On the H_* of the MCG of the Loch Ness monster

Joint work with Xiaolei Wu

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2212.11942
2405.03512

Topology seminar
Sheffield
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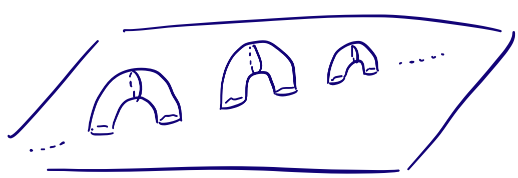
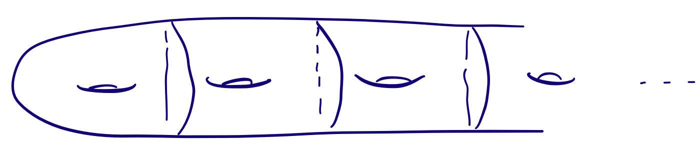
Loch Ness monster:

$L =$ 

$=$ 

$$\mathrm{Mod}(L) = \pi_0 \mathrm{Homeo}^+(L)$$

$$\mathrm{Mod}_c(L) = \{[\varphi] \in \mathrm{Mod}(L) \mid \varphi \text{ has compact support}\}$$

$$\cong \operatorname{colim}_g \mathrm{Mod}(\Sigma_{g,1})$$

<u>Thm</u> (Madsen-Weiss) · $H_* \mathrm{Mod}_c(L) \cong H_*(\Omega_0^\infty MTSO(2))$

· rationally gen. by duals of monomials in MMM classes $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]^*$

Q1: What is $H_* \mathrm{Mod}(L)$?

Q2: What is the image of $H_* \mathrm{Mod}_c(L)$ in $H_* \mathrm{Mod}(L)$?

(Partial) **A1:**

Thm (A) (P.-Wu '22)

$$H_i \mathrm{Mod}(L) \supseteq \bigoplus_{2^{\aleph_0}} \mathbb{Z} \qquad \forall i > 0$$

Extends **Thm** (Domat '20) $H_1 \mathrm{Mod}(L) \supseteq \bigoplus_{2^{\aleph_0}} \mathbb{Q}$

Rmk: No known torsion.

A2:

Thm (B) (P.-Wu '24)

With field coefficients, the image is zero.

Coro: dual MMM-classes vanish in $H_* \mathrm{Mod}(L)$

Rmk: $\not\Rightarrow$ image is zero with $\mathbb{Z}$ coeffs

(consider $S^1 \longrightarrow B(\mathbb{Q}/\mathbb{Z})$)

More general surfaces

S any connected, orientable surface

$\overline{S} :=$ its Freudenthal compactification ("maximal compact" with 0-dim remainder)

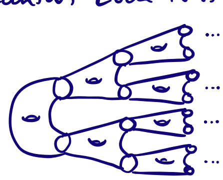
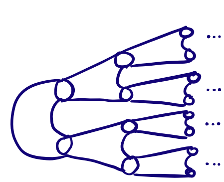
[Kerékjártó, Richards]

S is classified up to homeo. by

- $g(S) = \max\{ g \mid \Sigma_{g,1} \hookrightarrow S \} \in \mathbb{N}_0 \cup \{\infty\}$
- $\mathcal{E}(S) = \overline{S} - S$ $\qquad$ ($\cong$ subspace of $\mathcal{C}$ = Cantor)
- $\mathcal{E}_{np}(S) = \{ e \in \mathcal{E}(S) \mid S \cup \{e\} \text{ is not a mfld} \}$

Rmk: $\Rightarrow$ Every ∞-genus surface with a single end is homeomorphic to L.
E.g. $\partial \mathrm{Tub}(\Gamma)$ for Γ = any graph $\subset \mathbb{R}^3$ with infinite β_1.

Examples

S	$g(S)$	$\mathcal{E}(S)$	$\mathcal{E}_{np}(S)$
L	∞	$\{*\}$	$\{*\}$
$L_{\mathcal{C}}$ "Cantor's Loch Ness monster" 	∞	$\mathcal{C}$	$\mathcal{C}$
$S^2 \setminus \mathcal{C} \cong$ 	0	$\mathcal{C}$	$\emptyset$

Q1': What is $H_* \mathrm{Mod}(S)$?

