Homological stability for configuration spaces on closed manifolds

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Series abstract.

A classical result in algebraic topology — due to Arnol'd, McDuff and Segal in the 1970s — says that unordered configuration spaces on connected, non-compact manifolds are homologically stable. Here, the nth *unordered configuration space* $C_n(M)$ on a manifold M is the space of finite subsets $c \subset M$ of size n, topologised as a subquotient of the product M^n, and *homologically stable* means that $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ when n is sufficiently large as a function of i.

If M is instead a connected, *closed* manifold then its configuration spaces are generally *not* homologically stable: considering spherical braid groups one may check that it fails already in homological degree 1 for $M = S^2$. However, a number of more subtle stability or periodicity patterns have been discovered in the homology of configuration spaces on closed manifolds, depending in particular on the characteristic of the coefficient ring R that we use for homology. For example, in an appropriate range $n \gg i$, the homology $H_i(C_n(M); R)$ is:

- stable if $R = \mathbb{Q}$ or $R = \mathbb{F}_2$,
- stable if M is odd-dimensional and $R = \mathbb{Z}$,
- p-periodic if M is even-dimensional and R is a field of odd prime characteristic p.

A variety of different techniques have been used to prove these results, including *representation stability*, *scanning maps*, *replication maps*, *homology operations*, *factorisation homology* and *semi-simplicial resolutions*. The goal of this series of talks is to explain these different approaches.

Talks.

1. **10 April 2025** — We will follow the article
   O. Randal-Williams, *Homological stability for unordered configuration spaces*,
   Quart. J. Math. 64 (2013), pp. 303–326
   and prove homological stability with coefficients in $\mathbb{Q}$ or $\mathbb{F}_2$ using semi-simplicial resolutions and transfer maps.

2. **28 April 2025** — We will follow the article
   M. Bendersky, J. Miller, *Localization and homological stability of configuration spaces*,
   Quart. J. Math. 65 (2014), pp. 807–815
   and prove rational homological stability using scanning maps, localisations and rational homotopy theory.

3,4. **TBC** — We will follow the articles
   F. Cantero, M. Palmer, *On homological stability for configuration spaces on closed background manifolds*, Doc. Math. 20 (2015) pp. 753–805
   and
   A. Kupers, J. Miller, *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*, Proc. Amer. Math. Soc. 144 (2016) pp. 5457–5468
   and prove homological stability with coefficients in $\mathbb{Z}$ when M is odd-dimensional and certain homological periodicity results when M is even-dimensional, using scanning maps, replication maps and homology operations for E_n-algebras.

5. **TBC** — We will follow the article
   T. Church, *Homological stability for configuration spaces of manifolds*,
   Invent. Math. 188 (2012) pp. 465–504
   and prove rational homological stability using the concept of *representation stability* invented by T. Church and B. Farb.

Updated 23 April 2025.

6. **TBC** — We will follow the article
   B. Knudsen, *Betti numbers and stability for configuration spaces via factorization homology*, Algebr. Geom. Topol. 17 (2017) pp. 3137–3187
   and prove rational homological stability using factorisation homology and Lie algebra models. We will also see some explicit calculations of rational homology that this method affords.

7. **TBC** — We will follow the article
   O. Randal-Williams, *Configuration spaces as commutative monoids*, Bull. Lond. Math. Soc. 56 (2024) pp. 2847–2862
   and prove rational homological stability by considering a commutative monoid structure on one-point compactifications of configuration spaces.