Homological stability for configuration spaces on closed manifolds I

GeMAT seminar
IMAR
10 April 2025

M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M
topologised as a subquotient of M^n

Theorem [Arnol'd, McDuff, Segal, 70's]

"M is an open manifold"

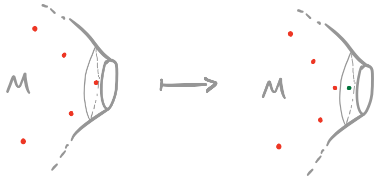
If $M = \operatorname{int}(\bar{M})$ where $\partial\bar{M} \neq \emptyset$

then $\exists$ maps $C_n(M) \xrightarrow{s} C_{n+1}(M)$

inducing $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.

$C_n(M) \xrightarrow{s} C_{n+1}(M)$

- Push the configuration away from a collar neighbourhood of $\partial\bar{M}$.
- Add a new point to the configuration in this collar.



Q: What about when M is closed (i.e. compact)?

$\hookrightarrow$ There are no stabilisation maps.

A: Homological stability is false!

E.g. $\pi_1(C_n(S^2)) = B_n(S^2)$

[Fadell - van Buskirk '62] ⟶ presentation

$\Rightarrow \quad H_1(C_n(S^2)) \cong \mathbb{Z}/(2n-2) \quad$ for $n \geq 2$

More generally, [Cantero-P. '15]

$$H_{2d-1}(C_n(S^{2d})) \cong \underset{\substack{\uparrow \\ \text{depending} \\ \text{only on } d}}{\boxed{\text{▨}}} \oplus \mathbb{Z}/(2n-2) \qquad \text{for } n \geq 4d-2$$

BUT there are some more delicate stable patterns:

(1) Stability holds with coeffs in $\mathbb{F}_2$ [ML'88] [BCT'89] [RW'13]

(2) Stability holds with coeffs in $\mathbb{Q}$ [C'12] [RW'13] [BM'14] [K'17] [RW'24]

(3) Stability holds if $\dim(M)$ is odd $\underbrace{\text{[BCT'89] [RW'13]}}_{\text{field coeff}}$ $\underbrace{\text{[CP'15]}}_{\frac{1}{2} \in \text{coeff}}$ $\underbrace{\text{[KM'16]}}_{\text{all coeffs}}$

(4) Eventual periodicity holds with coeffs in $\mathbb{F}_p$ [CP'15] [N'15] [KM'16]

[ML'88] = Milgram - Löffler
[BCT'89] = Bödigheimer - Cohen - Taylor
[C'12] = Church → Talk ④
[RW'13] = Randal - Williams → Talk ①
[BM'14] = Bendersky - Miller → Talk ②
[CP'15] = Cantero - Palmer → Talk ③
[N'15] = Nagpal
[KM'16] = Kupers - Miller → Talk ③
[K'17] = Knudsen → Talk ⑤
[RW'24] = Randal - Williams → Talk ⑥

Today:

Theorem [Randal-Williams '13]

M — any connected manifold
$\mathbb{F}$ — field

Suppose either (1) $\mathbb{F} = \mathbb{F}_2$
(2) $\mathbb{F} = \mathbb{Q}$
(3) $\dim(M)$ is odd

Then $\dim_{\mathbb{F}} H_i(C_n(M); \mathbb{F})$ is independent of n when $n \geq 2i$.

Proof

- If $C(f)$ is the mapping cone of $X \xrightarrow{f} Y$, then there is a long exact sequence

$$\cdots \longrightarrow H_i X \longrightarrow H_i Y \longrightarrow H_i C(f) \longrightarrow H_{i-1} X \longrightarrow H_{i-1} Y \longrightarrow \cdots$$

(Write $H_i(-) = H_i(-;\mathbb{F})$)

- **Lemma**: The mapping cone of $C_n(M \setminus \text{point}) \hookrightarrow C_n(M)$ is homology equivalent to $S^d \wedge C_{n-1}(M \setminus \text{point})_+$.