- Unknown in most cases.
- Very sensitive to the structure of $\mathcal{E}(S)$.

Thm Ⓒ (P.-Wu '22)

$$H_* \mathrm{Mod}(L_{\mathcal{E}} - pt) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases} \cong H_* \mathrm{Mod}(\mathbb{R}^2 \setminus \mathcal{C})$$

Coro: $H_1 \mathrm{Mod}(S^2 \setminus \mathcal{C}) = 0$ [Vlamis, Calegari-Chen]
$H_1 \mathrm{Mod}(L_{\mathcal{E}}) = 0$

(because $\mathrm{Mod}(L_{\mathcal{E}} - pt) \twoheadrightarrow \mathrm{Mod}(L_{\mathcal{E}})$
$\mathrm{Mod}(\mathbb{R}^2 \setminus \mathcal{C}) \twoheadrightarrow \mathrm{Mod}(S^2 \setminus \mathcal{C})$)

Q2': In general, $\mathrm{Mod}_c(S) \not\cong \underset{\Sigma \text{ cpt} \subseteq S}{\mathrm{colim}} \big(\mathrm{Mod}(\Sigma)\big) =: \widetilde{\mathrm{Mod}}_c(S)$

Example: $S =$ 1-punctured disc.

Lemma: central extension

$$0 \to \bigoplus_{P(S)} \mathbb{Z} \longrightarrow \widetilde{\mathrm{Mod}}_c(S) \longrightarrow \mathrm{Mod}_c(S) \to 1$$

punctures of S
= planar, isolated ends

Q: What is image $\big(H_* \widetilde{\mathrm{Mod}}_c(S) \to H_* \mathrm{Mod}(S) \big) = \tilde{I}(S)$?
image $\big(H_* \mathrm{Mod}_c(S) \to H_* \mathrm{Mod}(S) \big) = I(S)$?

Thm (B) (gen.) (P-Wu '24)

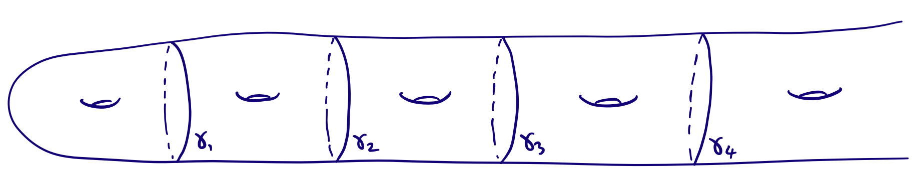
- If $g(S) = \infty$ then $\tilde{\mathcal{I}}(S) = 0$ (with field coeffs).
- If $g(S) = \infty$ and $\#P(S) \in [1,\infty)$ then $\mathcal{I}(S) \supseteq \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$

Uses work of [Bödigheimer–Tillmann]. See BONUS on pp. 16–17...

Proof of Thm (A) $\left(H_i \operatorname{Mod}(L) \supseteq \bigoplus_{\mathfrak{c}} \mathbb{Z} \quad \forall i > 0 \right)$, $\mathfrak{c} = 2^{\aleph_0}$

Idea: factor $\bigoplus_{\mathfrak{c}} \mathbb{Z} \hookrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q}$ through $\operatorname{Mod}(L)$

$$\begin{tikzcd}
H_*\left(\bigoplus_{\mathfrak{c}} \mathbb{Z}\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_*(\operatorname{Mod}(L)) \arrow[r] & H_*\left(\bigoplus_{\mathfrak{c}} \mathbb{Q}\right) \arrow[d, phantom, "\cong" rotate=90] \\
\Lambda^*_{\mathbb{Z}}\left(\bigoplus_{\mathfrak{c}} \mathbb{Z}\right) \arrow[rr, hook, "\text{injective}"'] & & \Lambda^*_{\mathbb{Z}}\left(\bigoplus_{\mathfrak{c}} \mathbb{Q}\right)
\end{tikzcd}$$



$$X \subseteq \mathbb{N} \rightsquigarrow f(X) = \prod_{i \in X} (T_{\gamma_i})^{i!}$$

$$f_n(X) = \prod_{\substack{i \in X \\ i \geq n}} (T_{\gamma_i})^{i!/n}$$

Exercise : $\exists$ a continuum collection $\{ X_a \subseteq \mathbb{N} \mid a \in \mathbb{R} \}$ so that

$$(*) \begin{cases} \cdot \text{ all } X_a \text{ are infinite} \\ \cdot \text{ all } X_a \cap X_b \text{ are finite} \end{cases}$$

