[l, bend right, dashed] & 1 \\
1 \arrow[r] & Q \arrow[r] & B_{m,n}(S)/\Gamma_i \arrow[r] & B_n(S)/\Gamma_i \arrow[r] \arrow[l, bend right, dashed] & 1
\end{tikzcd}$$

$\leadsto$ local system over $\mathbb{Z}[Q]$ on $C_m(S \setminus n)$

$\rightsquigarrow$ (BM) homology with local coeffs

sequence of (twisted) representations of $B_n(S)$

- Complexity / "richness" depends on length of $B_{m,n}(S)$
- Ex $\begin{array}{l} S = D^2 \\ m = 2 \end{array}$ $\leadsto$ sequence starting with the LKB representation

  Lawrence - Krammer - Bigelow

Proof of finite length

Def $$\mathcal{L}(G) = \bigoplus_{k=1}^{\infty} \mathcal{L}_k(G)$$

$$\mathcal{L}_k(G) = \Gamma_k(G) \big/ \Gamma_{k+1}(G)$$

Fact
- Graded Lie ring via $[\bar{g}, \bar{h}] = \overline{ghg^{-1}h^{-1}}$.
- As such it is generated by $\mathcal{L}_1(G) = G^{ab}$.
- $\ell(G) \leqslant k$ iff $\mathcal{L}_k(G) = 0$

Lemma Let S be a gen. set for $G^{ab} \neq 0$

Suppose: $\forall s,t \in S$ we can lift $s = \bar{g}$ $g \in G$, $t = \bar{h}$ $h \in G$ where g,h commute.

Then $\ell(G) = 2$.

Proof: $\mathcal{L}_2(G)$ is generated by $[s,t]$ for $s,t \in S$

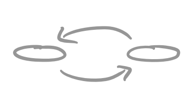
$$[s,t] = [\bar{g}, \bar{h}] = \overline{ghg^{-1}h^{-1}} = 1$$ □.

Coro $\ell(B_n) = 2$

$\ell(LB_n) = 2$ if $n \geqslant 4$

Proof $(B_n)^{ab} \cong \mathbb{Z}$ ✓

$(LB_n)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/2$



choose rep's with disjoint support □.

More generally:

$\ell(B_\lambda) = 2$ if all blocks of λ have size ≥ 3

Idea: $\lambda = (\lambda_1, \dots, \lambda_k)$

$$(B_\lambda)^{ab} \cong \mathbb{Z}^k \oplus \mathbb{Z}^{\binom{k}{2}}$$

Exercise

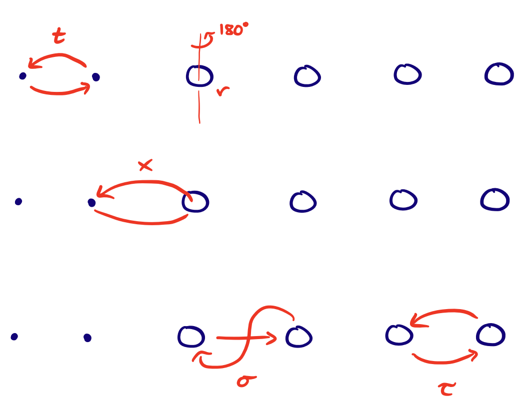
$\ell(wB(\lambda,\mu,\nu)) = 2$ if all blocks of λ have size ≥ 3
& ------ μ, ν ----- ≥ 4.

(Hint: consider the generating set on page 3)

Pattern:
large blocks $\longleftrightarrow$ finite length (often 2)
small blocks $\longleftrightarrow$ infinite length
intermediate $\longleftrightarrow$ .... difficult ...

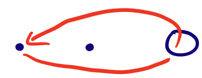
Case study: $G = wB(2,0,n)$ for $n \geq 4$

$$\mathcal{L}_1(G) = G^{ab} = \underset{\bar{x}}{\mathbb{Z}} \oplus \underset{\bar{t}}{\mathbb{Z}/2} \oplus \underset{\bar{r}}{\mathbb{Z}/2} \oplus \underset{\bar{\sigma}}{\mathbb{Z}} \oplus \underset{\bar{\tau}}{\mathbb{Z}/2}$$

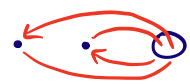


$\mathcal{L}_2(G)$: All Lie brackets between generators vanish due to disjoint support, except:

$$[\bar t, \bar x] = \overline{t x t^{-1} x^{-1}}$$

$t x t^{-1} =$  ...

$x^{-1} =$  ...

Let $z := t x t^{-1} x^{-1} =$  ....

$\mathcal{L}_2(G)$ is cyclic, generated by $\bar z$.

$\mathcal{L}_3(G)$: Generated by

$$\left.\begin{array}{l}[\bar z, \bar \sigma] \\ [\bar z, \bar \tau] \\ [\bar z, \bar r]\end{array}\right\} = 0 \qquad \text{(disjoint support)}$$

$$[\bar z, \bar x]$$
$$[\bar z, \bar t]$$

$$[\bar z, \bar x] = \overline{(t x t^{-1}) x (t x^{-1} t^{-1}) x^{-1}}$$
$$= 0$$

$t x t^{-1}$ commutes with x

$$[\bar z, \bar t] = \overline{z (t z^{-1} t^{-1})}$$
$$= \overline{z^2}$$

$t z^{-1} t^{-1} = z$

Trick:

$$0 = [\bar z, \bar r] = \overline{z r z^{-1} r^{-1}}$$
$$= \overline{z^2}$$

$r z^{-1} r^{-1} = z$

can choose different representative for $\bar r$ with disjoint support

Hence $\mathcal{L}_3(G) = 0$. So $\ell(wB(2,0,n)) = 3$.

