# Vanishing and uncountability in the homology of MCGs of infinite-type surfaces

Joint work with
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$S$ — surface of infinite type

$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S))$ id on $\partial S$

Infinite type $\overset{\text{def.}}{\iff}$ $S \not\cong$ compact surface minus some $\partial$-components

$\iff \pi_1(S)$ not finitely generated

$\iff \mathrm{Mod}(S)$ is uncountable

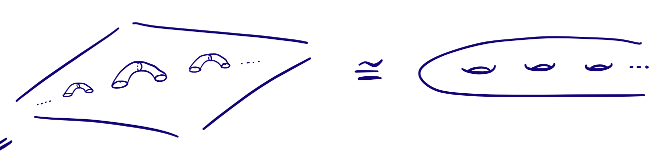
## Examples

(1) $\Sigma$ finite type surface $\rightsquigarrow$ $\Sigma \setminus \mathcal{C}$ — Cantor set

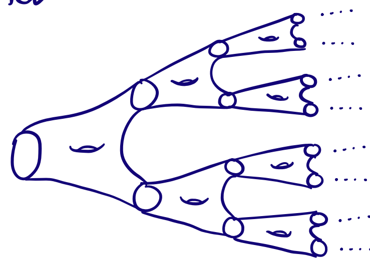
— Related to dynamical systems on $\Sigma$.

— $\mathbb{R}^2 \setminus \mathcal{C}$ $\rightsquigarrow$ key motivation for D. Calegari when he popularized "big MCGs" in 2009.

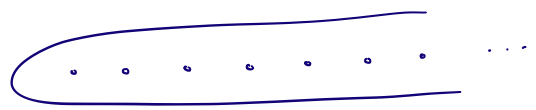
(2) Loch Ness monster surface :



(3) Cantor's Loch Ness monster

$L_{\mathcal{C}}$ = 

(4) Flute surface

$F$ = 

[Kerékjártó, Richards]

In general connected, orientable surfaces with $\partial = \emptyset$ are classified by

- genus$(S)$ $\qquad \mathbb{N} \cup \{\infty\}$
- $(E, E_{np})$ $\qquad$ (space that is $\cong$ closed subset of $\mathcal{C}$, closed subspace)

  $E_{np} \neq \emptyset \iff \text{genus}(S) = \infty$

$\bar{S}$ = Freudenthal compactification of $S$

$E_{np}$ = space of non-planar ends of $S$ = non-manifold points of $\bar{S} \setminus S$

Examples

| $S$ | genus$(S)$ | $E$ | $E_{np}$ |
|---|---|---|---|
| $\mathbb{R}^2 \setminus \mathcal{C}$ | $0$ | $\mathcal{C} \sqcup *$ | $\emptyset$ |
| $L$ | $\infty$ | $*$ | $*$ |
| $L_{\mathcal{C}} \cup D^2$ | $\infty$ | $\mathcal{C}$ | $\mathcal{C}$ |
| $F$ | $0$ | $(\mathbb{N})^+$ | $\emptyset$ |

## Vanishing

**Thm A** (P.–Wu '22)

$$\forall i \geq 1, \quad H_i(\mathrm{Mod}(D^2 \setminus \mathcal{C}); \mathbb{Z}) = 0 = H_i(\mathrm{Mod}(L_{\mathcal{C}}); \mathbb{Z})$$

$\hookrightarrow$ Coro $\quad H^*(\mathrm{Mod}(\mathbb{R}^2 \setminus \mathcal{C}); \mathbb{Z}) \cong \mathbb{Z}[x] \qquad \deg(x) = 2$

**Thm B** (P.–Wu '24)

If $\mathrm{genus}(S) = \infty$ and $\Sigma \subset S$ is a compact subsurface, then $\forall i \geq 1$, $H_i(\mathrm{Mod}(\Sigma); \mathbb{F}) \xrightarrow[0]{} H_i(\mathrm{Mod}(S); \mathbb{F})$ for any field $\mathbb{F}$.

