Homological stability for configuration spaces on closed manifolds

Martin Palmer-Anghel // GeMAT seminar, IMAR // April–July 2025

Series abstract.

A classical result in algebraic topology — due to Arnol'd, McDuff and Segal in the 1970s — says that unordered configuration spaces on connected, non-compact manifolds are homologically stable. Here, the nth *unordered configuration space* $C_n(M)$ on a manifold M is the space of finite subsets $c \subset M$ of size n, topologised as a subquotient of the product M^n, and *homologically stable* means that $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ when n is sufficiently large as a function of i.

If M is instead a connected, *closed* manifold then its configuration spaces are generally *not* homologically stable: considering spherical braid groups one may check that it fails already in homological degree 1 for $M = S^2$. However, a number of more subtle stability or periodicity patterns have been discovered in the homology of configuration spaces on closed manifolds, depending in particular on the characteristic of the coefficient ring R that we use for homology. For example, in an appropriate range $n \gg i$, the homology $H_i(C_n(M); R)$ is:

- stable if $R = \mathbb{Q}$ or $R = \mathbb{F}_2$,
- stable if M is odd-dimensional and $R = \mathbb{Z}$,
- p-periodic if M is even-dimensional and R is a field of odd prime characteristic p.

A variety of different techniques have been used to prove these results, including *representation stability*, *scanning maps*, *replication maps*, *homology operations*, *factorisation homology* and *semi-simplicial resolutions*. The goal of this series of talks is to explain these different approaches.

Pages:

Talks.

2–16

1. **10 April 2025** — We will follow the article
O. Randal-Williams, *Homological stability for unordered configuration spaces*,
Quart. J. Math. 64 (2013), pp. 303–326
and prove homological stability with coefficients in $\mathbb{Q}$ or $\mathbb{F}_2$ using semi-simplicial resolutions and transfer maps.

17–36
37–55

2,3. **28 April + 9 May 2025** — We will follow the article
M. Bendersky, J. Miller, *Localization and homological stability of configuration spaces*,
Quart. J. Math. 65 (2014), pp. 807–815
and use scanning maps, localisations and obstruction theory to prove rational homological stability, as well as more complicated stability-like patterns in p-local homology.

56–74
75–88

4,5. **16 May + 23 May 2025** — We will follow the articles
F. Cantero, M. Palmer, *On homological stability for configuration spaces on closed background manifolds*, Doc. Math. 20 (2015) pp. 753–805
and
A. Kupers, J. Miller, *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*, Proc. Amer. Math. Soc. 144 (2016) pp. 5457–5468
and prove homological stability with coefficients in $\mathbb{Z}$ when M is odd-dimensional and certain homological periodicity results when M is even-dimensional, using scanning maps, replication maps and homology operations for E_n-algebras.

89–97

6. **11 July 2025** — We will follow the article
T. Church, *Homological stability for configuration spaces of manifolds*,
Invent. Math. 188 (2012) pp. 465–504
and prove rational homological stability using the concept of *representation stability* invented by T. Church and B. Farb.

98–107

7. **15 July 2025** — We will follow the article
B. Knudsen, *Betti numbers and stability for configuration spaces via factorization homology*,
Algebr. Geom. Topol. 17 (2017) pp. 3137–3187
and prove rational homological stability using factorisation homology and Lie algebra models. We will also see some explicit calculations of rational homology that this method affords.

108–119

8. **16 July 2025** — We will follow the article
O. Randal-Williams, *Configuration spaces as commutative monoids*,
Bull. Lond. Math. Soc. 56 (2024) pp. 2847–2862
and prove rational homological stability by considering a commutative monoid structure on one-point compactifications of configuration spaces.

Homological stability for configuration spaces on closed manifolds I

GeMAT seminar
IMAR
10 April 2025

M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M topologised as a subquotient of M^n

Theorem [Arnol'd, McDuff, Segal, 70's]

"M is an open manifold"

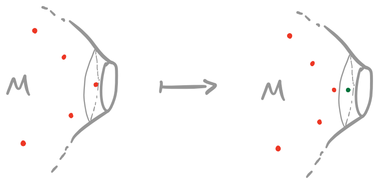
If $M = \mathrm{int}(\bar{M})$ where $\partial \bar{M} \neq \emptyset$

then $\exists$ maps $C_n(M) \xrightarrow{s} C_{n+1}(M)$

inducing $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.

$C_n(M) \xrightarrow{s} C_{n+1}(M)$

- Push the configuration away from a collar neighbourhood of $\partial \bar{M}$.
- Add a new point to the configuration in this collar.



Q: What about when M is closed (i.e. compact)?

↳ There are no stabilisation maps.

A: Homological stability is **false**!

E.g. $\pi_1(C_n(S^2)) = B_n(S^2)$

[Fadell - van Buskirk '62] ⟶ presentation

$\Rightarrow \quad H_1(C_n(S^2)) \cong \mathbb{Z}/(2n-2)$ for $n \geq 2$

More generally, [Cantero - P. '15]

$$H_{2d-1}(C_n(S^{2d})) \cong \underset{\substack{\uparrow \\ \text{depending} \\ \text{only on } d}}{\boxed{\text{▨}}} \oplus \mathbb{Z}/(2n-2) \qquad \text{for } n \geq 4d-2$$

BUT there are some more delicate stable patterns:

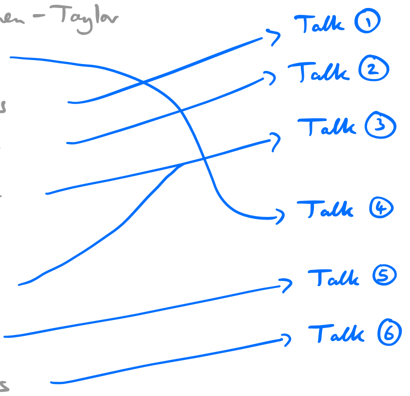
(1) Stability holds with coeffs in $\mathbb{F}_2$ [ML'88] [BCT'89] [RW'13]

(2) Stability holds with coeffs in $\mathbb{Q}$ [C'12] [RW'13] [BM'14] [K'17] [RW'24]

(3) Stability holds if dim (M) is odd $\underbrace{\text{[BCT'89] [RW'13]}}_{\text{field coeff}}$ $\underbrace{\text{[CP'15]}}_{\frac{1}{2} \in \text{coeff}}$ $\underbrace{\text{[KM'16]}}_{\text{all coeffs}}$

(4) Eventual periodicity holds with coeffs in $\mathbb{F}_p$ [CP'15] [N'15] [KM'16]

[ML'88] = Milgram - Löffler
[BCT'89] = Bödigheimer - Cohen - Taylor
[C'12] = Church
[RW'13] = Randal - Williams
[BM'14] = Bendersky - Miller
[CP'15] = Cantero - Palmer
[N'15] = Nagpal
[KM'16] = Kupers - Miller
[K'17] = Knudsen
[RW'24] = Randal - Williams



Today:

Theorem [Randal-Williams '13]

M — any connected manifold
$\mathbb{F}$ — field

Suppose either (1) $\mathbb{F}=\mathbb{F}_2$
(2) $\mathbb{F}=\mathbb{Q}$
(3) $\dim(M)$ is odd

Then $\dim_{\mathbb{F}} H_i(C_n(M);\mathbb{F})$ is independent of n when $n \geqslant 2i$.

Proof

- If $C(f)$ is the mapping cone of $X \xrightarrow{f} Y$, then there is a long exact sequence

$$\cdots \longrightarrow H_i X \longrightarrow H_i Y \longrightarrow H_i C(f) \longrightarrow H_{i-1} X \longrightarrow H_{i-1} Y \longrightarrow \cdots$$

(Write $H_i(-) = H_i(-;\mathbb{F})$)

- **Lemma**: The mapping cone of $C_n(M \setminus \text{point}) \hookrightarrow C_n(M)$ is homology equivalent to $S^d \wedge C_{n-1}(M \setminus \text{point})_+$.

$d = \dim(M)$

- Hence we have a LES :

$$H_i C_n(M\setminus pt) \longrightarrow H_i C_n(M) \longrightarrow H_{i-d}\big(C_{n-1}(M\setminus pt)\big) \xrightarrow{\delta_n^i} H_{i-1} C_n(M\setminus pt) \longrightarrow H_{i-1} C_n(M)$$

Note: These do stabilise, since $M\setminus pt$ is open.

- We can extract a SES :

$$0 \to \operatorname{coker}(\delta_n^{i+1}) \longrightarrow H_i C_n(M) \longrightarrow \ker(\delta_n^i) \to 0$$

$$\text{Hence} \quad \dim H_i C_n(M) = \dim \ker(\delta_n^i) + \dim \operatorname{coker}(\delta_n^{i+1}) \qquad (\dagger)$$

- Now suppose that the following square commutes :

$$\begin{tikzcd}
H_{i-d}\big(C_{n-1}(M\setminus pt)\big) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1} C_n(M\setminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}\big(C_n(M\setminus pt)\big) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt)
\end{tikzcd} \qquad (\square)$$

Then it extends to a map of exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & \ker(\delta_n^i) \arrow[r] \arrow[d, "(*)"'] & H_{i-d}\big(C_{n-1}(M\setminus pt)\big) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1} C_n(M\setminus pt) \arrow[r] \arrow[d, "\text{stabilisation}"] & \operatorname{coker}(\delta_n^i) \arrow[r] \arrow[d, "(*)"] & 0 \\
0 \arrow[r] & \ker(\delta_{n+1}^i) \arrow[r] & H_{i-d}\big(C_n(M\setminus pt)\big) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt) \arrow[r] & \operatorname{coker}(\delta_{n+1}^i) \arrow[r] & 0
\end{tikzcd}$$

And then homological stability for $C_n(M\setminus pt)$ + 5-lemma

$\Rightarrow$ stability for $(*)$

$\Rightarrow$ stability for $\ker(\delta_n^i)$ & for $\operatorname{coker}(\delta_n^i)$.

$\underset{(\dagger)}{\Rightarrow}$ stability for $\dim H_i C_n(M)$.