Construction :

$$\begin{tikzcd} \bigoplus_{a \in \mathbb{R}} \mathbb{Z} \arrow[r, "1_a \,\mapsto\, f(X_a)"] \arrow[d, hook] & \mathrm{Mod}(L) \arrow[d, two heads] \\ \bigoplus_{a \in \mathbb{R}} \mathbb{Q} \arrow[r, "\Theta", "\left(\frac{1}{n}\right)_a \,\mapsto\, {[f_n(X_a)]}"'] & \mathrm{Mod}(L)^{ab} \end{tikzcd}$$

check :

- $f(X_a) - f_n(X_a)^n$ cpt support
- [Birman, Powell '70s] $\Rightarrow$

$$\left[f(X_a) - f_n(X_a)^n \right] = 0$$

$(*)$ + [Domat '20] $\leftarrow$ [Bestvina - Bromberg - Fujiwara '15]

$\Rightarrow$ Θ is injective

But every injection $\bigoplus_I \mathbb{Q} \longrightarrow A$ admits a retraction.

$\longrightarrow \square$

Rmk

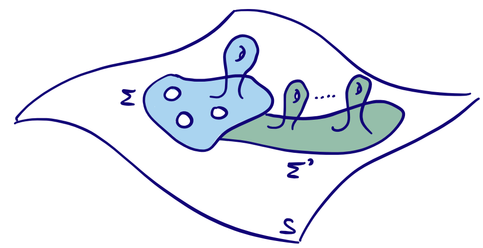
$$\begin{array}{ccc} H^i\left(\bigoplus_c \mathbb{Z}\right) & \xleftarrow{\quad 0 \quad} & H^i\left(\bigoplus_c \mathbb{Q}\right) \\ \cong & & \cong \\ \prod_c \mathbb{Z} & & \begin{Bmatrix} 0 & i = 1 \\ \bigoplus_{2^c} \mathbb{Q} & i \geq 2 \end{Bmatrix} \end{array}$$

Proof of Thm (B)

$\left(\begin{array}{l} \Sigma \text{ compact } \subseteq S \ \ \infty\text{-genus} \\ \Rightarrow \ H_* \mathrm{Mod}(\Sigma) \longrightarrow H_* \mathrm{Mod}(S) \\ \qquad \text{is zero with field coeffs} \end{array} \right)$

① find $\Sigma' \hookrightarrow S$ with $\Sigma' \cap \Sigma = \text{interval}$ for any $h \geq 0$

$\Sigma' \cong \Sigma_{h,1}$

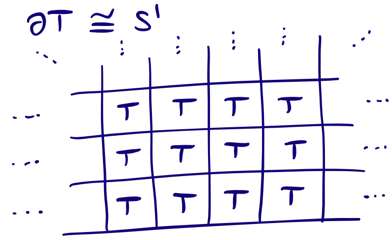


② extend $\Sigma' \hookrightarrow S$

to $\mathrm{Grid}(\Sigma') \hookrightarrow S$ (proper embedding)

Def T surface with $\partial T \cong S^1$

$\mathrm{Grid}(T) :=$



This induces $(i \geq 1)$:

$$\begin{tikzcd} H_i \mathrm{Mod}(\Sigma) \arrow[d] \arrow[drr] & & \\ H_i \mathrm{Mod}(\Sigma \natural \Sigma') \arrow[rr] & & H_i \mathrm{Mod}(S) \\ H_i \mathrm{Mod}(\Sigma') \arrow[u, two heads] \arrow[urr] \arrow[rr, "(*)"'] & & H_i \mathrm{Mod}(\mathrm{Grid}(\Sigma')) \arrow[u] \end{tikzcd}$$

surjective for $h \gg i$

[Harer]

$\Rightarrow$ enough to show that $(*)$ is 0.

③ prove this via a 2-dim. "infinite iteration trick"

Rmks

- [Varadarajan, Berrick] prove that pseudo-mitotic (binate) groups are acyclic via a 1-dim. "infinite iteration trick".
- Here we require field coeffs — need natural splitting in the Künneth SES.