$\hookrightarrow$ Coro

$$\underset{\Sigma \text{ cpt}}{\mathrm{colim}} \left( H_*(\mathrm{Mod}(\Sigma); \mathbb{Q}) \right) \xrightarrow{0} H_*(\mathrm{Mod}(S); \mathbb{Q})$$

$$\cong \quad \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]^*$$

Mumford conjecture [Madsen–Weiss '02]

Miller–Morita–Mumford classes

Can this be non-zero?

**Rmks** (a) With $\mathbb{Z}$ coeffs instead of $\mathbb{F}$ $\longrightarrow$ open.

$\left( \mathbb{Z} \xrightarrow{1/2} \mathbb{Q}/\mathbb{Z} \right)$

(b) We also partially answer the question for $\mathrm{genus}(S) < \infty$, which is more complicated.

## Uncountability

**Thm C** (P.-Wu '22)

(1) $\forall i \geq 1 \quad H_i(\mathrm{Mod}(L); \mathbb{Z}) \supseteq \bigoplus_{|\mathbb{R}|} \mathbb{Z}$

(2) Same for $\mathrm{Mod}(F)$.

Rmks (a) Also true for coeffs in any field $\mathbb{F}$ of characteristic zero, and more generally any abelian group $A$ such that $A \to A \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective.

(b) This generalises [Domat '20] who studied $H_1$.

(c) There is no known torsion ...

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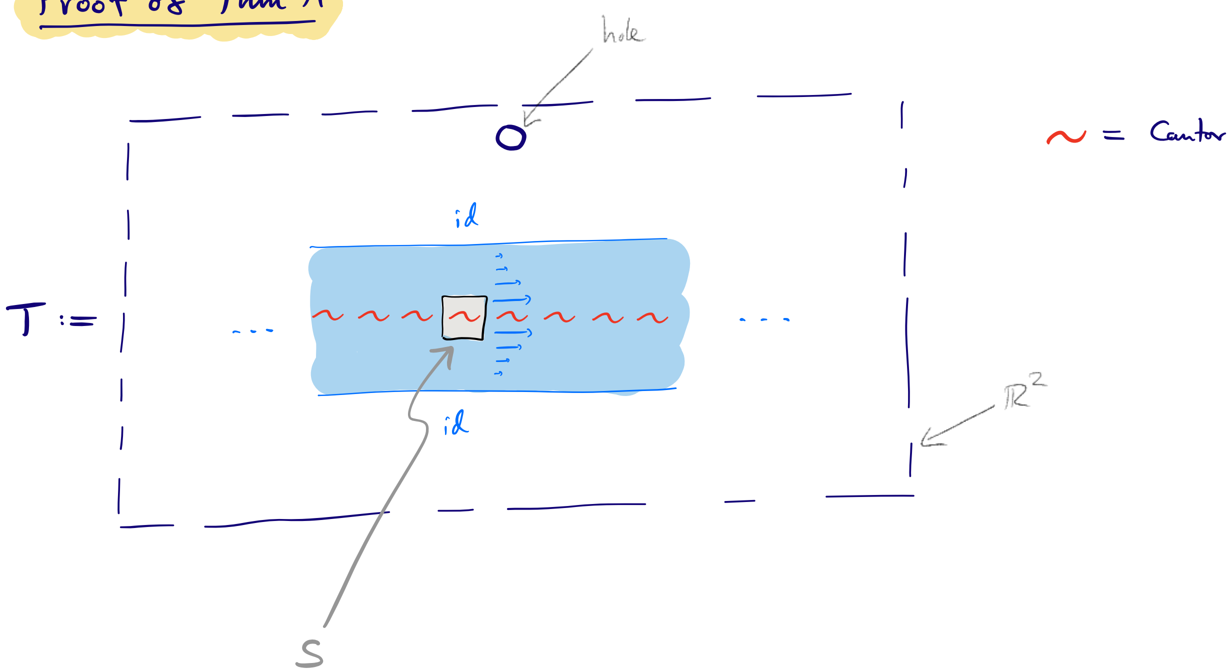
### Exercise for the audience

(Will be needed in 20-30 minutes...)