Plan :

- Proof of the mapping cone lemma.
- Calculate the obstruction to commutativity of $(\square)$
  - $\hookrightarrow$ $2[\mathbb{RP}^{d-1}] \in H_{d-1}(\mathbb{RP}^{d-1})$
  - $\Rightarrow$ ☺ if $\mathbb{F}=\mathbb{F}_2$ or d is odd.
- Modify the strategy for $\mathbb{F}=\mathbb{Q}$ ....

Proof of the mapping cone lemma

Reminder: we want to prove that the mapping cone of the inclusion $C_n(M\setminus \text{point}) \hookrightarrow C_n(M)$ is homology equivalent to $S^d \wedge C_{n-1}(M\setminus \text{point})_+$.

$\uparrow$ homeomorphic to $\dfrac{D^d \times C_{n-1}(M\setminus \text{point})}{S^{d-1}\times C_{n-1}(M\setminus \text{point})}$

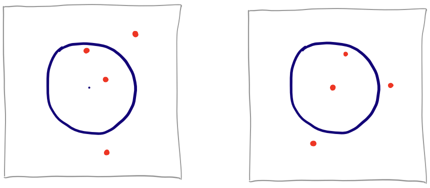
Choose $D^d \hookrightarrow M$

such that $0 \longmapsto$ the point that we remove.

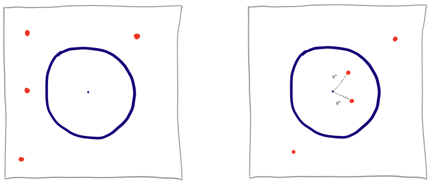
Let $U \subseteq C_n(M)$ be

$$U := \left\{ c = \{p_1, \dots, p_n\} \in C_n(M) \;\middle|\; \begin{array}{l} c \cap D^d \neq \emptyset \\ c \cap D^d \text{ has a } \underline{\text{unique}} \text{ closest} \\ \text{point to } 0 \in D^d \end{array} \right\}$$

Eg. ✓:



X:

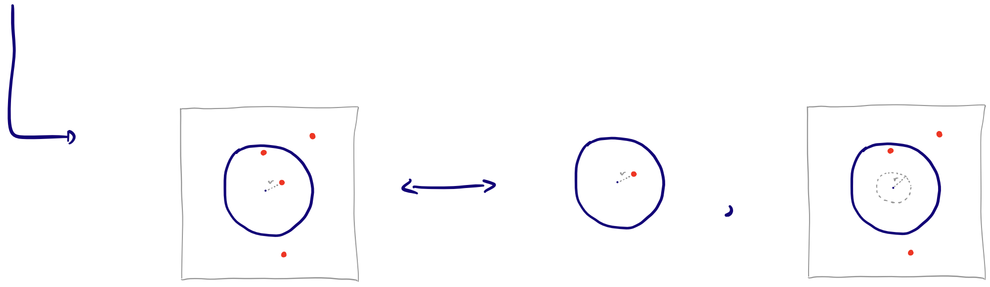


Observation: $C_n(M \setminus 0)$ and U form an open cover of $C_n(M)$.

Excision $\Rightarrow$

$$\begin{tikzcd} \text{mapping cone of } C_n(M\setminus 0) \hookrightarrow C_n(M) \arrow[d, leftrightarrow, "H_*\text{-equivalent}"] \\ \text{mapping cone of } C_n(M\setminus 0) \cap U \hookrightarrow U \end{tikzcd}$$

Claim: $U \cong D^d \times C_{n-1}(M \setminus 0)$



$$M \setminus B_r(0) \cong M \setminus 0$$

Moreover, this homeomorphism restricts to

$$U \cap C_n(M\setminus 0) \cong (D^d \setminus 0) \times C_{n-1}(M \setminus 0)$$
$$\simeq S^{d-1} \times C_{n-1}(M\setminus 0)$$

Hence:

$$\begin{tikzcd}
\text{mapping cone of } C_n(M\setminus 0) \hookrightarrow C_n(M) \arrow[d, leftrightarrow, "H_*\text{-equivalent}"] & \text{(excision)} \\
\text{mapping cone of } C_n(M\setminus 0) \cap U \hookrightarrow U \arrow[d, leftrightarrow, "H_*\text{-equivalent}"] & \text{(identification above)} \\
\text{mapping cone of } S^{d-1} \times C_{n-1}(M\setminus 0) \hookrightarrow D^d \times C_{n-1}(M \setminus 0) \arrow[d, phantom, "\cong" rotate=90] & \\
\dfrac{D^d \times C_{n-1}(M\setminus 0)}{S^{d-1}\times C_{n-1}(M\setminus 0)} . &
\end{tikzcd}$$

closed cofibration (NDR)

$\square$

Obstruction to commutativity of $(\square)$

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] \arrow[dr, phantom, "(\square)"] & H_{i-1}C_n(M\setminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1}C_{n+1}(M\setminus pt)
\end{tikzcd}$$

This is induced by a certain square

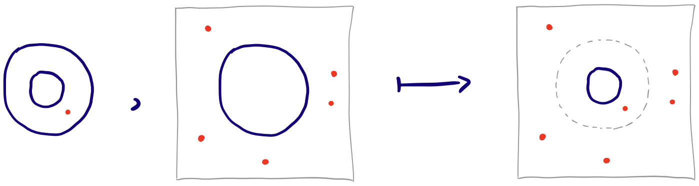
$$\begin{tikzcd}
S^{d-1}\times C_{n-1}(M\setminus pt) \arrow[r, "\Delta"] \arrow[d, "id\times s"'] & C_n(M\setminus pt) \arrow[d, "s"] \\
S^{d-1}\times C_n(M\setminus pt) \arrow[r, "\Delta"] & C_{n+1}(M\setminus pt)
\end{tikzcd}$$

by taking $H_{i-1}(-)$ and restricting to one summand in the Künneth decomposition on the LHS.

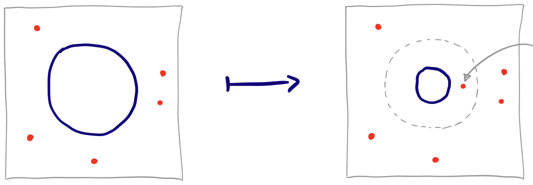
To describe this, replace $C_n(M\setminus pt) \simeq C_n(M\setminus D^d)$
$S^{d-1} \simeq S^{d-1}\times[0,1]$

Then $\Delta : (S^{d-1}\times[0,1])\times C_{n-1}(M\setminus D^d) \longrightarrow C_n(M\setminus D^d)$

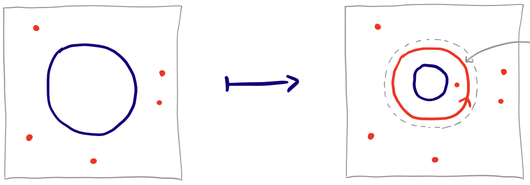
Part of an H-module structure on $\coprod_n C_n(M\setminus D^d)$ over the H-space $\coprod_n C_n(S^{d-1}\times[0,1])$.



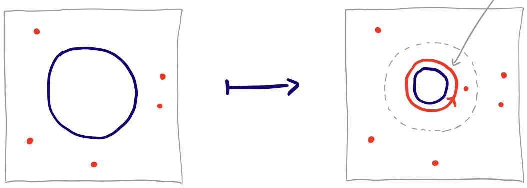
And $s: C_{n-1}(M\setminus D^d) \longrightarrow C_n(M\setminus D^d)$


Fixed point in $S^{d-1}\times[0,1]$.

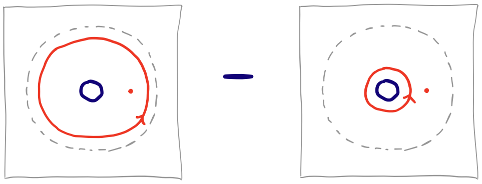
Hence ↴ of ($\square$) is given by


$(d-1)$-dimensional cycle parametrised by S^{d-1}

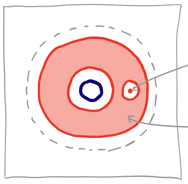
and ↳ of ($\square$) is given by



Their difference is

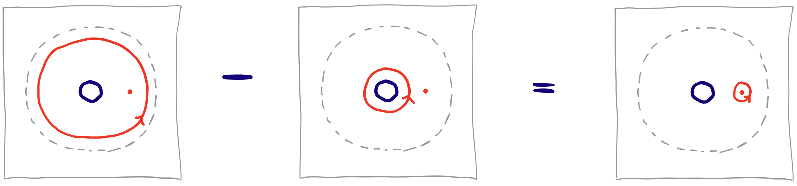

$\in H_{d-1}C_2(M\setminus D^d)$

Consider the d-dim singular chain:


one point fixed here

one point moving in this "d-dim pair of pants"

Its boundary gives the relation :



Hence this is the obstruction to commutativity of (□).

It is the image of the element  $\in H_{d-1} C_2(\mathbb{R}^d)$

$$\begin{tikzcd} \arrow[d, phantom, "\shortparallel"] & \arrow[d, phantom, "\cong" rotate=90] \\ 2[\mathbb{R}P^{d-1}] \in & H_{d-1}(\mathbb{R}P^{d-1}) . \end{tikzcd} \quad (*)$$

By the previous argument, if this vanishes,
then (□) commutes and we get homological stability.

If $\mathrm{char}(\mathbb{F}) = 2$ then $2 = 0$ so $(*)$ vanishes.

If $d = \dim(M)$ is odd then :

- either $\mathrm{char}(\mathbb{F}) = 2$ ⟶ ✓ by above
- or $\mathrm{char}(\mathbb{F}) \neq 2$ ⟶ $H_{d-1}(\mathbb{R}P^{d-1}; \mathbb{F}) \underset{(\mathrm{UCT})}{\cong} \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{F}) = 0$ ⟶ ✓

However, if $\mathbb{F} = \mathbb{Q}$ and $d = \dim(M)$ is even,
then $(*) = 2 \in \mathbb{Q} = H_{d-1}(\mathbb{R}P^{d-1}; \mathbb{Q})$,
so (□) does <u>not</u> commute.