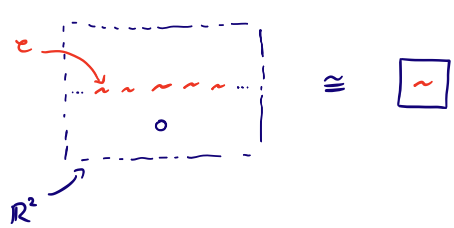
Proof of Thm (c)

$$\left(H_* \, \mathrm{Mod}(\mathbb{R}^2 \smallsetminus \mathcal{C}) \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases} \right)$$

$$0 \to \mathbb{Z} \longrightarrow \mathrm{Mod}(\mathbb{D}^2 \smallsetminus \mathcal{C}) \longrightarrow \mathrm{Mod}(\mathbb{R}^2 \smallsetminus \mathcal{C}) \to 1$$

$\rightsquigarrow$ Enough to prove $\mathrm{Mod}(\mathbb{D}^2 \smallsetminus \mathcal{C})$ is acyclic.

Note that $\left(\coprod_{\mathbb{N}} \mathcal{C}\right)^+ \cong \mathcal{C}$.



Simpler version of the 2-dim one used above. Inspired by [Mather].

$\rightsquigarrow$ Can use 1-dim "infinite iteration trick".

Key input needed is that $\mathrm{Mod}(\boxed{\sim}) \longrightarrow \mathrm{Mod}(\boxed{\sim|\sim})$ is a H_*-isomorphism.

Thm: $\mathrm{Mod}\left(\boxed{\sim}\right) \longrightarrow \mathrm{Mod}\left(\boxed{\sim}\boxed{\sim}\right) \longrightarrow \mathrm{Mod}\left(\boxed{\sim}\boxed{\sim}\boxed{\sim}\right) \longrightarrow \cdots$

is homologically stable.

↳ Use strategy of proof following [Szymik-Wahl].

Note that $\mathcal{C} \amalg \mathcal{C} \cong \mathcal{C}$.

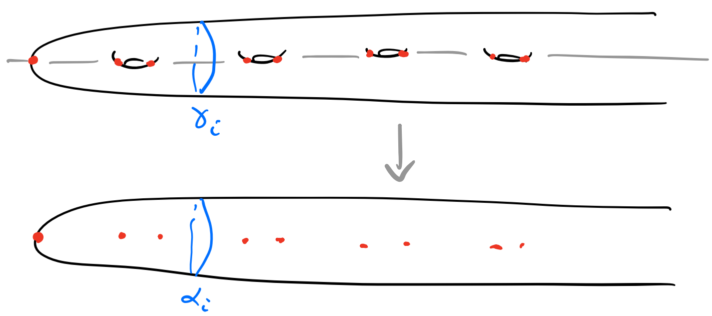
$$\boxed{\sim}\boxed{\sim} \cong \boxed{\sim}$$

Hence hom. stability $\Rightarrow$ H_*-isomorphism.

BONUS': Addendum to Thm Ⓐ

$$H_* \mathrm{Mod}(\mathbb{R}^2 \setminus \mathbb{N}) \supseteq \Lambda^*_{\mathbb{Z}}\Big(\bigoplus_c \mathbb{Z}\Big)$$

Adapt [Malestein–Tao '21]:



$$\begin{array}{c} L \setminus \mathbb{N} \\ \downarrow \pi \\ \mathbb{R}^2 \setminus \mathbb{N} \end{array}$$

Lemma Every based homeo. of $\mathbb{R}^2 \setminus \mathbb{N}$ preserves $\pi_1(L \setminus \mathbb{N}) \trianglelefteq \pi_1(\mathbb{R}^2 \setminus \mathbb{N})$ and hence lifts uniquely to a based homeo. of $L \setminus \mathbb{N}$.