$\exists$ a continuum collection $\{ X_a \subseteq \mathbb{N} \mid a \in \mathbb{R} \}$ so that

$$\begin{cases} \cdot \text{ all } X_a \text{ are infinite} \\ \cdot \text{ all } X_a \cap X_b \text{ are finite} \end{cases}$$

## Proof of Thm A




$S \cong D^2 \setminus \mathcal{C}$

$T$ : genus $= 0$
space of ends $= \left(\coprod_{\mathbb{N}} \mathcal{C}\right)^+ \cong \mathcal{C}$.
$\partial = S^1$

$\Rightarrow \quad T \cong D^2 \setminus \mathcal{C}$.

$\varphi \in \mathrm{Homeo}_\partial(T)$

- Iterated images of $S$ are pairwise disjoint
- Their union is closed

Prop. — adapted from [Mather '71]
— related to dissipated (binate) groups

extend by id on $T \setminus S$

If $\mathrm{Map}(S) \longrightarrow \mathrm{Map}(T)$ is a $H_*$-isomorphism

then $H_i(\mathrm{Map}(S)) = H_i(\mathrm{Map}(T)) = 0$ for all $i \geq 1$.

**Obs**

$$\begin{array}{ccc} S & \hookrightarrow & T \\ \cong & & \cong \\ \boxed{\sim} & \xhookrightarrow{i_1} & \boxed{\sim}\boxed{\sim} \\ \cong & & \cong \\ \boxed{\sim}\boxed{\sim} & \xhookrightarrow{i_2} & \boxed{\sim}\boxed{\sim}\boxed{\sim} \\ \vdots & & \vdots \end{array}$$

**Thm** (P.-Wu)

$$\mathrm{Mod}\left(\boxed{\sim}\right) \xrightarrow{(i_1)_*} \mathrm{Mod}\left(\boxed{\sim}\boxed{\sim}\right) \xrightarrow{(i_2)_*} \mathrm{Mod}\left(\boxed{\sim}\boxed{\sim}\boxed{\sim}\right) \longrightarrow \cdots$$

is homologically stable : $(i_n)_*$ induces $\cong$ on $H_k(-)$ if $k \leq n/2$.

$\hookrightarrow$ Coro $(i_1)_*$ is a $H_*$-isomorphism (in all degrees).

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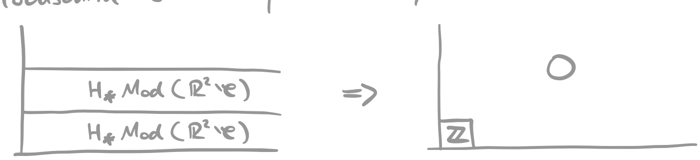
Notes:

- Proof of hom. stab.^y uses techniques of [Szymik-Wahl '17], who proved the long-standing conjecture that Thompson's group $V$ is acyclic.

- Central extension $0 \to \mathbb{Z} \to \mathrm{Mod}(D^2 \setminus e) \to \mathrm{Mod}(\mathbb{R}^2 \setminus e) \to 1$
  Lyndon-Hochschild-Serre spectral sequence

$E^2 =$ 

$= E^\infty$

## Proof of Thm B

**Step 1**

<u>Def.</u> $T$ surface with $\partial T \cong S^1$

$$\mathrm{Grid}(T) := \begin{array}{c|c|c|c|c|c} & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & T & T & T & T & \cdots \\ \hline \cdots & T & T & T & T & \cdots \\ \hline \cdots & T & T & T & T & \cdots \\ \hline \end{array}$$

<u>Proposition</u>

For any such $T$ and any field $\mathbb{F}$,

$$H_i(\mathrm{Mod}(T);\mathbb{F}) \longrightarrow H_i(\mathrm{Mod}(\mathrm{Grid}(T));\mathbb{F})$$

is $0$ for all $i \geq 1$.