Strategy for $\mathbb{F} = \mathbb{Q}$

Transfer maps $H_i C_n(M) \longrightarrow H_i C_{n-1}(M)$

Def.

$$\begin{tikzcd} C_n(M) & \underbrace{C_{n-1,1}}_{\substack{n-1 \text{ blue points}\\ 1 \text{ red point}}}(M) \arrow[l, "\text{forget colours}"', "\text{n-sheeted covering}"] \arrow[r, "\text{forget red point}"] & C_{n-1}(M) \end{tikzcd}$$

$$\begin{tikzcd} H_i C_n(M) \arrow[r, "\text{transfer}"] \arrow[rr, bend right, "\text{formal sum of all ways of forgetting a point}"'] & H_i C_{n-1,1}(M) \arrow[r] & H_i C_{n-1}(M) \end{tikzcd}$$

Lemma (Dold '62)

With coefficients in $\mathbb{Q}$, if M is an <u>open</u> manifold, then

$$H_i(C_n(M);\mathbb{Q}) \xrightarrow{(\text{stab})_*} H_i(C_{n+1}(M);\mathbb{Q}) \xrightarrow{\text{transfer}} H_i(C_n(M);\mathbb{Q})$$

is an automorphism.

Corollary: With $\mathbb{Q}$ coefficients, there is homological stability with respect to the transfer maps.

Idea: Run the same argument, but with <u>transfer</u> maps instead of stabilisation maps.

$\Rightarrow$ It is enough to prove that (with $\mathbb{Q}$ coeffs) the following commutes:

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] & H_{i-1} C_n(M\setminus pt) \\
H_{i-d}(C_n(M\setminus pt)) \arrow[u, "\text{transfer}"] \arrow[r, "\delta_{n+1}^i"'] & H_{i-1} C_{n+1}(M\setminus pt) \arrow[u, "\text{transfer}"']
\end{tikzcd} \qquad (\square')$$

Proof:

"Stacking" thickened spheres $S^{d-1}\times[0,1]$ gives $\coprod_n C_n(S^{d-1}\times[0,1])$ the structure of an H-space, and gluing $S^{d-1}\times[0,1]$ to the boundary of $M\setminus\mathring{D}^d$ gives $\coprod_n C_n(M\setminus\mathring{D}^d)$ the structure of an H-module over it. On homology, we therefore have:

$H_*(\coprod_n C_n(S^{d-1}\times[0,1]))$ —— bigraded ring

$H_*(\coprod_n C_n(M\setminus\mathring{D}^d))$ —— bigraded module over it

The horizontal maps of $(\square')$ are induced by this module structure: they are both multiplication by the element

$$[S^{d-1}]\in H_{d-1}(C_1(S^{d-1}\times[0,1])) \qquad \text{(in bigrading } (d-1,1))$$

↱ of $(\square')$ is given by

$$\alpha \longmapsto [S^{d-1}]\cdot \text{transfer}(\alpha)$$

$$= \sum_{i=1}^{n} [S^{d-1}]\cdot(\text{forget } i^{\text{th}} \text{ point in } \alpha)$$

$\lrcorner\!\!\uparrow$ of (□') is given by

$$\alpha \longmapsto \text{transfer}\left([S^{d-1}]\cdot\alpha\right)$$
$$= \sum_{i=1}^{n} [S^{d-1}]\cdot(\text{forget } i^{th} \text{ point in } \alpha)$$
$$+ [\emptyset]\cdot\alpha$$

where $[\emptyset]$ is the result of forgetting the unique point of $[S^{d-1}] \in H_{d-1}(C_1(S^{d-1}\times[0,1]))$.

(The fundamental class of the empty manifold.)

But this means that $[\emptyset] \in H_{d-1}(\underbrace{C_0(S^{d-1}\times[0,1])}_{=\,*}) = 0$

There is only one empty configuration!

Hence $[\emptyset] = 0$,

so $[\emptyset].\alpha = 0$.

$\Rightarrow$ (□') commutes, as claimed. □.

Summary :

Mapping cone lemma

$\Rightarrow$ with field coeffs, to prove hom. stab.^{y} for $H_i C_n(M)$, it is enough to prove stability for the kernel and cokernel of the map :

$$H_{i-d}(C_{n-1}(M\setminus pt)) \xrightarrow{S_n^i} H_{i-1}\, C_n(M\setminus pt)$$

When $\dim(M)$ is odd or $\mathrm{char}(\mathbb{F}) = 2$

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] \arrow[d, "\text{stabilisation}"'] & H_{i-1}\, C_n(M\setminus pt) \arrow[d, "\text{stabilisation}"] \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] & H_{i-1}\, C_{n+1}(M\setminus pt)
\end{tikzcd} \qquad (\square)$$

$(\square)$ commutes

open manifold

Hence hom. stab.^{y} for $C_n(M\setminus pt)$ implies stability for $\ker(\delta_n^i)$ and $\mathrm{coker}(\delta_n^i)$.

Transfer maps

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M\setminus pt)) \arrow[r, "\delta_n^i"] & H_{i-1}\, C_n(M\setminus pt) \\
H_{i-d}(C_n(M\setminus pt)) \arrow[r, "\delta_{n+1}^i"'] \arrow[u, "\text{transfer}"] & H_{i-1}\, C_{n+1}(M\setminus pt) \arrow[u, "\text{transfer}"']
\end{tikzcd} \qquad (\square')$$

$(\square')$ commutes with any coefficients.

When $\mathrm{char}(\mathbb{F}) = 0$

[Dold] $\Rightarrow$ transfer is a one-sided inverse to stabilisation

Hence hom. stab.^{y} for $C_n(M\setminus pt)$ w.r.t. stabilisation maps

$\Rightarrow$ hom. stab.^{y} for $C_n(M\setminus pt)$ w.r.t. transfer maps

$\Rightarrow$ stability for $\ker(\delta_n^i)$ and $\mathrm{coker}(\delta_n^i)$.

by commutativity of $(\square')$

Remarks

(1) All of the arguments above *use* homological stability for $C_n(M \setminus pt)$ as an *input* — to deduce homological stability results for $C_n(M)$.

(2) The obstruction to commutativity of $(\square)$ is the element $2[\mathbb{RP}^{d-1}] \in H_{d-1}(\mathbb{RP}^{d-1})$. This vanishes also with $\mathbb{Z}$ coefficients when d is odd.

Hence in this case, the argument goes through as far as proving that $\ker(s_n^i)$ and $\mathrm{coker}(s_n^i)$ stabilise.

But then we just have a short exact sequence

$$0 \to \mathrm{coker}(s_n^{i+1}) \longrightarrow H_i C_n(M) \longrightarrow \ker(s_n^i) \to 0$$

of abelian groups, and stability of the outer terms does *not* imply stability of the middle term.

(Recall that we do *not* have maps $H_i C_n(M) \to H_i C_{n+1}(M)$.)

However, it *is* true that $C_n(M)$ is always homologically stable with $\mathbb{Z}$ coefficients when n is odd — this will appear in a later talk, using different methods...

Homological stability for configuration spaces on closed manifolds II

GeMAT seminar
IMAR
28 April 2025

Recall from last time

M —— connected manifold of $\dim(M) = d \geqslant 2$ with $\partial M = \phi$

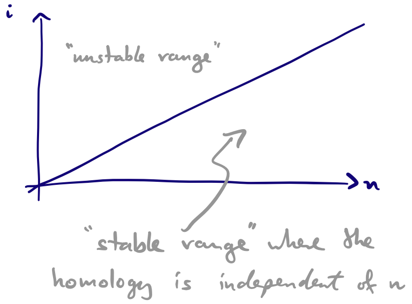
$C_n(M)$ = space of n-point subsets of M

(topologised as a subquotient of M^n)

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open —— $M \cong \mathrm{int}(\bar{M})$, $\partial \bar{M} \neq \phi$

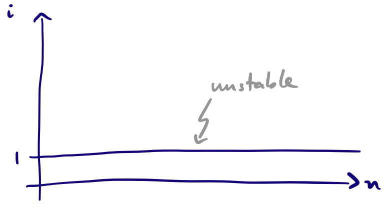
then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geqslant 2i$:



Counterexample [Fadell - van Buskirk '62]

For $M = S^2$, $H_1(C_n(M)) \cong \mathbb{Z}/(2n-2)$

closed —— closed & $\partial M = \phi$



Theorem [Randal-Williams '13]

For any M,

we have $H_i(C_n(M); R) \cong H_i(C_{n+1}(M); R)$ for $n \geqslant 2i$

as long as (1) R = field of char = 2

or (2) R = field of char = 0

or (3) R = any field and $\dim(M)$ is odd.

Aim for today :

Theorem [Bendersky - Miller '14]

For any M,

we have $H_i(C_m(M); R) \cong H_i(C_n(M); R)$ for $m, n \geqslant 2i$

as long as

- when $\dim(M)$ is odd
  (1) $R = \mathbb{Q}$
  (2) $R = \mathbb{Z}_{(p)}$ with $p \geqslant \frac{1}{2}(\dim(M)+3)$
- when $\dim(M)$ is even
  (3) $R = \mathbb{Q}$
  and $m, n \neq \frac{1}{2}\chi(M)$ (if $\chi(M)$ is even)
  (4) $R = \mathbb{Z}_{(p)}$ with $p \geqslant \frac{1}{2}(\dim(M)+3)$
  and $\nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$

i.e. each subsequence $\{ C_n(M) \mid \nu_p(2n - \chi(M)) = k \}$

fixed

is homologically stable.

Tools / ideas of proof / plan of today's talk

(1) Scanning maps ~ will also be crucial in talks #3 and #4

(2) Localisations of spaces

(3) Bundle maps

(4) Obstruction theory

(5) Degree formula

(1) Scanning maps

Idea (Segal '73)

$M = \mathbb{R}^d$ "Electrostatic map"

$\underline{p} = \{p_1, \dots, p_n\} \in C_n(\mathbb{R}^d)$

Think of this as a collection of electrically charged particles of charge $+1$ (positrons).

$\rightsquigarrow$ electric field $E(\underline{p}) : \mathbb{R}^d \setminus \underline{p} \longrightarrow \mathbb{R}^d$.