Proof of infinite length

Lemma If $G \xrightarrow{\pi} H$ then $\ell(G) \geq \ell(H)$.

Proof $\pi(\Gamma_k(G)) = \Gamma_k(H)$

$$\Gamma_{\ell(G)+1}(H) = \pi(\Gamma_{\ell(G)+1}(G))$$
$$= \pi(\Gamma_{\ell(G)}(G))$$
$$= \Gamma_{\ell(G)}(H) \qquad \square$$

Examples

- $\mathbb{Z} \rtimes \mathbb{Z}/2$ (sign) $\rightsquigarrow \Gamma_k = 2^{k-1}\mathbb{Z}$
  $\rightsquigarrow$ length $= \infty$

- $\mathbb{Z}^2 \rtimes \mathbb{Z}/2$ (swap) $\rightsquigarrow \Gamma_k = 2^{k-2}(\delta\mathbb{Z})$ $\quad \delta\mathbb{Z} = \langle(1,-1)\rangle \subseteq \mathbb{Z}^2$
  $\rightsquigarrow$ length $= \infty$

- F_n $(n \geq 2)$ $\rightsquigarrow$ length $= \infty$ (+ res. nilpotent) [Magnus]

  ALTERNATIVE PROOF:
  $$F_n \twoheadrightarrow F_2 \twoheadrightarrow \mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$$

Proposition

$B_{n,2}$

$B_1(\Sigma_g) = \pi_1(\Sigma_g)$ $\quad (g \geq 2)$

$B_2(\Sigma_g)$ $\quad (g \geq 1)$

all have length $= \infty$

Proof

g

$(n \geq 3)$

$$B_{n,2} \twoheadrightarrow B_{n,2}/\Gamma_\infty \cong \mathbb{Z} \times (\mathbb{Z}^2 \rtimes \mathbb{Z})$$

in block of n

in block of 2

swap

$$\mathbb{Z} \times (\mathbb{Z}^2 \rtimes \mathbb{Z}) \twoheadrightarrow \mathbb{Z}^2 \rtimes \mathbb{Z}/2$$

sign

$$B_{2,2} \twoheadrightarrow \mathbb{Z} \rtimes \mathbb{Z}/2$$

$$\begin{array}{c} B_{1,2} \cong F_2 \rtimes \mathbb{Z} \\ \big\downarrow\!\!\!\downarrow \\ \mathbb{Z}^2 \rtimes \mathbb{Z} \\ \big\downarrow\!\!\!\downarrow \\ \mathbb{Z}^2 \rtimes \mathbb{Z}/2 \end{array}$$

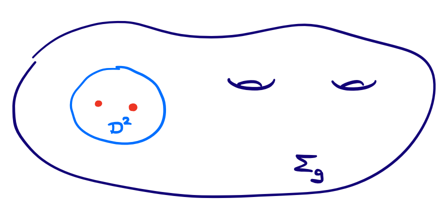
$$1 \to B_1(D^2 - 2\text{pts}) \to B_{1,2} \to B_2 \to 1, \qquad B_1(D^2-2\text{pts}) \cong F_2, \quad B_2 \cong \mathbb{Z}$$

$(g \geq 2)$

$$B_1(\Sigma_g) = \pi_1(\Sigma_g) = \langle a_1 \ldots a_g, b_1 \ldots b_g \mid [a_1,b_1] \cdots [a_g,b_g] = 1 \rangle \xrightarrow{a_i = b_i} F_g$$

$(g \geq 1)$

$$B_2(\Sigma_g) \twoheadrightarrow \pi_1(\Sigma_g)^2 \rtimes \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}^2 \rtimes \mathbb{Z}/2$$



$$\Sigma_g \twoheadrightarrow \Sigma_g / D^2$$

BONUS

LB_2
LB_3
$B_n(\Sigma_g) \quad (n \geq 3)$

$$LB_2 \cong \mathbb{Z} * \mathbb{Z}/2 \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$$

$$LB_3 \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$$

quotient by Γ_∞
and then by the
centre of the result

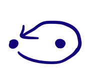
(both) (both)

$B_n(\Sigma_g)$:

Fact: $\ell(G) \leq k$ iff G/Γ_∞ is $(k-1)$-nilpotent

Thm: For $n \geq 3$ and <u>any</u> surface S,
there is a central extension

$$1 \to \langle \sigma^2 \rangle \to B_n(S)/\Gamma_\infty \longrightarrow\!\!\!\!\rightarrow H_1(S) \times \mathbb{Z}/2 \to 1$$

$\sigma^2 =$ 

Ψ
b
$\updownarrow$
collection of closed loops in S $\longmapsto$ (fundamental class, $\mathrm{sgn}(\beta)$)

Coro: $B_n(S)/\Gamma_\infty$ is 2-nilpotent

$\longrightarrow$ length ≤ 3

$$\left(\text{length} = \begin{cases} 2 & S \subseteq S^2 \\ 2 & \text{non-orientable} \\ 3 & \text{o/w} \end{cases} \right)$$