$$\begin{tikzcd}
(T_{\alpha_i})^2 \in \arrow[ddd, mapsto] & \mathrm{Mod}_*(\mathbb{R}^2 \setminus \mathbb{N}) \arrow[d, "\text{lift}"] & \\
& \mathrm{Mod}_*(L \setminus \mathbb{N}) \arrow[d, "\text{forget basept}"] & \\
& \mathrm{Mod}(L \setminus \mathbb{N}) \arrow[d, "\text{fill in punctures}"] & \\
T_{\gamma_i} \in & \mathrm{Mod}(L) \arrow[dr] & \\
\bigoplus_c \mathbb{Z} \arrow[ur] \arrow[uuuur, bend left] \arrow[rr, hook] & & \bigoplus_c \mathbb{Q}
\end{tikzcd}$$

Then pass to $\mathrm{Mod}(\mathbb{R}^2 \setminus \mathbb{N})$ via

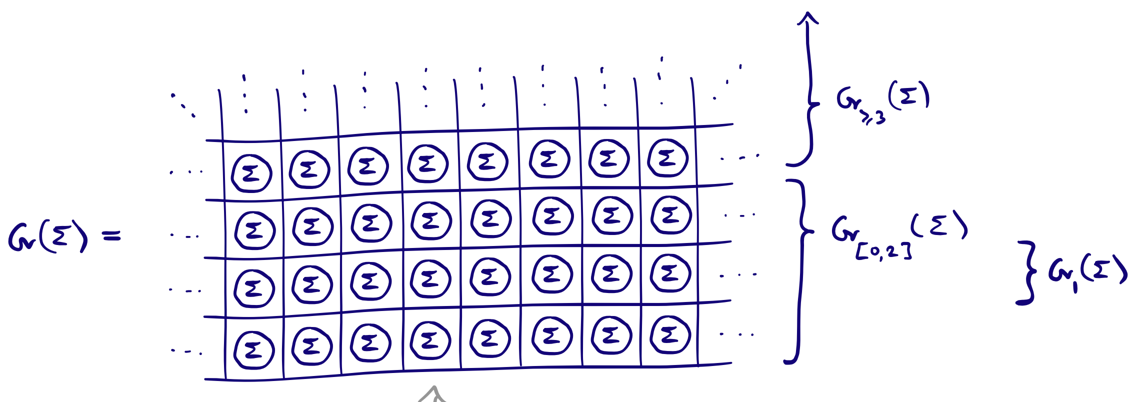
$$1 \to \underbrace{\pi_1(\mathbb{R}^2 \setminus \mathbb{N})}_{\substack{\cong \\ F_\infty \\ \text{countable}}} \to \mathrm{Mod}_*(\mathbb{R}^2 \setminus \mathbb{N}) \to \mathrm{Mod}(\mathbb{R}^2 \setminus \mathbb{N}) \to 1$$

BONUS^2: 2-dim. ∞ iteration trick (from proof of Thm Ⓑ)

$\mathrm{genus}(S) = \infty$

$\Sigma_{g,1} \cong \Sigma \hookrightarrow S$ compact subsurface.

Definition Grid surface:



denote the copies of Σ by $\Sigma_{i,j}$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$

Since $\mathrm{genus}(S) = \infty$, the inclusion $\Sigma \hookrightarrow S$ extends to a proper embedding $Gr(\Sigma) \hookrightarrow S$.

Hence we have a factorisation of $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ into

$$\mathrm{Mod}(\Sigma) \xrightarrow{(*)} \mathrm{Mod}(Gr(\Sigma)) \longrightarrow \mathrm{Mod}(S)$$

The proof reduces to:

Prop. The map (*) induces the zero map on $\tilde{H}_j$ for all $j \geq 0$.

Proof — by induction on j.

$\mathcal{H}(n,j)$: The map $\mathrm{Mod}(\Sigma) \xrightarrow{i_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$

$(n \geq 0,\ j \geq 0)$ $\qquad [\varphi] \longmapsto \begin{bmatrix} \varphi & \text{on } \Sigma_{0,n} \\ id & \text{elsewhere} \end{bmatrix}$

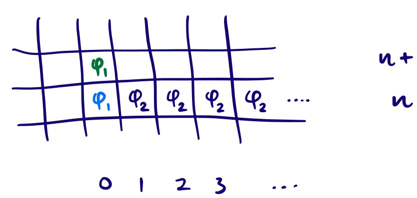
induces the zero map on $\tilde{H}_j$.