Extend to $S^d = \mathbb{R}^d \cup \{\infty\} \longrightarrow S^d = \mathbb{R}^d \cup \{\infty\}$

via $p_1, \dots, p_n \longmapsto \infty$

$\infty \longmapsto 0$

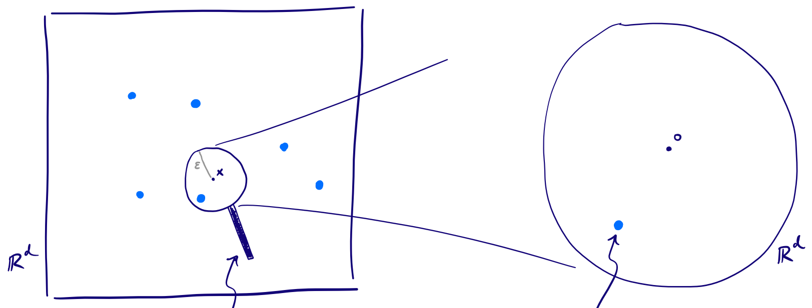
Reflect the target sphere so that $\infty \leftrightarrow 0$.

$\rightsquigarrow$ Based map $S^d \to S^d$

Element of $\mathrm{Map}_*(S^d, S^d) =: \Omega^d S^d$.

This is the <u>scanning map</u> $C_n(\mathbb{R}^d) \longrightarrow \Omega^d S^d$.

Another description (magnifying glasses)



call this $\mathrm{Mag}_x(\underline{p})$ (if it exists)

ε-magnifying glass centred at $x \in \mathbb{R}^d$

i.e. $B_x(\varepsilon)$

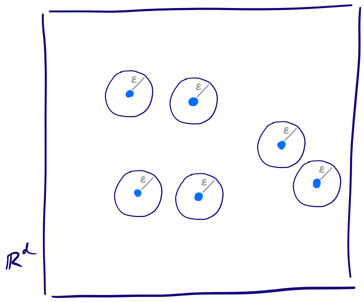
choose continuously depending on $\underline{p}$
(e.g. $\varepsilon = \frac{1}{4} \min\{|p_i - p_j| : i \neq j\}$)
so that it only ever contains <u>at most one particle</u>

$$\underline{p} \rightsquigarrow \left(S^d \xrightarrow{s(\underline{p})} S^d \right)$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if} \quad |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if} \quad | \quad \cdots \quad | = 0 \end{cases}$$

$$C_n(\mathbb{R}^d) \xrightarrow{s} \Omega^d S^d$$

Yet another description (collapsing)



(choose ε as above)

$$\xrightarrow{s(\underline{p})} B_1(0) \big/ \partial B_1(0) \cong S^d$$

- $B_\varepsilon(p_i) \xrightarrow{\cong} B_1(0)$ rescale & translate
- complement $\longrightarrow \{\infty\}$

Rmk $s(\underline{p})$ clearly has degree n:

- $0 \in \mathbb{R}^d \cup \{\infty\} = S^d$ is a regular value
- its pre-image is $\{p_1, \dots, p_n\}$

Notation $\Omega^d_n S^d \subseteq \Omega^d S^d$ space of degree-n based maps $S^d \to S^d$

Rmk In fact $\pi_0(\Omega^d S^d) \xrightarrow[\cong]{\text{degree}} \mathbb{Z}$,

and so $\Omega^d S^d = \coprod_{n \in \mathbb{Z}} \Omega^d_n S^d$.

Target of the scanning map in general

M any smooth manifold

TM = tangent bundle
fibres $= \mathbb{R}^d$

$\dot{T}M$ = "fibrewise 1-point compactified" tangent bundle
fibres $= S^d$

$\Gamma(\dot{T}M)$ = space of sections of $\dot{T}M \longrightarrow M$.
$\rotatebox{90}{$\subseteq$}$
$\Gamma^c(\dot{T}M)$ = subspace of compactly-supported sections, i.e. those that agree with the ∞ section outside a compact subset.

The scanning map will be

$$\begin{tikzcd}
C_n(M) \arrow[r] \arrow[dr] & \Gamma^c(\dot{T}M) \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
 & \Gamma^c_n(\dot{T}M)
\end{tikzcd}$$

sections of degree n

Degree of a section

α, β sections of $\dot{T}M \xrightarrow{\pi} M$ such that $\alpha(p) \neq \beta(p)$ for all $p \in M \setminus K$, where $K \subseteq M$ is compact (*)

Borel-Moore

orientation local system

(1) $[M] \in H_d^{BM}(M;\mathcal{O})$

(2) $\alpha_*[M], \beta_*[M] \in H_d^{BM}(\dot{T}M;\pi^*\mathcal{O})$

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[d] & \pi^*\mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[d] \\
M \arrow[r, "\alpha"] & \dot{T}M \arrow[r, "\pi"] & M
\end{tikzcd}$$

(3) Apply Poincaré duality: —— $\dot{T}M$ is always orientable!

$$\alpha_*[M]^\vee, \beta_*[M]^\vee \in H^d(\dot{T}M;\pi^*\mathcal{O})$$

(4) Cup product

$$\left(\alpha_*[M]^\vee\right) \cup \left(\beta_*[M]^\vee\right) \in H^{2d}_c(\dot{T}M;\mathbb{Z})$$

$\pi^*\mathcal{O} \otimes \pi^*\mathcal{O} \cong \pi^*(\mathcal{O}\otimes\mathcal{O}) \cong \mathbb{Z}$

compactly-supported cohomology because of (*)

(5) Poincaré duality again:

$$\left(\left(\alpha_*[M]^\vee\right) \cup \left(\beta_*[M]^\vee\right)\right)^\vee \in H_0(\dot{T}M;\mathbb{Z}) \cong \mathbb{Z}$$

$\rightsquigarrow$ This is the relative degree $\underline{\mathrm{vdeg}(\alpha,\beta) \in \mathbb{Z}}$.

$$\begin{tikzcd}
{[M]} \arrow[d, phantom, "\in"] & & H_d^{BM}(\dot{T}M;\pi^*\mathcal{O}) \cong H^d(\dot{T}M;\pi^*\mathcal{O}) & & \\
H_d^{BM}(M;\mathcal{O}) \arrow[urr, "\alpha_*"] \arrow[drr, "\beta_*"'] & & & \otimes \arrow[r, "\cup"] & H^{2d}_c(\dot{T}M;\mathbb{Z}) \arrow[d, phantom, "\cong"] \\
 & & H_d^{BM}(\dot{T}M;\pi^*\mathcal{O}) \cong H^d(\dot{T}M;\pi^*\mathcal{O}) & & H_0(\dot{T}M;\mathbb{Z}) \arrow[d, phantom, "\cong"] \\
 & & & & \mathbb{Z}
\end{tikzcd}$$

Rmk: Steps (3)-(5) $\longleftrightarrow$ intersection product

$\longleftrightarrow$ counting (with sign) intersection points between $\alpha(M)$ and $\beta(M)$ after homotoping α and β so that these are transverse submanifolds

Rmk: If α, β don't satisfy $(*)$ then we land in $H^{2d}(\dot{T}M;\mathbb{Z}) \cong H_0^{BM}(\dot{T}M;\mathbb{Z}) = 0$ instead of $H_c^{2d}(\dot{T}M;\mathbb{Z}) \cong H_0(\dot{T}M;\mathbb{Z}) \cong \mathbb{Z}$.

($\longleftrightarrow$ there are ∞ many intersection points, so the intersection # is not well-defined)

Def^n: Let β be a compactly-supported section of $\dot{T}M \to M$.

Set z = zero section.

Then (z,β) satisfy $(*)$, and

$$\deg(\beta) := \mathrm{vdeg}(z,\beta).$$

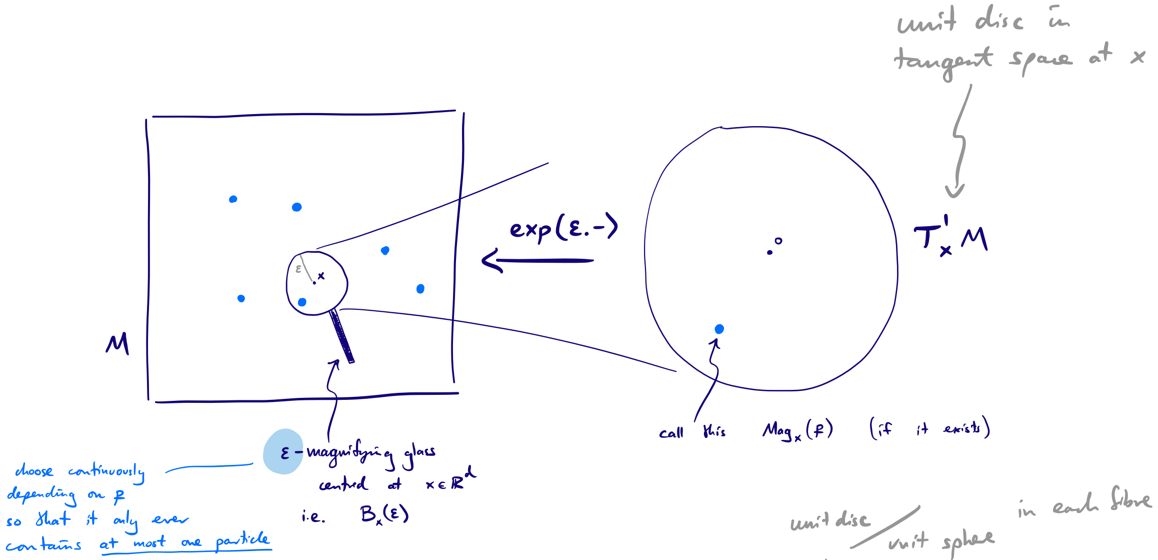
Rmk Similarly to above, $\pi_0(\Gamma^c(\dot{T}M)) \xrightarrow[\cong]{\text{degree}} \mathbb{Z}$,

and so $\Gamma^c(\dot{T}M) = \coprod_{n\in\mathbb{Z}} \Gamma_n^c(\dot{T}M)$.

Scanning any manifold M

Idea Same picture as before.