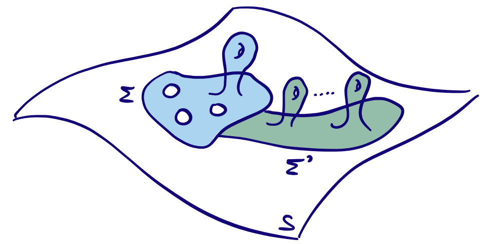
<u>Rmks</u> (a) The proof uses a similar dynamical trick to the proof of the Proposition in the proof of Thm A above — but using two dimensions instead of just one.

(b) [Varadarajan, Berrick] prove that pseudo-mitotic (binate) groups are acyclic via a one-dim. infinite iteration trick, as in the proof of Thm A.

(c) Why coeffs in a field $\mathbb{F}$: we need a <u>natural</u> splitting in the Künneth SES.

## $\boxed{\text{Step 2}}$

(i) For any $g \geq 0$, find $\Sigma' \hookrightarrow S$ with $\Sigma' \cap \Sigma = $ interval
$\Sigma' \cong \Sigma_{g,1}$




(ii) Extend

$$\begin{tikzcd} \Sigma' \arrow[r, hook] \arrow[d, phantom, "\cap"] & S \\ \mathrm{Grid}(\Sigma') \arrow[ur] & \end{tikzcd}$$

proper embedding

(iii) For any $i \geq 1$, choose any $g \geq \frac{3}{2} i$.   $H_i(-) = H_i(-;\mathbb{F})$

$$\begin{tikzcd} H_i\,\mathrm{Mod}(\Sigma) \arrow[d] \arrow[drr] & & \\ H_i\,\mathrm{Mod}(\Sigma \natural \Sigma') \arrow[rr] & & H_i\,\mathrm{Mod}(S) \\ H_i\,\mathrm{Mod}(\Sigma') \arrow[u, two heads] \arrow[urr] \arrow[rr, "0"'] & & H_i\,\mathrm{Mod}(\mathrm{Grid}(\Sigma')) \arrow[u] \end{tikzcd}$$

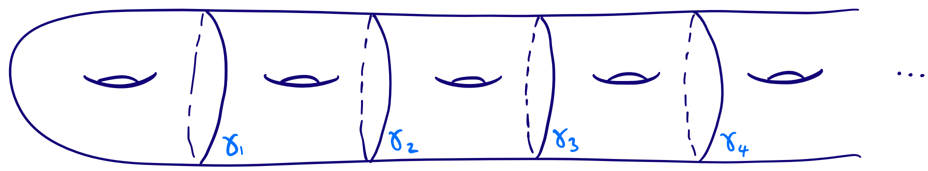
surjective
$(\mathrm{genus}(\Sigma') = g \geq \frac{3}{2} i)$
[Harer '85, .....]

0 by Step 1

$\mathrm{Mod}(\Sigma_{g,b}) \longrightarrow \mathrm{Mod}(\Sigma_{g+g', b+b'})$ induces $\cong$ on $H_i(-)$ if $g \gg i$.

Important point: we have stability w.r.t. $g$ and $b$ as long as $g$ is sufficiently large (with no condition on $b$).

## Proof of Thm C (1)

$L =$ 



$$H_i(\mathrm{Mod}(L);\mathbb{Z}) \supseteq \bigoplus_{|\mathbb{R}|} \mathbb{Z} \quad \text{for } i \geq 1$$

$H_i(-) = H_i(-;\mathbb{Z})$

**Idea**: Factor $\bigoplus_{\mathbb{R}} \mathbb{Z} \hookrightarrow \bigoplus_{\mathbb{R}} \mathbb{Q}$ through $\mathrm{Mod}(L)$.

