Homological stability for configuration spaces on closed manifolds II

GeMAT seminar
IMAR
28 April 2025

Recall from last time

M —— connected manifold of $\dim(M) = d \geq 2$ with $\partial M = \phi$

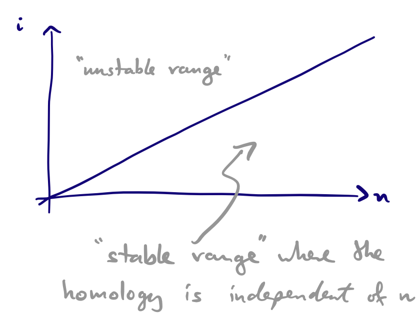
$C_n(M)$ = space of n-point subsets of M

(topologised as a subquotient of M^n)

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open —— $M \cong \mathrm{int}(\bar{M})$, $\partial \bar{M} \neq \phi$

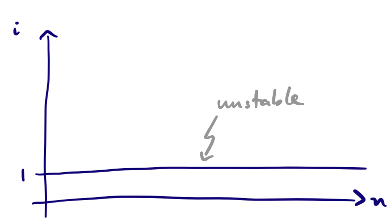
then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$:



Counterexample [Fadell - van Buskirk '62]

For $M = S^2$, $H_1(C_n(M)) \cong \mathbb{Z}/(2n-2)$

closed —— closed & $\partial M = \phi$



Theorem [Randal-Williams '13]

For any M,
we have $H_i(C_n(M);R) \cong H_i(C_{n+1}(M);R)$ for $n \geq 2i$

as long as (1) R = field of char = 2
or (2) R = field of char = 0
or (3) R = any field and $\dim(M)$ is odd.

Aim for today:

Theorem [Bendersky-Miller '14]

For any M,
we have $H_i(C_m(M);R) \cong H_i(C_n(M);R)$ for $m,n \geq 2i$

as long as

- when $\dim(M)$ is odd
  (1) $R = \mathbb{Q}$
  (2) $R = \mathbb{Z}_{(p)}$ with $p \geq \frac{1}{2}(\dim(M)+3)$
- when $\dim(M)$ is even
  (3) $R = \mathbb{Q}$
  and $m,n \neq \frac{1}{2}\chi(M)$ (if $\chi(M)$ is even)
  (4) $R = \mathbb{Z}_{(p)}$ with $p \geq \frac{1}{2}(\dim(M)+3)$
  and $\nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$

i.e. each subsequence $\{C_n(M) \mid \nu_p(2n-\chi(M)) = k\}$ fixed
is homologically stable.

Tools / ideas of proof / plan of today's talk

(1) Scanning maps — will also be crucial in talks #3 and #4

(2) Localisations of spaces

(3) Bundle maps

(4) Obstruction theory

(5) Degree formula

(1) Scanning maps

Idea (Segal '73)

$M = \mathbb{R}^d$ "Electrostatic map"

$\underline{p} = \{p_1, \dots, p_n\} \in C_n(\mathbb{R}^d)$

Think of this as a collection of electrically charged particles of charge $+1$ (positrons).

$\rightsquigarrow$ electric field $E(\underline{p}) : \mathbb{R}^d \setminus \underline{p} \longrightarrow \mathbb{R}^d$.

Extend to $S^d = \mathbb{R}^d \cup \{\infty\} \longrightarrow S^d = \mathbb{R}^d \cup \{\infty\}$

via $p_1, \dots, p_n \longmapsto \infty$

$\infty \longmapsto 0$

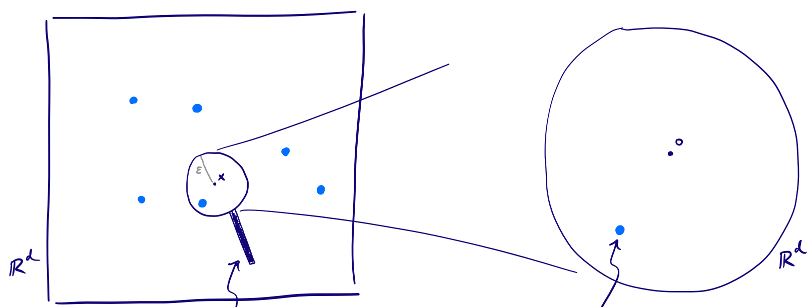
Reflect the target sphere so that $\infty \leftrightarrow 0$.