Choose Riemannian metric on M.



choose continuously depending on $\underline{p}$ so that it only ever contains at most one particle

unit disc / unit sphere in each fibre

$$\underline{p} \rightsquigarrow M \longrightarrow \dot{T}M \cong T'M / \partial T'M$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if} \quad |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if} \quad | \quad \cdots \quad | = 0 \end{cases}$$

Agrees with ∞ outside $\bigcup_i B_\varepsilon(p_i)$ (compact)

Intersection with 0 section is $p_1, \ldots, p_n$ => degree n.

$$C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$$

Note: Rigorous definition (sketch)

- Choose Riemannian metric with injectivity radius bounded below by $\varepsilon > 0$. (Exists by [Greene '78])
- $C_n^\varepsilon(M) :=$ configurations of pairwise distance $> \varepsilon$.
- Decreasing ε if necessary, $C_n^\varepsilon(M) \hookrightarrow C_n(M)$ is a hty equivalence.
- $$\begin{tikzcd} C_n^\varepsilon(M) \arrow[r, "\text{scan}"] \arrow[d, phantom, sloped, "\cong"] & \Gamma_n^c(\dot{T}M) \\ C_n(M) & \end{tikzcd}$$ defined as on the previous page, for ε fixed.

Theorem (McDuff '75 + Segal '79)

The scanning map $C_n(M) \longrightarrow \Gamma_n^c(\dot{T}M)$ induces isomorphisms on $H_i(-;\mathbb{Z})$ for all $n \geq 2i$.

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(2) Localisations of spaces

[Sullivan '70]

$P = \{\text{primes}\}$

Choose $T \subseteq P$

reduced form

Def $\mathbb{Z}_T := \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; b = \text{product of primes in } P \setminus T \right\}$

Ex $\mathbb{Z}_\emptyset = \mathbb{Q}$

Notation $\mathbb{Z}_{(0)} := \mathbb{Z}_\emptyset = \mathbb{Q}$

$\mathbb{Z}_{(p)} := \mathbb{Z}_{\{p\}}$

Def An abelian group A is T-local if it has a structure of a $\mathbb{Z}_T$-module.

Equivalently : Every $a \in A$ may be uniquely divided by p, for each $p \in P \setminus T$.

Rmk Any A has at most one $\mathbb{Z}_T$-module structure.

Def A simply-connected space X is T-local if $H_i(X;\mathbb{Z})$ is T-local $\forall i$.

Equivalently (Theorem) : $\pi_i(X)$ is T-local $\forall i$.

Rmk Can be defined more generally for nilpotent spaces.

Def A T-localisation of a simply-connected space X is a map $f : X \longrightarrow Y$, where Y is T-local and $f_* : H_i(X; \mathbb{Z}_T) \xrightarrow{\cong} H_i(Y; \mathbb{Z}_T)$ $\forall i$.

Equivalently (Theorem) :

- $$\begin{tikzcd} H_i(X;\mathbb{Z}) \arrow[r, "f_*"'] \arrow[dr, "- \otimes \mathbb{Z}_T"'] & H_i(Y;\mathbb{Z}) \arrow[d, phantom, "\cong"] \\ & H_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_T \end{tikzcd}$$

- $$\begin{tikzcd} \pi_i(X) \arrow[r, "f_*"'] \arrow[dr, "- \otimes \mathbb{Z}_T"'] & \pi_i(Y) \arrow[d, phantom, "\cong"] \\ & \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_T \end{tikzcd}$$

Theorem (Sullivan '70)

$\exists$ unique (up to $\simeq$) T-localisation of any simply-connected space X.

(can generalise to nilpotent spaces)

Notation X_T := the T-localisation of X

In particular,

$(T = \phi)$ $X_{(0)}$:= rationalisation of X

$(T = \{p\})$ $X_{(p)}$:= p-localisation of X

Fibrewise localisation

$$\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}$$ fibre bundle with fibre F

$$\longrightarrow \quad \begin{tikzcd} E_T \arrow[d] \\ B \end{tikzcd}$$ fibre bundle with fibre F_T

In particular, for p prime or $p=0$, we have

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] \\ M \end{tikzcd}$$ fibre bundle with fibre $S^d_{(p)}$

From now on, assume that M is compact, so $\Gamma^c(\dot{T}M) = \Gamma(\dot{T}M)$.

Thm (Møller '87)

$\forall n \in \mathbb{Z}$, $\quad \Gamma_n(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M)_{(p)}$ $\quad \left[\pi_0(\Gamma(\dot{T}M_{(p)})) \cong \mathbb{Z}_{(p)}\right]$

Coro:

$\forall n \geq 2i$, $\quad H_i(C_n(M); \mathbb{Z}_{(p)})$

$\cong H_i(\Gamma_n(\dot{T}M); \mathbb{Z}_{(p)})$ [McDuff-Segal]

$\cong H_i(\Gamma_n(\dot{T}M)_{(p)}; \mathbb{Z}_{(p)})$ [Sullivan]

$\cong H_i(\Gamma_n(\dot{T}M_{(p)}); \mathbb{Z}_{(p)})$ [Møller]

Hence it will suffice to prove that

$$\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$$

under the appropriate conditions:

- when $\dim(M)$ is odd:
  - (1) $p = 0$
  - (2) $p \geq \frac{1}{2}(\dim(M)+3)$
- when $\dim(M)$ is even:
  - (3) $p = 0$ and $m, n \neq \frac{1}{2}\chi(M)$
  - (4) $p \geq \frac{1}{2}(\dim(M)+3)$ and $\nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$

Idea: construct bundle self-homotopy-equivalences

$$\begin{tikzcd}
\dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\
& M &
\end{tikzcd}$$

such that the induced self-homotopy-equivalences

$$\Gamma(\dot{T}M_{(p)}) \longrightarrow \Gamma(\dot{T}M_{(p)})$$

send degree-m sections to degree-n sections under the above conditions.

(3) Bundle maps

Def.

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

:= bundle with fibre over $x \in M$ given by self-maps $\dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$ of degree $r \in \mathbb{Z}_{(p)}$

Lemma.

$$\begin{tikzcd} \text{bundle endomorphisms of } \dot{T}M_{(p)} \arrow[d] \arrow[r, leftrightarrow] & \text{sections of } \mathrm{end}(\dot{T}M_{(p)}) \arrow[d] \\ M & M \end{tikzcd}$$

Now fix $\sigma, \tau \in \Gamma(\dot{T}M_{(p)})$:

Def.

$$\begin{tikzcd} \mathrm{end}^{\sigma,\tau}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

:= bundle with fibre over $x \in M$ given by self-maps $\dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$ of degree $r \in \mathbb{Z}_{(p)}$ sending $\sigma(x)$ to $\tau(x)$

Lemma.

$$\begin{tikzcd} \text{bundle endomorphisms } \phi \text{ of } \dot{T}M_{(p)} \arrow[d] \arrow[r, leftrightarrow] & \text{sections of } \mathrm{end}^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M & M \end{tikzcd}$$

such that $\phi \circ \sigma = \tau$

Theorem (Dold '63)

If M is a paracompact manifold,

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$

is a bundle endomorphism

and $\phi_x : E|_x \longrightarrow E|_x$ is a hty equivalence $\forall x \in M$,

then ϕ admits a <u>fibrewise homotopy inverse</u>.

Coro. If $r \in \mathbb{Z}_{(p)}$ is invertible, then

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left] \end{tikzcd} \rightsquigarrow \quad \text{bundle self-homotopy-equivalence} \quad \begin{tikzcd} \dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

So we need to:

(1) Find sections $\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left] \end{tikzcd}$ —— obstruction theory

(2) Understand how the induced $\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$ acts on degrees of sections. —— degree formula

(4) Obstruction theory

$\dim(M) = \text{odd}$

Claim: Assume $p = 0$ or $p \geq \frac{1}{2}(\dim(M)+3)$. Then:

$\forall \sigma, \tau \in \Gamma(\dot{T}M_{(p)})$,

$$\exists \text{ a section of } \begin{tikzcd} \operatorname{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

Proof of main theorem :

We need to prove that $\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$ for any $m, n \in \mathbb{Z}$.

Choose σ of degree m

τ n.

$$\text{Claim} \Rightarrow \text{section of } \begin{tikzcd} \operatorname{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

Coro $\Rightarrow$ bundle self-homotopy-equivalence

previous page

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

such that $\phi \circ \sigma = \tau$

$\Rightarrow$ $\Gamma(\dot{T}M_{(p)}) \simeq \Gamma(\dot{T}M_{(p)})$

restricting to $\Gamma_m(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M_{(p)})$.

$\square$

Proof of Claim:

Assume $p=0$ or $p \geq \frac{1}{2}(\dim(M)+3)$.

Obstruction theory $\Rightarrow$ $\exists$ section as long as certain obstruction classes vanish, which live in the groups

$$H^i(M;\pi_{i-1}(\text{fibre}))$$

of $\mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)})$

$$\begin{tikzcd} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \ni x \end{tikzcd}$$

for $i = 1, \ldots, d = \dim(M)$.

$$\text{fibre} \simeq \left\{ \dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x \ \begin{array}{l}\text{of degree } 1\\ \text{sending } \sigma(x) \text{ to } \tau(x)\end{array} \right\}$$

based

$$\simeq \mathrm{Map}_1^*(S^d_{(p)}, S^d_{(p)})$$

$$\simeq \mathrm{Map}_1^*(S^d, S^d_{(p)}) = \Omega^d_1 S^d_{(p)}$$

path-conn.

Hence for $i \geq 2$,

$$\pi_{i-1}(\text{fibre}) \cong \pi_{d+i-1}(S^d_{(p)})$$

$$\cong \pi_{d+i-1}(S^d)_{(p)}$$

sufficient to prove that this vanishes for $2 \leq i \leq d$.