$d = \dim(M)$

- Hence we have a LES :

$$H_i C_n(M\setminus pt) \longrightarrow H_i C_n(M) \longrightarrow H_{i-d}(C_{n-1}(M\setminus pt)) \xrightarrow{\delta_n^i} H_{i-1} C_n(M\setminus pt) \longrightarrow H_{i-1} C_n(M)$$

Note: These do stabilise, since $M\setminus pt$ is open.

- We can extract a SES :

$$0 \to \operatorname{coker}(\delta_n^{i+1}) \longrightarrow H_i C_n(M) \longrightarrow \ker(\delta_n^i) \to 0$$

Hence $\dim H_i C_n(M) = \dim \ker(\delta_n^i) + \dim \operatorname{coker}(\delta_n^{i+1})$ (†)

- Now suppose that the following square commutes :

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1} C_n(M\setminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt)
\end{tikzcd}$$

(□)

Then it extends to a map of exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & \ker(\delta_n^i) \arrow[r] \arrow[d, "(*)"'] & H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1} C_n(M\setminus pt) \arrow[r] \arrow[d, "\text{stabilisation}"] & \operatorname{coker}(\delta_n^i) \arrow[r] \arrow[d, "(*)"] & 0 \\
0 \arrow[r] & \ker(\delta_{n+1}^i) \arrow[r] & H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt) \arrow[r] & \operatorname{coker}(\delta_{n+1}^i) \arrow[r] & 0
\end{tikzcd}$$

And then homological stability for $C_n(M \setminus pt)$ + 5-lemma

$\Rightarrow$ stability for $(*)$

$\Rightarrow$ stability for $\ker(\delta_n^i)$ & for $\operatorname{coker}(\delta_n^i)$.

$\underset{(\dagger)}{\Rightarrow}$ stability for $\dim H_i C_n(M)$.

Plan :

- Proof of the mapping cone lemma.
- Calculate the obstruction to commutativity of (□)

  ↳ $2[\mathbb{RP}^{d-1}] \in H_{d-1}(\mathbb{RP}^{d-1})$

  $\Rightarrow$ ☺ if $\mathbb{F} = \mathbb{F}_2$ or d is odd.
- Modify the strategy for $\mathbb{F} = \mathbb{Q}$ ....

Proof of the mapping cone lemma

Reminder: we want to prove that the mapping cone of the inclusion $C_n(M \setminus \text{point}) \hookrightarrow C_n(M)$ is homology equivalent to $S^d \wedge C_{n-1}(M \setminus \text{point})_+$.

homeomorphic to $\dfrac{D^d \times C_{n-1}(M \setminus \text{point})}{S^{d-1} \times C_{n-1}(M \setminus \text{point})}$

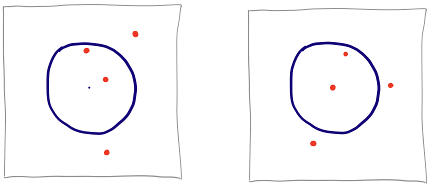
Choose $D^d \hookrightarrow M$

such that $0 \longmapsto$ the point that we remove.

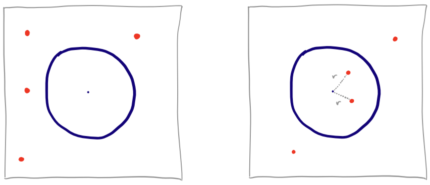
Let $U \subseteq C_n(M)$ be

$$U := \left\{ c = \{p_1, \dots, p_n\} \in C_n(M) \;\middle|\; \begin{array}{l} c \cap D^d \neq \emptyset \\ c \cap D^d \text{ has a } \underline{\text{unique}} \text{ closest} \\ \text{point to } 0 \in D^d \end{array} \right\}$$

Eg. ✓:



X:

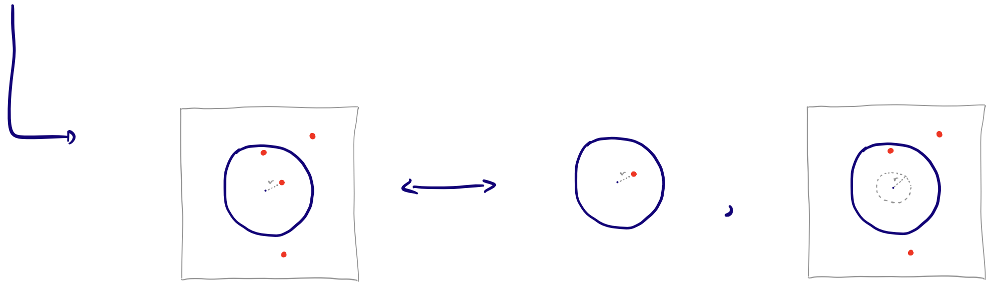


Observation: $C_n(M \setminus 0)$ and U form an <u>open cover</u> of $C_n(M)$.

Excision $\Rightarrow$

$$\begin{tikzcd} \text{mapping cone of } C_n(M\setminus 0) \hookrightarrow C_n(M) \arrow[d, leftrightarrow, "H_*\text{-equivalent}"] \\ \text{mapping cone of } C_n(M\setminus 0) \cap U \hookrightarrow U \end{tikzcd}$$

Claim: $\quad U \cong D^d \times C_{n-1}(M \setminus 0)$



$$M \setminus B_r(0) \cong M \setminus 0$$

Moreover, this homeomorphism restricts to

$$U \cap C_n(M \setminus 0) \cong (D^d \setminus 0) \times C_{n-1}(M \setminus 0)$$
$$\simeq S^{d-1} \times C_{n-1}(M \setminus 0)$$

Hence:

mapping cone of $C_n(M \setminus 0) \hookrightarrow C_n(M)$

$\updownarrow$ H_*-equivalent (excision)

mapping cone of $C_n(M \setminus 0) \cap U \hookrightarrow U$

$\updownarrow$ H_*-equivalent (identification above)

mapping cone of $S^{d-1} \times C_{n-1}(M \setminus 0) \hookrightarrow D^d \times C_{n-1}(M \setminus 0)$

closed cofibration (NDR)

$$\cong$$

$$\frac{D^d \times C_{n-1}(M \setminus 0)}{S^{d-1} \times C_{n-1}(M \setminus 0)} .$$

$\square$

Obstruction to commutativity of (□)

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M \smallsetminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] \arrow[dr, phantom, "(\square)"] & H_{i-1} C_n(M \smallsetminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}(C_n(M \smallsetminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M \smallsetminus pt)
\end{tikzcd}$$

This is induced by a certain square

$$\begin{tikzcd}
S^{d-1} \times C_{n-1}(M \smallsetminus pt) \arrow[r, "\Delta"] \arrow[d, "id \times s"'] & C_n(M \smallsetminus pt) \arrow[d, "s"] \\
S^{d-1} \times C_n(M \smallsetminus pt) \arrow[r, "\Delta"] & C_{n+1}(M \smallsetminus pt)
\end{tikzcd}$$

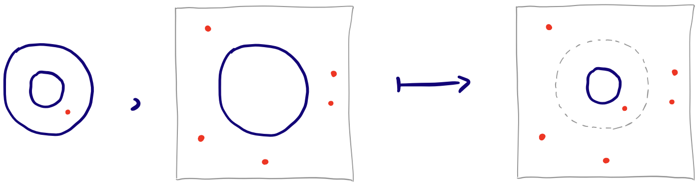
by taking $H_{i-1}(-)$ and restricting to one summand in the Künneth decomposition on the LHS.