$\rightsquigarrow$ Based map $S^d \to S^d$

Element of $\mathrm{Map}_*(S^d, S^d) =: \Omega^d S^d$.

This is the <u>scanning map</u> $C_n(\mathbb{R}^d) \longrightarrow \Omega^d S^d$.

Another description (magnifying glasses)



call this $\mathrm{Mag}_x(\underline{p})$ (if it exists)

ε-magnifying glass centred at $x \in \mathbb{R}^d$

i.e. $B_x(\varepsilon)$

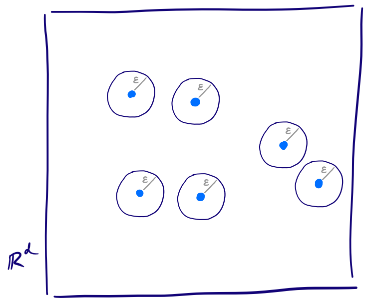
choose continuously depending on $\underline{p}$
(e.g. $\varepsilon = \frac{1}{4} \min\{|p_i - p_j| : i \neq j\}$)
so that it only ever contains <u>at most one particle</u>

$$\underline{p} \rightsquigarrow S^d \xrightarrow{s(\underline{p})} S^d$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if } |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if } |\ \cdots\ | = 0 \end{cases}$$

$$C_n(\mathbb{R}^d) \xrightarrow{s} \Omega^d S^d$$

Yet another description (collapsing)



$$\xrightarrow{s(\underline{p})} \quad B_1(0) \big/ \partial B_1(0) \cong S^d$$

- $B_\varepsilon(p_i) \xrightarrow{\cong} B_1(0)$ rescale & translate
- complement $\longrightarrow \{\infty\}$

(choose ε as above)

Rmk $s(\underline{p})$ clearly has degree n:

- $0 \in \mathbb{R}^d \cup \{\infty\} = S^d$ is a regular value
- its pre-image is $\{p_1, \dots, p_n\}$

Notation $\Omega^d_n S^d \subseteq \Omega^d S^d$ space of degree-n based maps $S^d \to S^d$

Rmk In fact $\pi_0(\Omega^d S^d) \xrightarrow[\cong]{\text{degree}} \mathbb{Z}$,

and so $\Omega^d S^d = \coprod_{n \in \mathbb{Z}} \Omega^d_n S^d$.

Target of the scanning map in general

M any smooth manifold

TM = tangent bundle

fibres = $\mathbb{R}^d$

$\dot{T}M$ = "fibrewise 1-point compactified" tangent bundle

fibres = S^d

$\Gamma(\dot{T}M)$ = space of sections of $\dot{T}M \longrightarrow M$.

$\cup|$

$\Gamma^c(\dot{T}M)$ = subspace of compactly-supported sections, i.e. those that agree with the ∞ section outside a compact subset.

The scanning map will be

$$\begin{array}{ccc} C_n(M) & \longrightarrow & \Gamma^c(\dot{T}M) \\ & \searrow & \cup| \\ & & \Gamma^c_n(\dot{T}M) \end{array}$$

sections of degree n

Degree of a section

α, β sections of $\dot{T}M \xrightarrow{\pi} M$ such that $\alpha(p) \neq \beta(p)$ for all $p \in M \setminus K$, where $K \subseteq M$ is compact (*)