Theorem (Serre '51) If d is odd,

$$\pi_*(S^d)_{(0)} = 0 \quad \forall * \geq d+1$$

$$\pi_*(S^d)_{(p)} = 0 \quad \forall d+1 \leq * \leq d+2p-4$$

Hence for $2 \leqslant i \leqslant d$,

$$\pi_{i-1}(\text{fibre}) \cong \pi_{d+i-1}(S^d)_{(p)} = 0$$

$p \geqslant \frac{1}{2}(d+3) \longrightarrow d \leqslant 2p-3$

$\Rightarrow \quad d-1+i \leqslant d-1+(2p-3) = d+2p-4$

$\square$

Next time:

dim(M) = even

- obstructions don't vanish!
- instead $\leadsto$ $\exists$ bundle self-htpy-equiv ϕ_r of any fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$

  but we cannot force it to send σ to τ for any two given sections σ, τ
- understand how ϕ_r acts on degrees of sections ......

Homological stability for configuration spaces on closed manifolds III

GeMAT seminar
IMAR
9 May 2025

Reminder of last week's talk

M —— connected manifold of $\dim(M) = d \geq 2$ with $\partial M = \emptyset$

$C_n(M)$ = space of n-point subsets of M

Theorem [Bendersky – Miller '14]

For any M as above, we have:

$$H_i(C_m(M);\mathbb{Q}) \cong H_i(C_n(M);\mathbb{Q}) \quad \text{for } m,n \geq 2i$$

if $m,n \neq \frac{1}{2}\chi(M)$ if $\dim(M)$ and $\chi(M)$ are even;

$$H_i(C_m(M);\mathbb{Z}_{(p)}) \cong H_i(C_n(M);\mathbb{Z}_{(p)}) \quad \text{for } m,n \geq 2i$$

if $p \geq \frac{1}{2}(\dim(M)+3)$

& $\nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$ if $\dim(M)$ is even.

↑

i.e. each subsequence

$\mathcal{C}_k = \{ C_n(M) \mid n = \frac{1}{2}(\chi(M)+p^k a) \text{, } a \text{ coprime to } p \}$

is homologically stable with $\mathbb{Z}_{(p)}$ coefficients.

Steps of the proof

(1) Scanning maps
(2) Localisations of spaces
(3) Bundle maps
(4) Obstruction theory

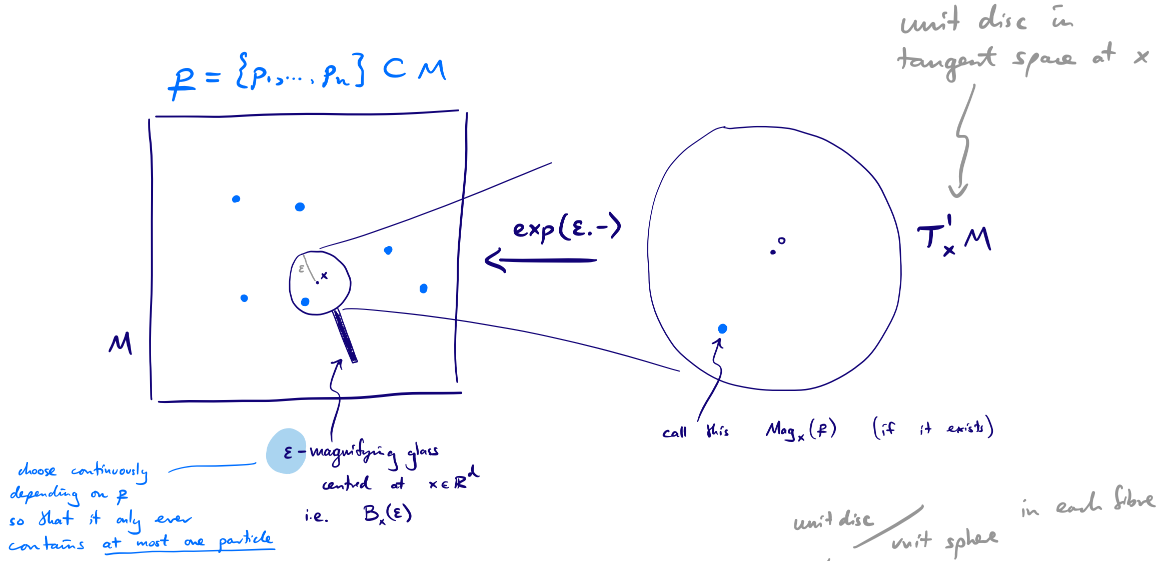
last week (+ reminder today)

(5) Degree formula

today

(1) Scanning maps

Construction (Segal '73, McDuff '75)



unit disc / unit sphere in each fibre

$$\underline{p} \rightsquigarrow \left(M \longrightarrow \dot{T}M \cong T^1M / \partial T^1M \right)$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if } |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if } | \cdots | = 0 \end{cases}$$

Agrees with ∞ outside $\bigcup_i B_\varepsilon(p_i)$ (compact)

Intersection with 0 section is $p_1, \dots, p_n \Rightarrow$ degree n.

$$C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$$

Reminder (Degree)

$$\Gamma^c(\dot{T}M) = \text{space of } \underbrace{\text{compactly-supported}}_{\text{equal to } \infty \text{ section outside cpt subset}} \text{ sections of } \begin{tikzcd} \dot{T}M \arrow[d] \\ M \end{tikzcd}$$

$$= \coprod_{n \in \mathbb{Z}} \Gamma^c_n(\dot{T}M)$$

degree = alg. intersection # of $s_*[M]$ and $\overset{\text{zero-section}}{0_*}[M]$

Theorem (McDuff '75 + Segal '79)

The map $C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$ induces isomorphisms on $H_i(-;\mathbb{Z})$ for $n \geq 2i$.

and hence with coefficients in any abelian group

⟶ Can study $H_*(\Gamma^c_n(\dot{T}M))$ instead of $H_*(C_n(M))$.

From now on, assume that M is compact, so $\Gamma^c(\dot{T}M) = \Gamma(\dot{T}M)$.

(2) Localisations

X simply-connected space

p prime or 0 $\longrightarrow$ $\mathbb{Z}_{(0)} = \mathbb{Q}$

$(p \geq 2)$ $\mathbb{Z}_{(p)} = \{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ coprime to } p \}$

$\downarrow$

p-localisation of X

$$X \longrightarrow X_{(p)}$$

induces isom.'s on $H_i(-;\mathbb{Z}_{(p)})$

simply-connected & integral homology is a $\mathbb{Z}_{(p)}$-module

$\Updownarrow$

$(p=0)$ $\forall$ element is uniquely q-divisible, $\forall$ primes q

$(p \geq 2)$ $\forall$ element is uniquely q-divisible, $\forall$ primes $q \neq p$

Can also be done fibrewise:

$$\begin{tikzcd}
\dot{T}M \arrow[d] \arrow[r] & \dot{T}M_{(p)} \arrow[d] \\
M & M
\end{tikzcd}$$

fibres $= S^d$ fibres $= S^d_{(p)}$

Theorem (Møller '87)

$$\forall n \in \mathbb{Z}, \quad \Gamma_n(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M)_{(p)}$$

Rmk: Path-components of $\Gamma(\dot{T}M_{(p)})$ are indexed by $n \in \mathbb{Z}_{(p)}$.

Corollary

$$\forall n \geq 2i, \quad H_i(C_n(M);\mathbb{Z}_{(p)}) \cong H_i(\Gamma_n(\dot{T}M_{(p)});\mathbb{Z}_{(p)})$$

(3) Bundle maps

Lemma:

$$\begin{tikzcd} \text{Bundle self-maps} & & & \text{Sections} \\ \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] & \mathrm{end}^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ & M \arrow[ul, dashed, bend left, "\sigma"] \arrow[ur, dashed, bend right, "\tau"'] & & M \arrow[u, bend right, ""] \end{tikzcd}$$

such that $\phi \circ \sigma = \tau$ $\quad \longleftrightarrow \quad$ Sections

Def:

$$\begin{tikzcd} \mathrm{end}^{\sigma,\tau}_{r}(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

$:=$ bundle whose fibre over $x \in M$ is the space of self-maps $\dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$

$$\begin{tikzcd} & \simeq S^d_{(p)} \arrow[dl] \arrow[d] \\ \dot{T}M_{(p)}|_x \arrow[r] & \dot{T}M_{(p)}|_x \end{tikzcd}$$

- of degree $r \in \mathbb{Z}_{(p)}$
- sending $\sigma(x)$ to $\tau(x)$

Theorem (Dold '63)

If ϕ restricts to a hty equivalence on each fibre, then ϕ is a fibrewise hty equivalence.

($\Rightarrow$ induces a hty equivalence $\Gamma(\dot{T}M_{(p)}) \to \Gamma(\dot{T}M_{(p)})$.
$\sigma \longmapsto \phi \circ \sigma$)

Upshot

$$\begin{tikzcd} \mathrm{end}^{\sigma,\tau}_{r}(\dot{T}M_{(p)}) \arrow[d] & & \Gamma(\dot{T}M_{(p)}) \arrow[r, "\simeq"] & \Gamma(\dot{T}M_{(p)}) \\ M \arrow[u, bend right] \arrow[rr, squiggly, "\text{induces}", phantom] & & \sigma \arrow[u, phantom, sloped, "\in"] & \tau \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$\mathrm{end}^{\sigma,\tau}_{r}(\dot{T}M_{(p)}) \rightsquigarrow$ induces $\Gamma(\dot{T}M_{(p)}) \xrightarrow{\simeq} \Gamma(\dot{T}M_{(p)})$

$r \in \mathbb{Z}_{(p)}^{\times}$ ← invertible, i.e.

$$\begin{cases} (p=0) & \text{non-}0 \\ (p\geq 2) & \frac{a}{b} \in \mathbb{Q} \quad \text{with } a,b \text{ coprime to } p \end{cases}$$

(4) Obstruction theory

In the talk last week we proved:

Proposition

If $\dim(M) = d$ is odd and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

then $\forall \sigma, \tau \in \Gamma(\dot{T}M_{(p)})$, $\exists$ section

$$\begin{tikzcd} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$

(Idea of proof — Obstruction theory

+ Theorem (Serre '51) If d is odd,

$\pi_*(S^d)_{(0)} = 0 \quad \forall * \geq d+1$

$\pi_*(S^d)_{(p)} = 0 \quad \forall\, d+1 \leq * \leq d+2p-4$)

Corollary

Under these conditions on d and p,

all path-components of $\Gamma(\dot{T}M_{(p)})$ are homotopy equivalent.