$\underline{j=0}$ ✓

$\underline{j \geq 1}$:

$$\mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \xrightarrow{\eta_n \ \ \nu_n} \mathrm{Mod}(Gr_{\geq n}(\Sigma))$$

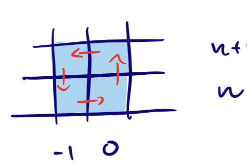
$$(\varphi_1, \varphi_2) \longmapsto$$



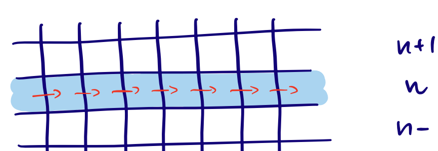
$$\psi_{0,n}(\varphi) := \eta_n(\varphi, \varphi)$$

$$\psi_{1,n}(\varphi) := \eta_n(id, \varphi)$$

Obs : $\qquad \eta_n \sim \nu_n$



$$\psi_{0,n} \sim \psi_{1,n}$$



$$\begin{tikzcd}
 & & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
\mathrm{Mod}(\Sigma) \arrow[urr, bend left, "\psi_{1,n}"] \arrow[r, "\Delta"] & \mathrm{Mod}(\Sigma) \times \mathrm{Mod}(\Sigma) \arrow[r, "\eta_n"] \arrow[d, "i_{n+1} \times \psi_{1,n}"] \arrow[dr, "\nu_n"] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)) \arrow[d, leftrightarrow, "\text{conj.}"] \\
 & \mathrm{Mod}(\mathrm{Gr}_{\geq n+1}(\Sigma)) \times \mathrm{Mod}(\mathrm{Gr}_n(\Sigma)) \arrow[r, "\text{glue}"'] & \mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma))
\end{tikzcd}$$

$\alpha \in H_j(\mathrm{Mod}(\Sigma))$

[Künneth]

$$\Delta_*(\alpha) \in H_j(\mathrm{Mod}(\Sigma)) \cong \bigoplus_{k+\ell=j} H_k(\mathrm{Mod}(\Sigma)) \otimes H_\ell(\mathrm{Mod}(\Sigma))$$

$$\| $$

$$\alpha \otimes 1 + \underbrace{\cdots\cdots}_{} + 1 \otimes \alpha$$

vanish under the map $(i_{n+1})_* \otimes (\psi_{1,n})_*$ by induction.

Image of α in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$ is $(i_{n+1})_*(\alpha) + (\psi_{1,n})_*(\alpha)$

But its image is also $(\psi_{1,n})_*(\alpha)$.

Hence $(i_{n+1})_*(\alpha) = 0$ in $H_j(\mathrm{Mod}(\mathrm{Gr}_{\geq n}(\Sigma)))$.

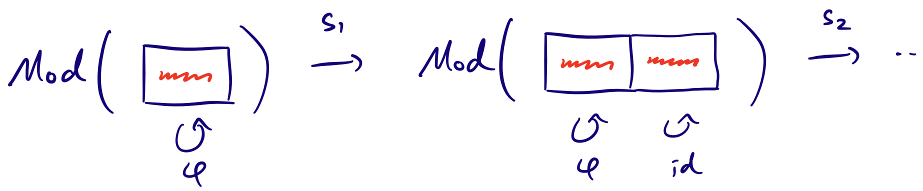
$$\| $$

$(i_n)_*(\alpha)$ since $i_n \sim i_{n+1}$

$\square$

BONUS^3: More details on the homological stability proof.

(for Thm ©)



Idea of proof that this is homologically stable:

Apply the machine of [RWW] to:

$\mathcal{M}_2^\infty$ = groupoid of conn., orientable surfaces S with $I \hookrightarrow \partial S$ & homeomorphisms

braided monoidal under $\natural$

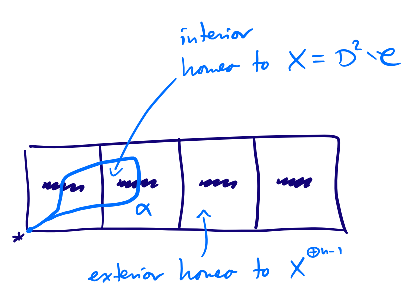
fails cancellation $\longrightarrow$ fix formally by passing to another category with objects = words on the alphabet of surfaces

Set $A = D^2$
$X = D^2 \setminus e$

Machine of [RWW] + tricks $\Rightarrow$ enough to prove high-connectivity of:

$TC_n(A, X)$

vertex:



$\{\alpha_0, \dots, \alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X^{\oplus n-p-1}$

(this is $(n-1)$-dim.)