Borel-Moore

orientation local system

(1) $[M] \in H_d^{BM}(M; \mathcal{O})$

(2) $\alpha_*[M], \beta_*[M] \in H_d^{BM}(\dot{T}M; \pi^*\mathcal{O})$

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[d] & \pi^*\mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[d] \\ M \arrow[r, "\alpha"] & \dot{T}M \arrow[r, "\pi"] & M \end{tikzcd}$$

(3) Apply Poincaré duality: $\dot{T}M$ is always orientable!

$$\alpha_*[M]^\vee, \beta_*[M]^\vee \in H^d(\dot{T}M; \pi^*\mathcal{O})$$

(4) Cup product

$$(\alpha_*[M]^\vee) \cup (\beta_*[M]^\vee) \in H_c^{2d}(\dot{T}M; \mathbb{Z})$$

$\pi^*\mathcal{O} \otimes \pi^*\mathcal{O} \cong \pi^*(\mathcal{O} \otimes \mathcal{O}) \cong \mathbb{Z}$

compactly-supported cohomology because of (*)

(5) Poincaré duality again:

$$\left((\alpha_*[M]^\vee) \cup (\beta_*[M]^\vee)\right)^\vee \in H_0(\dot{T}M; \mathbb{Z}) \cong \mathbb{Z}$$

$\rightsquigarrow$ This is the relative degree $\underline{\mathrm{rdeg}(\alpha, \beta) \in \mathbb{Z}}$.

$$\begin{tikzcd}[column sep=small] {[M]} \\ H_d^{BM}(M; \mathcal{O}) \arrow[u, phantom, "\cap" description] \arrow[r, "\alpha_*"] \arrow[dr, "\beta_*"'] & H_d^{BM}(\dot{T}M; \pi^*\mathcal{O}) \cong H^d(\dot{T}M; \pi^*\mathcal{O}) & & \\ & \otimes \arrow[r, "\cup"] & H_c^{2d}(\dot{T}M; \mathbb{Z}) \cong H_0(\dot{T}M; \mathbb{Z}) \cong \mathbb{Z} \\ & H_d^{BM}(\dot{T}M; \pi^*\mathcal{O}) \cong H^d(\dot{T}M; \pi^*\mathcal{O}) & \end{tikzcd}$$

Rmk: Steps (3)-(5) $\longleftrightarrow$ intersection product

$\longleftrightarrow$ counting (with sign) intersection points between $\alpha(M)$ and $\beta(M)$ after homotoping α and β so that these are transverse submanifolds

Rmk: If α, β <u>don't</u> satisfy (*) then we land in $H^{2d}(\dot{T}M;\mathbb{Z}) \cong H_0^{BM}(\dot{T}M;\mathbb{Z}) = 0$ instead of $H^{2d}_c(\dot{T}M;\mathbb{Z}) \cong H_0(\dot{T}M;\mathbb{Z}) \cong \mathbb{Z}$.

($\longleftrightarrow$ there are ∞ many intersection points, so the intersection # is not well-defined)

Def^{n}: Let β be a compactly-supported section of $\dot{T}M \to M$.

Set z = zero section.

Then (z,β) satisfy (*), and

$$\deg(\beta) := \mathrm{vdeg}(z,\beta).$$

Rmk Similarly to above, $\pi_0(\Gamma^c(\dot{T}M)) \xrightarrow[\cong]{\text{degree}} \mathbb{Z}$,

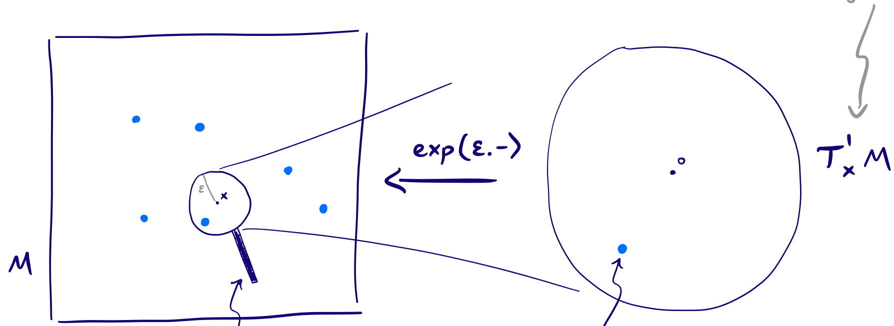
and so $\Gamma^c(\dot{T}M) = \coprod_{n\in\mathbb{Z}} \Gamma^c_n(\dot{T}M)$.