Hence if

- $\dim(M) = d$ is odd,
- $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

we have, for all $m, n \geq 2i$:

scanning [McDuff-Segal]
+ localisation [Møller]

$$\begin{tikzcd} H_i(C_m(M); \mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_m(\dot{T}M_{(p)}); \mathbb{Z}_{(p)}) \arrow[d, phantom, "\cong" rotate=90] \\ H_i(C_n(M); \mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_n(\dot{T}M_{(p)}); \mathbb{Z}_{(p)}) \end{tikzcd}$$

$\longrightarrow$ complete the proof when $d = \dim(M)$ is odd.

When $\dim(M) = d$ is even:

- The vanishing results of Serre for $\pi_*(S^d)_{(p)}$ do not hold for even d.

  $\longrightarrow$ can't prove that $\exists$
  $$\begin{tikzcd} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$
  for every $\sigma, \tau \in \Gamma(\dot{T}M_{(p)})$

- Instead:

Proposition:

If $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

then $\forall\, r \in \mathbb{Z}_{(p)}^{\times}$, $\exists$ section
$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$

When $p = 0$, it is unique up to homotopy.

Proof

unbased mapping space

- The fibre of
$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$
is
$$\begin{tikzcd} \mathrm{Map}_r(S^d_{(p)}, S^d_{(p)}) \arrow[d, "\simeq"', "\text{compose with self-map of degree } 1/r"] \\ \mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}) \end{tikzcd}$$

- Obstructions to existence of a section lie in

$$H^i(M; \pi_{i-1}(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))) \qquad (*)$$

But [Serre '51] + some calculations imply that

$\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected.

Hence (*) vanishes for all i, so $\exists$ section.

- Obstructions to uniqueness (up to $\simeq$) of a section lie in

$$H^i(M; \pi_i(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))) \qquad (**)$$

But [Møller-Raussen '85] prove (using minimal models) that

$$\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)}) \simeq S^{2d-1}_{(0)}$$

so in the case $p=0$, $\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)})$ is d-connected.

(because $2d-2 \geq d$)

Hence (**) vanishes for all i, so the section is unique up to $\simeq$.

$\square$

Upshot:

In the even-dimensional case, if $p=0$ or $p \geq \frac{1}{2}(d+3)$,

$\forall$ invertible $r \in \mathbb{Z}_{(p)}$,

$\exists$ bundle self-hty-equivalence

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

of fibrewise degree r.

(Unique when $p=0$.)

(5) Degree formula

Any bundle self-htpy equivalence
$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$
of fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$

induces a self-htpy equivalence $\Gamma(\dot{T}M_{(p)}) \xrightarrow[\phi\circ -]{\simeq} \Gamma(\dot{T}M_{(p)})$.

Its effect on degrees of sections is

$$\begin{tikzcd} \pi_0(\Gamma(\dot{T}M_{(p)})) \arrow[r, "(\phi\circ -)_*", "\cong"'] \arrow[d, "\deg"', "\cong"] & \pi_0(\Gamma(\dot{T}M_{(p)})) \arrow[d, "\deg"', "\cong"] \\ \mathbb{Z}_{(p)} \arrow[r, "\phi_\#"'] & \mathbb{Z}_{(p)} \end{tikzcd}$$

Proposition (degree formula)

If $\dim(M) = d$ is even and $p=0$ or $p \geq \frac{1}{2}(d+3)$,

and $\phi : \dot{T}M_{(p)} \longrightarrow \dot{T}M_{(p)}$ has fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$,

then

$$\phi_\#(k) = rk + \tfrac{1}{2}(1-r)\chi(M)$$

Rmk

This is false for $\dim(M) = d$ odd.

In that case we can find ϕ of fibrewise degree $r=1$

such that $\phi_\#(k) = \ell$ for any two specified $k, \ell \in \mathbb{Z}_{(p)}$.

How this finishes the proof

Assuming that $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

$$\left.\begin{array}{c}\text{obstruction theory}\\ +\\ \text{degree formula}\end{array}\right\} \Longrightarrow \boxed{\begin{array}{ll}\forall r \in \mathbb{Z}_{(p)}^{\times} & \Gamma_k(\dot{T}M_{(p)}) \simeq \Gamma_\ell(\dot{T}M_{(p)}) \\ \forall k, \ell \in \mathbb{Z}_{(p)} & \quad \text{if } \ell = rk + \tfrac{1}{2}(1-r)\chi(M)\end{array}} \quad (*)$$

Now let $m, n \in \mathbb{Z}$, $m, n \geq 2i$

and assume that $\nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$.

when $p = 0$ this is interpreted as $2m - \chi(M) \neq 0 \neq 2n - \chi(M)$.

Then $r := \dfrac{2n - \chi(M)}{2m - \chi(M)} \in \mathbb{Z}_{(p)}^{\times}$

and
$$rm + \frac{1}{2}(1-r)\chi(M) = \frac{(2n - \cancel{\chi(M)})m + \frac{1}{2}(\cancel{2m} - 2n)\chi(M)}{2m - \chi(M)} = \frac{2nm - n\chi(M)}{2m - \chi(M)} = n$$

Hence:

$$\begin{tikzcd}
H_i(C_m(M); \mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_m(\dot{T}M_{(p)}); \mathbb{Z}_{(p)}) \arrow[d, phantom, "\cong" rotate=90] \\
H_i(C_n(M); \mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_n(\dot{T}M_{(p)}); \mathbb{Z}_{(p)})
\end{tikzcd}$$

scanning [McDuff-Segal] + localisation [Møller]

$(*)$ with $r = \dfrac{2n - \chi(M)}{2m - \chi(M)}$

$\square$

Proof of the degree formula

If $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

and $\phi : \dot{T}M_{(p)} \longrightarrow \dot{T}M_{(p)}$ has fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$,

then $\phi_{\#}(k) = rk + \frac{1}{2}(1-r)\,\chi(M)$ (*)

Steps

(1) M orientable $\Rightarrow$ $\forall r \in \mathbb{Z}_{(p)}^{\times}$ $\exists \phi$ of fibrewise degree r satisfying (*)

(2) M orientable $\Rightarrow$ $\phi_{\#}$ depends only on the fibrewise degree of ϕ

$\rightsquigarrow$ degree formula when M orientable

(3) Degree formula when M non-orientable

Rmks

- (3) is easy — just pass to orientation double covers.
- If we just care about orientable manifolds, (1) is sufficient.
- If we care about non-orientable manifolds, we need (2) to deduce (3).
- (2) is obvious when $p = 0$, because in that case ϕ is determined up to $\simeq$ by its fibrewise degree (uniqueness).

Plan for the remainder of the talk:

- Proof of (1),
- Proof of (2).

For the remainder of the talk we assume that M is orientable.

Step 1

$\forall r \in \mathbb{Z}_{(p)}^{\times}$ $\exists\, \phi$ of fibrewise degree r satisfying

$$\phi_{\#}(k) = rk + \tfrac{1}{2}(1-r)\chi(M) \qquad (*)$$

$$\begin{tikzcd} TM \arrow[d] \\ M \end{tikzcd}$$ is classified by a map $M \longrightarrow BSO(d)$

$$\begin{tikzcd} {} \arrow[d, squiggly] \\ {} \end{tikzcd}$$

$$\begin{tikzcd} \dot{T}M \arrow[d] \\ M \end{tikzcd}$$ is classified by a map $$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left] & BSO(d) \arrow[r] & BMap_1(S^d, S^d) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, squiggly] \\ {} \end{tikzcd}$$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] \\ M \end{tikzcd}$$ is classified by a map $$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "\tau"] & BSO(d) \arrow[r] & BMap_1(S^d_{(p)}, S^d_{(p)}) \end{tikzcd}$$

Recall from earlier that $Map_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected,

hence $BMap_1(S^d_{(p)}, S^d_{(p)})$ is d-connected.

$\Rightarrow$ τ is nullhomotopic

$\Rightarrow$ $$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\psi"] \arrow[dr] & & M \times S^d_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

Given $r \in \mathbb{Z}_{(p)}^{\times}$, let

$$\phi := \psi^{-1} \circ (id \times f_r) \circ \psi$$

where $f_r : S^d_{(p)} \to S^d_{(p)}$ is the (unique up to $\simeq$) map of degree r.

Composition with ψ induces

$$\Gamma(\dot{T}M_{(p)}) \cong \Gamma(M \times S^d_{(p)}) \cong \mathrm{Map}(M, S^d_{(p)})$$

Lemma

This restricts, on each path-component, to

$$\Gamma_k(\dot{T}M_{(p)}) \cong \mathrm{Map}_{k-\frac{1}{2}\chi(M)}(M, S^d_{(p)}).$$

End of proof of Step 1:

$$\begin{tikzcd}
\Gamma_k(\dot{T}M_{(p)}) \arrow[r, "\psi\circ -"] \arrow[d, "\phi\circ -"'] & \mathrm{Map}_{k-\frac{1}{2}\chi(M)}(M, S^d_{(p)}) \arrow[d, "f_r\circ -"] \\
\Gamma_\ell(\dot{T}M_{(p)}) & \mathrm{Map}_{r(k-\frac{1}{2}\chi(M))}(M, S^d_{(p)}) \arrow[l, "\psi^{-1}\circ -"']
\end{tikzcd}$$

$$\ell = r\left(k - \tfrac{1}{2}\chi(M)\right) + \tfrac{1}{2}\chi(M)$$
$$= rk + \tfrac{1}{2}(1-r)\chi(M).$$

Proof of Lemma.