**Consequence**:

$$\begin{array}{ccccc}
H_i(\bigoplus_{\mathbb{R}} \mathbb{Z}) & \longrightarrow & H_i(\mathrm{Mod}(L)) & \longrightarrow & H_i(\bigoplus_{\mathbb{R}} \mathbb{Q}) \\
\cong & & & & \cong \\
\Lambda^i(\bigoplus_{\mathbb{R}} \mathbb{Z}) & & \xhookrightarrow[\text{injective}]{\qquad\qquad} & & \Lambda^i(\bigoplus_{\mathbb{R}} \mathbb{Q}) \\
\cong & & & & \\
\bigoplus_{\mathbb{R}} \mathbb{Z} & & & &
\end{array}$$

**Construction**:

$$X \subseteq \mathbb{N} \longrightarrow \begin{cases} f(X) := \prod_{i \in X} (T_{\gamma_i})^{i!} \\ f_n(X) := \prod_{\substack{i \in X \\ i \geq n}} (T_{\gamma_i})^{i!/n} \end{cases}$$

Exercise from earlier:

$\exists$ a continuum collection $\{X_a \subseteq \mathbb{N} \mid a \in \mathbb{R}\}$ so that

$$(*) \begin{cases} \cdot \text{ all } X_a \text{ are infinite} \\ \cdot \text{ all } X_a \cap X_b \text{ are finite} \end{cases}$$

$$\begin{tikzcd} \bigoplus_{a\in\mathbb{R}} \mathbb{Z} \arrow[r, "1_a \mapsto f(x_a)"] \arrow[d, hook] & \mathrm{Mod}(L) \arrow[d, two heads] \\ \bigoplus_{a\in\mathbb{R}} \mathbb{Q} \arrow[r, "\theta"'] & \mathrm{Mod}(L)^{ab} \end{tikzcd}$$

$$\left(\tfrac{1}{n}\right)_a \mapsto [f_n(x_a)]$$

__Check:__

- $f(x_a) - f_n(x_a)^n$ cpt support
- [Birman, Powell '70s] $\Rightarrow$

$$[f(x_a) - f_n(x_a)^n] = 0$$

$(*)$ + [Domat '20] $\leftarrow$ [Bestvina - Bromberg - Fujiwara '15]

$\Rightarrow$ $\theta$ is injective

But $\bigoplus_{\mathbb{R}} \mathbb{Q}$ is an injective $\mathbb{Z}$-module

$\Rightarrow$ every injection $\bigoplus_{\mathbb{R}} \mathbb{Q} \hookrightarrow A$ admits a retraction

$$\Rightarrow \quad \begin{tikzcd} \bigoplus_{a\in\mathbb{R}} \mathbb{Z} \arrow[r] \arrow[d, hook] & \mathrm{Mod}(L) \arrow[d, two heads] \\ \bigoplus_{a\in\mathbb{R}} \mathbb{Q} \arrow[r, "\theta"] & \mathrm{Mod}(L)^{ab} \arrow[l, bend left, dashed] \end{tikzcd} \qquad \square$$

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__Rmk__

$$\begin{array}{ccc} H^i\left(\bigoplus_c \mathbb{Z}\right) & \xleftarrow{\;0\;} & H^i\left(\bigoplus_c \mathbb{Q}\right) \\ \cong & & \cong \\ \prod_c \mathbb{Z} & & \begin{cases} 0 & i=1 \\ \bigoplus_{2^c} \mathbb{Q} & i\geq 2 \end{cases} \end{array}$$