Scanning any manifold M

Idea Same picture as before.

Choose Riemannian metric on M.

unit disc in tangent space at x



call this $\mathrm{Mag}_x(\underline{p})$ (if it exists)

ε-magnifying glass centred at $x \in \mathbb{R}^d$

i.e. $B_x(\varepsilon)$

choose continuously depending on $\underline{p}$ so that it only ever contains <u>at most one particle</u>

unit disc / unit sphere in each fibre

$$\underline{p} \rightsquigarrow M \longrightarrow \dot{T}M \cong T'M/\partial T'M$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if } |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if } |\quad \cdots \quad| = 0 \end{cases}$$

Agrees with ∞ outside $\bigcup_i B_\varepsilon(p_i)$ (compact)

Intersection with 0 section is $p_1, \dots, p_n \Rightarrow$ degree n.

$$C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$$

Note: Rigorous definition (sketch)

- Choose Riemannian metric with injectivity radius bounded below by $\varepsilon > 0$. (Exists by [Greene '78])
- $C_n^\varepsilon(M) :=$ configurations of pairwise distance $> \varepsilon$.
- Decreasing ε if necessary, $C_n^\varepsilon(M) \hookrightarrow C_n(M)$ is a hty equivalence.
- $\begin{array}{c} C_n^\varepsilon(M) \xrightarrow{\text{scan}} \Gamma_n^c(\dot{T}M) \\ \cong \\ C_n(M) \end{array}$ defined as on the previous page, for ε fixed.

Theorem (McDuff '75 + Segal '79)

The scanning map $C_n(M) \longrightarrow \Gamma_n^c(\dot{T}M)$ induces isomorphisms on $H_i(-;\mathbb{Z})$ for all $n \geq 2i$.

(2) Localisations of spaces [Sullivan '70]

$P = \{\text{primes}\}$

Choose $T \subseteq P$

reduced form

Def $\mathbb{Z}_T := \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; b = \text{product of primes in } P \setminus T \right\}$

Ex $\mathbb{Z}_\emptyset = \mathbb{Q}$

Notation $\mathbb{Z}_{(0)} := \mathbb{Z}_\emptyset = \mathbb{Q}$

$\mathbb{Z}_{(p)} := \mathbb{Z}_{\{p\}}$

Def An abelian group A is T-local if it has a structure of a $\mathbb{Z}_T$-module.

Equivalently: Every $a \in A$ may be uniquely divided by p, for each $p \in P \setminus T$.

Rmk Any A has at most one $\mathbb{Z}_T$-module structure.

Def A simply-connected space X is T-local if $H_i(X;\mathbb{Z})$ is T-local $\forall i$.

Equivalently (Theorem): $\pi_i(X)$ is T-local $\forall i$.

Rmk Can be defined more generally for nilpotent spaces.

Def A <u>T-localisation</u> of a simply-connected space X is a map $f : X \longrightarrow Y$, where Y is T-local and $f_* : H_i(X; \mathbb{Z}_T) \xrightarrow{\cong} H_i(Y; \mathbb{Z}_T)$ $\forall i$.