To describe this, replace $C_n(M \smallsetminus pt) \simeq C_n(M \smallsetminus D^d)$

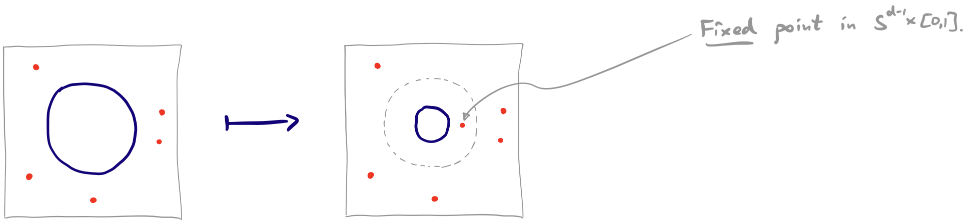
$S^{d-1} \simeq S^{d-1} \times [0,1]$

Then $\Delta : (S^{d-1} \times [0,1]) \times C_{n-1}(M \smallsetminus D^d) \longrightarrow C_n(M \smallsetminus D^d)$

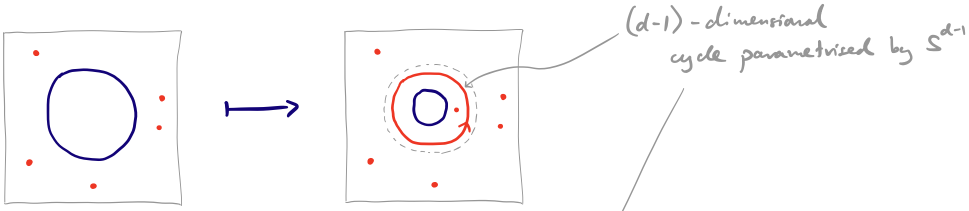
Part of an H-module structure on $\coprod_n C_n(M \smallsetminus D^d)$ over the H-space $\coprod_n C_n(S^{d-1} \times [0,1])$.



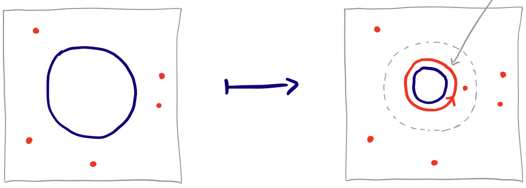
And $s: C_{n-1}(M \setminus D^d) \longrightarrow C_n(M \setminus D^d)$

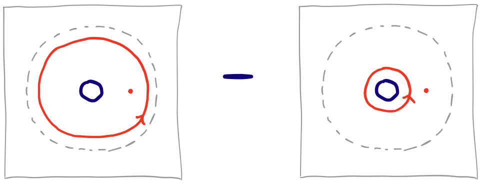


Hence ↴ of (□) is given by

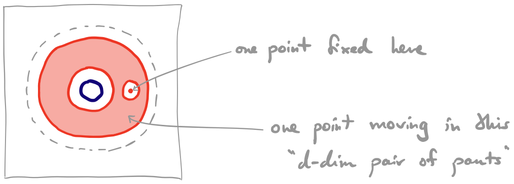


and ↳ of (□) is given by

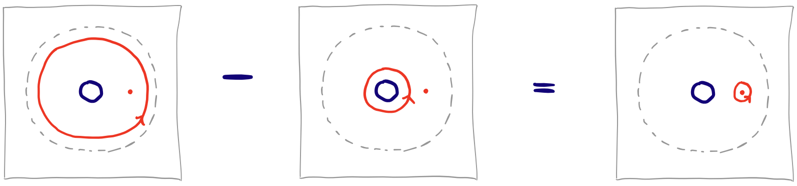


Their difference is  $\in H_{d-1} C_2(M \setminus D^d)$

Consider the d-dim singular chain :



Its boundary gives the relation :



Hence this is the obstruction to commutativity of $(\square)$.