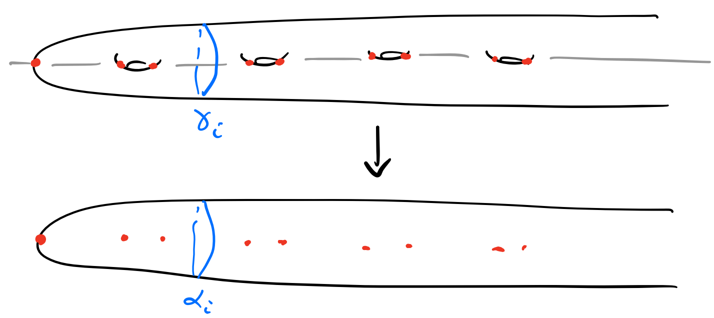
BONUS (if time)

## Proof of Thm C (2)

$$H_i(\mathrm{Mod}(F);\mathbb{Z}) \supseteq \bigoplus_{|\mathbb{R}|} \mathbb{Z} \quad \text{for } i \geq 1$$

flute surface

**Idea** (adapted from [Malestein–Tao '21]):




$$L \setminus N \overset{\pi}{\twoheadrightarrow} F$$

**Lemma** Every based homeo. of $F$ preserves $\pi_1(L \setminus N) \overset{(2)}{\triangleleft} \pi_1(F)$ and hence lifts uniquely to a based homeo. of $L \setminus N$.

$$\begin{tikzcd}
 & T_{\alpha_i}^2 \in \mathrm{Mod}_*(F) \arrow[d, "\text{lift}"] & \\
 & \mathrm{Mod}_*(L \setminus N) \arrow[d, "\text{forget basepoint}"] & \\
 & \mathrm{Mod}(L \setminus N) \arrow[d, "\text{fill in punctures}"] & \\
 & T_{\gamma_i} \in \mathrm{Mod}(L) \arrow[dr] & \\
\bigoplus_{\mathbb{R}} \mathbb{Z} \arrow[uuuur, bend left, "1_a \mapsto \hat{f}(x_a)"] \arrow[ur, "1_a \mapsto f(x_a)"'] \arrow[rr] & & \bigoplus_{\mathbb{R}} \mathbb{Q}
\end{tikzcd}$$

$$f(X) := \prod_{i \in X} (T_{\gamma_i})^{i!}$$

$$\hat{f}(X) := \prod_{i \in X} (T_{\alpha_i}^2)^{i!}$$

Then pass to $\mathrm{Mod}(F)$ via the Birman exact sequence:

$$\begin{tikzcd}
 & K = \pi_1(F) \cong F_\infty \arrow[d, hook] & \\
\bigoplus_{\mathbb{R}} \mathbb{Z} \arrow[r] & \mathrm{Mod}_*(F) \arrow[r, "\varphi"] \arrow[d, two heads] & \bigoplus_{\mathbb{R}} \mathbb{Q} \\
 & \mathrm{Mod}(F) &
\end{tikzcd}$$

countable

$I \subseteq \mathbb{R}$

$I :=$ summands of $\bigoplus_{\mathbb{R}} \mathbb{Q}$ where some element of $\varphi(K)$ has non-zero coordinate.

countable

$\Rightarrow |\mathbb{R} \setminus I| = |\mathbb{R}|$.

$$\begin{tikzcd}
 & & K \arrow[d, hook] \arrow[rrr, "0"] & & & \bigoplus_{\mathbb{R}\setminus I} \mathbb{Q} \\
\bigoplus_{\mathbb{R}\setminus I} \mathbb{Z} \arrow[r, hook] \arrow[rrrrr, bend right=40, "\bigoplus_{\mathbb{R}\setminus I} (\mathbb{Z} \xhookrightarrow{\text{inc.}} \mathbb{Q})"'] & \bigoplus_{\mathbb{R}} \mathbb{Z} \arrow[r] & \mathrm{Mod}_*(F) \arrow[r, "\varphi"] \arrow[d, two heads] & \bigoplus_{\mathbb{R}} \mathbb{Q} \arrow[rr, two heads] & & \bigoplus_{\mathbb{R}\setminus I} \mathbb{Q} \\
 & & \mathrm{Mod}(F) \arrow[rrru] & & &
\end{tikzcd}$$

$\square$