<u>Equivalently</u> (Theorem) :

- $$\begin{tikzcd} f_* : H_i(X;\mathbb{Z}) \arrow[r] \arrow[dr, "- \otimes \mathbb{Z}_T"'] & H_i(Y;\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] \\ & H_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_T \end{tikzcd}$$

- $$\begin{tikzcd} f_* : \pi_i(X) \arrow[r] \arrow[dr, "- \otimes \mathbb{Z}_T"'] & \pi_i(Y) \arrow[d, phantom, "\cong" rotate=90] \\ & \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_T \end{tikzcd}$$

<u>Theorem</u> (Sullivan '70)

$\exists$ unique (up to $\simeq$) T-localisation of any simply-connected space X.

(can generalise to nilpotent spaces)

<u>Notation</u> X_T := the T-localisation of X

In particular,

$(T = \emptyset)$ $X_{(0)}$:= rationalisation of X

$(T = \{p\})$ $X_{(p)}$:= p-localisation of X

Fibrewise localisation

$$\begin{array}{c} E \\ \downarrow \\ B \end{array} \quad \text{fibre bundle with fibre } F$$

$$\longrightarrow \quad \begin{array}{c} E_T \\ \downarrow \\ B \end{array} \quad \text{fibre bundle with fibre } F_T$$

In particular, for p prime or $p=0$, we have

$$\begin{array}{c} \dot{T}M_{(p)} \\ \downarrow \\ M \end{array} \quad \text{fibre bundle with fibre } S^d_{(p)}$$

From now on, assume that M is compact, so $\Gamma^c(\dot{T}M) = \Gamma(\dot{T}M)$.

Thm (Møller '87)

$\forall n \in \mathbb{Z}$, $\quad \Gamma_n(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M)_{(p)}$ $\quad$ $[\pi_0(\Gamma(\dot{T}M_{(p)})) \cong \mathbb{Z}_{(p)}]$

Coro:

$\forall n \geqslant 2i$, $\quad H_i(C_n(M); \mathbb{Z}_{(p)})$

$\cong H_i(\Gamma_n(\dot{T}M); \mathbb{Z}_{(p)})$ $\quad$ [McDuff - Segal]

$\cong H_i(\Gamma_n(\dot{T}M)_{(p)}; \mathbb{Z}_{(p)})$ $\quad$ [Sullivan]

$\cong H_i(\Gamma_n(\dot{T}M_{(p)}); \mathbb{Z}_{(p)})$ $\quad$ [Møller]

Hence it will suffice to prove that

$$\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$$

under the appropriate conditions:

- when $\dim(M)$ is odd:
  - (1) $p = 0$
  - (2) $p \geq \frac{1}{2}(\dim(M)+3)$
- when $\dim(M)$ is even:
  - (3) $p = 0$ and $m, n \neq \frac{1}{2}\chi(M)$
  - (4) $p \geq \frac{1}{2}(\dim(M)+3)$ and $\nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$

Idea: construct bundle self-homotopy-equivalences

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

such that the induced self-homotopy-equivalences

$$\Gamma(\dot{T}M_{(p)}) \longrightarrow \Gamma(\dot{T}M_{(p)})$$

send degree-m sections to degree-n sections under the above conditions.

(3) Bundle maps

Def.

$$\begin{array}{c} \operatorname{end}_r(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array} := \text{bundle with fibre over } x \in M \text{ given by self-maps } \dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$$

of degree $r \in \mathbb{Z}_{(p)}$

Lemma.

$$\text{bundle endomorphisms of } \begin{array}{c} \dot{T}M_{(p)} \\ \downarrow \\ M \end{array} \longleftrightarrow \text{sections of } \begin{array}{c} \operatorname{end}(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array}$$

Now fix $\sigma, \tau \in \Gamma(\dot{T}M_{(p)})$:

Def.

$$\begin{array}{c} \operatorname{end}_r^{\sigma,\tau}(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array} := \text{bundle with fibre over } x \in M \text{ given by self-maps } \dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$$

of degree $r \in \mathbb{Z}_{(p)}$
sending $\sigma(x)$ to $\tau(x)$

Lemma.

$$\text{bundle endomorphisms } \phi \text{ of } \begin{array}{c} \dot{T}M_{(p)} \\ \downarrow \\ M \end{array} \longleftrightarrow \text{sections of } \begin{array}{c} \operatorname{end}^{\sigma,\tau}(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array}$$

such that $\phi \circ \sigma = \tau$

Theorem (Dold '63)

If M is a paracompact manifold,

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$ is a bundle endomorphism

and $\phi_x : E|_x \longrightarrow E|_x$ is a hty equivalence $\forall x \in M$,

then ϕ admits a <u>fibrewise homotopy inverse</u>.