It is the image of the element

$$\begin{array}{ccc} \boxed{Q} & \in & H_{d-1} C_2(\mathbb{R}^d) \\ \| & & \wr\| \\ 2[\mathbb{RP}^{d-1}] & \in & H_{d-1}(\mathbb{RP}^{d-1}) . \end{array} \qquad (*)$$

By the previous argument, if this vanishes,
then $(\square)$ commutes and we get homological stability.

If $\mathrm{char}(\mathbb{F}) = 2$ then $2 = 0$ so $(*)$ vanishes.

If $d = \dim(M)$ is odd then :

- either $\mathrm{char}(\mathbb{F}) = 2 \longrightarrow$ ✓ by above
- or $\mathrm{char}(\mathbb{F}) \neq 2 \longrightarrow H_{d-1}(\mathbb{RP}^{d-1}; \mathbb{F}) \underset{(\mathrm{UCT})}{\cong} \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{F}) = 0 \longrightarrow$ ✓

However, if $\mathbb{F} = \mathbb{Q}$ and $d = \dim(M)$ is even,
then $(*) = 2 \in \mathbb{Q} = H_{d-1}(\mathbb{RP}^{d-1}; \mathbb{Q})$,
so $(\square)$ does not commute.

Strategy for $\mathbb{F} = \mathbb{Q}$

Transfer maps $H_i\, C_n(M) \longrightarrow H_i\, C_{n-1}(M)$

Def.

$$\begin{tikzcd} C_n(M) & \underbrace{C_{n-1,1}}_{\substack{n-1 \text{ blue points} \\ 1 \text{ red point}}}(M) \arrow[l, "\text{forget colours}"', "\text{n-sheeted covering}"] \arrow[r, "\text{forget red point}"] & C_{n-1}(M) \end{tikzcd}$$

$$\begin{tikzcd} H_i\, C_n(M) \arrow[r, "\text{transfer}"] \arrow[rr, bend right, "\text{formal sum of all ways of forgetting a point}"'] & H_i\, C_{n-1,1}(M) \arrow[r] & H_i\, C_{n-1}(M) \end{tikzcd}$$

Lemma (Dold '62)

With coefficients in $\mathbb{Q}$, if M is an open manifold, then

$$H_i(C_n(M);\mathbb{Q}) \xrightarrow{(\text{stab})_*} H_i(C_{n+1}(M);\mathbb{Q}) \xrightarrow{\text{transfer}} H_i(C_n(M);\mathbb{Q})$$

is an automorphism.

Corollary: With $\mathbb{Q}$ coefficients, there is homological stability with respect to the transfer maps.

Idea: Run the same argument, but with transfer maps instead of stabilisation maps.

$\Rightarrow$ It is enough to prove that (with $\mathbb{Q}$ coeffs) the following commutes:

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "S_n^i"] & H_{i-1}C_n(M\setminus pt) \\
H_{i-d}(C_n(M\setminus pt)) \arrow[u, "transfer"] \arrow[r, "S_{n+1}^i"'] & H_{i-1}C_{n+1}(M\setminus pt) \arrow[u, "transfer"']
\end{tikzcd} \qquad (\square')$$

<u>Proof</u>:

"Stacking" thickened spheres $S^{d-1}\times[0,1]$ gives $\coprod_n C_n(S^{d-1}\times[0,1])$ the structure of an H-space, and gluing $S^{d-1}\times[0,1]$ to the boundary of $M\setminus\mathring{D}^d$ gives $\coprod_n C_n(M\setminus\mathring{D}^d)$ the structure of an H-module over it. On homology, we therefore have:

$H_*(\coprod_n C_n(S^{d-1}\times[0,1]))$ —— bigraded ring

$H_*(\coprod_n C_n(M\setminus\mathring{D}^d))$ —— bigraded module over it

The horizontal maps of $(\square')$ are induced by this module structure: they are both multiplication by the element

$$[S^{d-1}]\in H_{d-1}(C_1(S^{d-1}\times[0,1])) \qquad (\text{in bigrading } (d-1,1))$$