We can also define:

<u>$TC_\infty(A,X)$</u>

same vertices

$\{\alpha_0,\dots,\alpha_p\}$ simplex if
- disjoint except at $*$
- exterior $\cong X$

This is ∞-dim,
but has the <u>same</u> $(n-2)$-skeleton as $TC_n(A,X)$.

(only clopen subsets of $\mathcal{C}$ are $\emptyset$ and $\mathcal{C}$)

<u>Proposition</u> $TC_\infty(A,X)$ is <u>contractible</u>.

Follows ideas from the proof of [Szymik–Wahl], who proved that $\widetilde{H}_*(V)=0$ via hom. stab. for $V\to V\to V\to\cdots$.

Thompson group

BONUS[4]: Idea of proof of 2^{nd} part of Thm (B)

if $g(S) = \infty$ and $\# P(S) \in [1, \infty)$

then image $\left(H_* \mathrm{Map}_c(S) \to H_* \mathrm{Map}(S) \right)$

contains $H_*(\mathbb{CP}^\infty)$.

Def [Bödigheimer–Tillmann in fin. type case]

$$
\begin{aligned}
\mathrm{Mod}(S) &\cong \mathrm{Mod}(\hat{S}, P(S)) \qquad \hat{S} = S \cup P(S) \\
&= \pi_0 \mathrm{Homeo}^+(\hat{S}, P(S)) \\
&\simeq \mathrm{Homeo}^+(\hat{S}, P(S))
\end{aligned}
$$

- [Hamstrom '66] fin. type
  [Yagasaki '00] in general
  $\hookrightarrow$ contractible components
- $\mathrm{MCG}(S)$ is totally disconnected

$$\simeq \mathrm{Diff}^+(\hat{S}, P(S))$$

[Kirby–Siebenmann '77]

$$
\begin{gathered}
\mathrm{Diff}^+(\hat{S}, P(S)) \\
\downarrow \\
S^1 \wr \mathfrak{S}_p
\end{gathered}
$$

permutation of $P(S)$
rotations of tangent planes to $\hat{S}$ at $P(S)$
(fix framing on $P(S)$)

$$B\,\mathrm{Mod}(S) \longrightarrow B(S^1 \wr \mathfrak{S}_p)$$

Obs Restricting to $B\,\mathrm{Mod}_c(S)$:

$$
\begin{tikzcd}
B\,\mathrm{Mod}_c(S) \arrow[r, "(*)"'] \arrow[d] & (BS^1)^p \arrow[d] \\
B\,\mathrm{Mod}(S) \arrow[r] & B(S^1 \wr \mathfrak{S}_p)
\end{tikzcd}
$$

Thm [Bödigheimer – Tillmann '01 (+ Madsen – Weiss '07)]

$B\,\mathrm{Mod}_c(S)$ splits as $\Omega^\infty_0 MTSO(2) \times (BS^1)^p$ on H_* / + construction

(*) is the projection onto the second factor

Coro (*) has a section on H_*

$$\begin{tikzcd}
H_*(BS^1)^p \arrow[r] \arrow[rr, bend left, "id"] & H_*\,\mathrm{Mod}_c(S) \arrow[r, "(*)"'] \arrow[d, "(*)"] & H_*(BS^1)^p \arrow[d] \\
 & H_*\,\mathrm{Mod}(S) \arrow[r] & H_* B(S^1 \wr \mathfrak{S}_p) \arrow[d, "{\begin{array}{c} S^1 \wr \mathfrak{S}_p \to S^1 \\ (t_1,\dots,t_p,\sigma) \mapsto \sum_i t_i \end{array}}"] \\
H_*(BS^1) \arrow[uu, "{\begin{array}{c} S^1 \to (S^1)^p \\ t \mapsto (t,1,\dots,1) \end{array}}"] \arrow[rr, "id"'] & & H_*(BS^1)
\end{tikzcd}$$

$\Rightarrow$ image (*) $\supseteq_{\oplus}$ $H_*(BS^1)$.