Coro. If $r \in \mathbb{Z}_{(p)}$ is invertible, then

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left] \end{tikzcd} \rightsquigarrow \text{bundle self-homotopy-equivalence} \begin{tikzcd} \dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

So we need to:

(1) Find sections $\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left] \end{tikzcd}$ —— obstruction theory

(2) Understand how the induced $\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$

acts on degrees of sections. —— degree formula

(4) Obstruction theory

$\dim(M) = \text{odd}$

Claim: Assume $p = 0$ or $p \geq \frac{1}{2}(\dim(M)+3)$. Then:

$\forall \sigma, \tau \in \Gamma(\dot{T}M_{(p)})$,

$\exists$ a section of

$$\begin{array}{c} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array}$$

Proof of main theorem:

We need to prove that $\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$ for any $m, n \in \mathbb{Z}$.

Choose σ of degree m
τ n.

Claim $\Rightarrow$ section of
$$\begin{array}{c} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array}$$

Coro $\Rightarrow$ bundle self-homotopy-equivalence

previous page

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

such that $\phi \circ \sigma = \tau$

$\Rightarrow$ $\Gamma(\dot{T}M_{(p)}) \simeq \Gamma(\dot{T}M_{(p)})$

restricting to $\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$.

$\square$

Proof of Claim:

Assume $p=0$ or $p \geq \frac{1}{2}(\dim(M)+3)$.

Obstruction theory $\Rightarrow$ $\exists$ section as long as certain obstruction classes vanish, which live in the groups

$$H^i(M; \pi_{i-1}(\text{fibre}))$$

of $\text{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \to M \ni x$

for $i = 1, \ldots, d = \dim(M)$.

$$\text{fibre} \simeq \left\{ \dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x \text{ of degree } 1 \text{ sending } \sigma(x) \text{ to } \tau(x) \right\}$$

based

$$\simeq \mathrm{Map}_1^*(S^d_{(p)}, S^d_{(p)})$$

$$\simeq \mathrm{Map}_1^*(S^d, S^d_{(p)}) = \Omega^d_1 S^d_{(p)}$$

path-conn.

Hence for $i \geq 2$,

$$\pi_{i-1}(\text{fibre}) \cong \pi_{d+i-1}(S^d_{(p)})$$

$$\cong \pi_{d+i-1}(S^d)_{(p)}$$

sufficient to prove that this vanishes for $2 \leq i \leq d$.

Theorem (Serre '51) If d is odd,

$$\pi_*(S^d)_{(0)} = 0 \quad \forall * \geq d+1$$

$$\pi_*(S^d)_{(p)} = 0 \quad \forall\ d+1 \leq * \leq d+2p-4$$

Hence for $2 \leq i \leq d$,

$$\pi_{i-1}(\text{fibre}) \cong \pi_{d+i-1}(S^d)_{(p)} = 0$$

$p \geq \frac{1}{2}(d+3) \longrightarrow d \leq 2p-3$

$\Rightarrow \quad d-1+i \leq d-1+(2p-3) = d+2p-4$

$\square$

Next time:

dim(M) = even

- obstructions don't vanish!
- instead $\rightsquigarrow$ $\exists$ bundle self-htpy-equiv ϕ_r of any fibrewise degree $r \in \mathbb{Z}_{(p)}^\times$

  but we cannot force it to send σ to τ for any two given sections σ, τ
- understand how ϕ_r acts on degrees of sections ......