$\lceil\!\!\!\rightarrow$ of $(\square')$ is given by

$$\alpha \longmapsto [S^{d-1}]\cdot \text{transfer}(\alpha)$$
$$= \sum_{i=1}^{n} [S^{d-1}]\cdot(\text{forget } i^{th} \text{ point in } \alpha)$$

$\uparrow$ of (□') is given by

$$\alpha \longmapsto \text{transfer}\left([S^{d-1}]\cdot\alpha\right)$$
$$= \sum_{i=1}^{n} [S^{d-1}]\cdot\left(\text{forget } i^{th} \text{ point in } \alpha\right)$$
$$+ \; [\emptyset]\cdot\alpha$$

where $[\emptyset]$ is the result of forgetting the unique point of $[S^{d-1}] \in H_{d-1}(C_1(S^{d-1}\times[0,1]))$.

(The fundamental class of the empty manifold.)

But this means that $[\emptyset] \in H_{d-1}(\underbrace{C_0(S^{d-1}\times[0,1])}_{=\,*}) = 0$

There is only one empty configuration!

Hence $[\emptyset] = 0$,

so $[\emptyset].\alpha = 0$.

$\Rightarrow$ (□') commutes, as claimed. □.

Summary:

Mapping cone lemma

$\Rightarrow$ with <u>field</u> coeffs, to prove hom. stab.^{y} for $H_i C_n(M)$, it is enough to prove stability for the kernel and cokernel of the map:

$$H_{i-d}(C_{n-1}(M\setminus pt)) \xrightarrow{S_n^i} H_{i-1}\, C_n(M\setminus pt)$$

When dim(M) is odd or char($\mathbb{F}$) = 2

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1} C_n(M\setminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt)
\end{tikzcd} \qquad (\square)$$

($\square$) commutes

open manifold

Hence hom. stab.^{y} for $C_n(M\setminus pt)$ implies stability for $\ker(\delta_n^i)$ and $\operatorname{coker}(\delta_n^i)$.

Transfer maps

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] & H_{i-1} C_n(M\setminus pt) \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] \arrow[u, "\text{transfer}"] & H_{i-1} C_{n+1}(M\setminus pt) \arrow[u, "\text{transfer}"']
\end{tikzcd} \qquad (\square')$$

($\square'$) commutes with any coefficients.

When char($\mathbb{F}$) = 0

[Dold] ⇒ transfer is a one-sided inverse to stabilisation

Hence hom. stab.^{y} for $C_n(M\setminus pt)$ w.r.t. stabilisation maps

⇒ hom. stab.^{y} for $C_n(M\setminus pt)$ w.r.t. transfer maps

⇒ stability for $\ker(\delta_n^i)$ and $\operatorname{coker}(\delta_n^i)$.

by commutativity of ($\square'$)

Remarks

(1) All of the arguments above use homological stability for $C_n(M \setminus pt)$ as an input — to deduce homological stability results for $C_n(M)$.

(2) The obstruction to commutativity of $(\square)$ is the element $2[\mathbb{RP}^{d-1}] \in H_{d-1}(\mathbb{RP}^{d-1})$. This vanishes also with $\mathbb{Z}$ coefficients when d is odd.

Hence in this case, the argument goes through as far as proving that $\ker(\delta_n^i)$ and $\operatorname{coker}(\delta_n^i)$ stabilise.

But then we just have a short exact sequence

$$0 \to \operatorname{coker}(\delta_n^{i+1}) \longrightarrow H_i C_n(M) \longrightarrow \ker(\delta_n^i) \to 0$$

of abelian groups, and stability of the outer terms does not imply stability of the middle term.

(Recall that we do not have maps $H_i C_n(M) \to H_i C_{n+1}(M)$.)

However, it is true that $C_n(M)$ is always homologically stable with $\mathbb{Z}$ coefficients when n is odd — this will appear in a later talk, using different methods...