$$\psi : \dot{T}M_{(p)} \longrightarrow M \times S^d_{(p)}$$

$$\psi_* : H_d(\dot{T}M_{(p)}) \xrightarrow{\cong} H_d(M \times S^d_{(p)}) \cong \mathbb{Z} \times \mathbb{Z}_{(p)}$$

generated by $[M] \in \mathbb{Z}$
and $[F] \in \mathbb{Z}_{(p)}$

intersection number $\langle -,- \rangle : H_d(\dot{T}M_{(p)}) \times H_d(\dot{T}M_{(p)}) \longrightarrow \mathbb{Z}_{(p)}$

$$\begin{tikzcd}
\dot{T}M_{(p)} \arrow[d, shift left] \\
M \arrow[u, shift left, "0"]
\end{tikzcd}$$

$$\psi_*(0_*[M]) = [M] + k[F] \quad \text{for some } k \in \mathbb{Z}_{(p)}.$$

$$\chi(M) = \langle 0_*[M], 0_*[M] \rangle$$
$$= \langle \psi_*^{-1}[M] + k\psi_*^{-1}[F] , \psi_*^{-1}[M] + k\psi_*^{-1}[F] \rangle$$
$$= 2k$$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

$$\psi_*(\sigma_*[M]) = [M] + \ell[F] \quad \text{for some } \ell \in \mathbb{Z}_{(p)}.$$

$$\deg(\psi\circ\sigma) = \ell$$

$$\begin{aligned}\deg(\sigma) &= \langle 0_*[M], \sigma_*[M]\rangle \\ &= \left\langle \psi_*^{-1}[M] + \tfrac{\chi(M)}{2}\psi_*^{-1}[F] \,,\, \psi_*^{-1}[M] + \ell\psi_*^{-1}[F] \right\rangle \\ &= \tfrac{\chi(M)}{2} + \ell\end{aligned}$$

Hence $\deg(\psi\circ\sigma) = \deg(\sigma) - \frac{1}{2}\chi(M)$. □

Step 2

Any two bundle self-htpy equivalences

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "{\phi,\,\phi'}"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M \arrow[ul, bend left, "\sigma"] & \end{tikzcd}$$

of the same fibrewise degree $v \in \mathbb{Z}_{(p)}^{\times}$ act in the same way on degrees of sections:

$$\forall \text{ section } \sigma, \quad \deg(\phi'\circ\sigma) = \deg(\phi\circ\sigma).$$

Proof

Let ϕ^{-1} denote a fibrewise htpy inverse of ϕ.

⟶ Enough to show that $\underbrace{(\phi'\circ\phi^{-1})}_{\overset{\|}{\theta}}\circ -$ preserves degrees of sections.

$\theta \longleftarrow$ fibrewise degree $= 1$.

Degree of a section is calculated in $H_d(\dot{T}M_{(p)})$.

$\rightsquigarrow$ Enough to show that $\theta_* : H_d(\dot{T}M_{(p)}) \longrightarrow H_d(\dot{T}M_{(p)})$

is the <u>identity</u>.

Use a trivialisation

$$\begin{tikzcd}
\dot{T}M_{(p)} \arrow[rr, "\psi"] \arrow[dr] & & M \times S^d_{(p)} \arrow[dl] \\
& M &
\end{tikzcd}$$

to write

$$\begin{tikzcd}
H_d(\dot{T}M_{(p)}) \arrow[r, phantom, "\cong"] \arrow[d, "\theta_*"'] & H_d(M \times S^d_{(p)}) \arrow[r, phantom, "\cong"] & \mathbb{Z} \times \mathbb{Z}_{(p)} \arrow[d, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"] \\
H_d(\dot{T}M_{(p)}) \arrow[r, phantom, "\cong"] & H_d(M \times S^d_{(p)}) \arrow[r, phantom, "\cong"] & \mathbb{Z} \times \mathbb{Z}_{(p)}
\end{tikzcd}$$

Now θ is a bundle map $\rightsquigarrow$ $\alpha = 1$

$\beta = 0$

Also, δ = fibrewise degree of θ

$= 1$

Hence $\theta_* = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ for some $\gamma \in \mathbb{Z}_{(p)}$.

Lemma $\boxed{\exists N : \theta^N \simeq \text{id}.}$

Hence $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}^N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\quad\|$

$\begin{pmatrix} 1 & 0 \\ N\gamma & 1 \end{pmatrix}$

$\Rightarrow N\gamma = 0$

$\Rightarrow \gamma = 0 \quad \Rightarrow \theta_* = \text{id}.$ $\square$

Proof of lemma

(a bit more technical...)

$$\begin{array}{l} \theta \in \text{space of bundle maps of degree } 1 \\ \qquad \| \\ \qquad \Gamma(\mathrm{end}_1(\dot{T}M_{(p)})) \\ \qquad\qquad \uparrow \\ \qquad\qquad \text{topological monoid} \end{array}$$

$$[\theta] \in \pi_0(\Gamma(\mathrm{end}_1(\dot{T}M_{(p)}))) \longleftarrow \text{monoid}$$

Aim: $[\theta]$ has finite order.

Actually will prove that the whole monoid is a torsion group.

For short, write $\begin{array}{c} P = \mathrm{end}_1(\dot{T}M_{(p)}) \\ \downarrow \\ M \end{array}$, so our aim is

to show that the monoid $\pi_0(\Gamma(P))$ is a torsion group.

① Choose a CW-structure on M with a single d-cell.

$M^{d-1} :=$ $(d-1)$-skeleton of this CW-structure.

Denote the restriction of $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ to M^{d-1} by $\begin{array}{c} P^{d-1} \\ \downarrow \\ M^{d-1} \end{array}$.

② Obstructions to vanishing of $\pi_0(\Gamma(P^{d-1}))$ lie in

$$H^i(M^{d-1}; \pi_i(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))).$$

Since
- $\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected,
- M^{d-1} is $(d-1)$-dimensional,

these vanish and hence $\pi_0(\Gamma(P^{d-1})) = *$.

③ $M^{d-1} \hookrightarrow M$ is a cofibration

$\Rightarrow$ restriction : $\Gamma(P) \longrightarrow \Gamma(P^{d-1})$ is a fibration.

Fibre $= \left\{ \text{extensions of a section } \begin{tikzcd} P^{d-1} \arrow[d] \\ M^{d-1} \arrow[u, bend left, "s"] \end{tikzcd} \text{ to } \begin{tikzcd} P \arrow[d] \\ M \end{tikzcd} \right\}$

$$\simeq \Omega^d \left(\text{fibre of } \begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix} \right)$$

$$= \Omega^d \mathrm{Map}_1 (S^d_{(p)}, S^d_{(p)})$$

$$\begin{tikzcd} Q^{d-1} \arrow[r, phantom, "\subset"] \arrow[d] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P^{d-1} \arrow[d] \\ S^{d-1} \arrow[u, bend left, "s'"] \arrow[r, phantom, "\subset"] & D^d \arrow[r] & M^{d-1} \arrow[u, bend right, "s"'] \end{tikzcd}$$

characteristic map of the d-cell

extension of s to $\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}$

trivialisable $\cong D^d \times F$

$\longleftrightarrow$ extension of s' to $\begin{smallmatrix} Q \\ \downarrow \\ D^d \end{smallmatrix}$

$\longleftrightarrow$ extension of $s' : S^{d-1} \longrightarrow F$ to $D^d \longrightarrow F$

$\longleftrightarrow$ based maps $D^d / S^{d-1} \longrightarrow F$

$D^d / S^{d-1} \cong S^d$

Hence we have a fibre sequence:

$$\Omega^d \mathrm{Map}_1 (S^d_{(p)}, S^d_{(p)}) \longrightarrow \Gamma(P) \longrightarrow \Gamma(P^{d-1})$$

④ This induces an exact sequence of monoids:

$$\pi_0 \left(\Omega^d \mathrm{Map}_1 (S^d_{(p)}, S^d_{(p)}) \right) \xrightarrow{(*)} \pi_0 (\Gamma(P)) \longrightarrow \pi_0 (\Gamma(P^{d-1}))$$

⑤ This has <u>two</u> monoid structures:

- composition of maps,
- concatenation of (d-fold) loops.

This one makes $(*)$ into a monoid homomorphism

This one allows the identification with $\pi_d (\mathrm{Map}_1 (S^d_{(p)}, S^d_{(p)}))$

By the Eckmann-Hilton argument, they agree, and hence we have:

$$\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})) \longrightarrow \pi_0(\Gamma(P)) \longrightarrow \pi_0(\Gamma(P^{d-1}))$$

(exact sequence of monoids).

⑥ By above, $\pi_0(\Gamma(P^{d-1})) = *$,

so $\pi_0(\Gamma(P))$ = quotient of $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$

(in particular it is a group)

so it is enough to show that $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$ is a torsion group.

⑦ Recall from earlier that

$$\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)}) \simeq S^{2d-1}_{(0)}$$ [Møller-Raussen '85]

so it is d-connected (since $2d-2 \geq d$),

$$\text{so} \quad 0 = \pi_d(\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)}))$$

$$\cong \pi_d(\mathrm{Map}_1(S^d, S^d)_{(0)})$$

$$\cong \pi_d((\mathrm{Map}_1(S^d, S^d)_{(p)})_{(0)})$$

$$\cong \pi_d(\mathrm{Map}_1(S^d, S^d)_{(p)}) \otimes \mathbb{Q}$$

$$\cong \pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})) \otimes \mathbb{Q}$$

and hence $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$ is a torsion group. $\square$

Homological stability for configuration spaces on closed manifolds IV

GeMAT seminar
IMAR
16 May 2025

Reminder

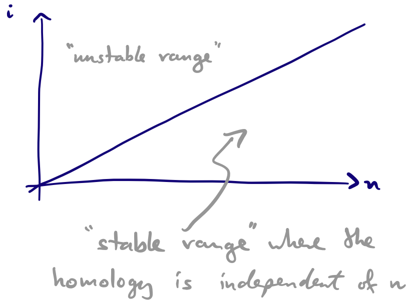
M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open — $M \cong \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$

then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.



"stable range" where the homology is independent of n

compact
$\partial M = \phi$

This is false for closed manifolds M.

E.g. $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/_{(2n-2)}$.

What we have seen so far:

Theorem (Combining [Randal-Williams '13] & [Bendersky-Miller '14])

Let $m, n \geq 2i$. Then

$$H_i(C_m(M); R) \cong H_i(C_n(M); R)$$

as long as:

- R is a field and either $\dim(M)$ is odd or $\mathrm{char}(R) \in \{0,2\}$;
- $R = \mathbb{Z}_{(p)}$ with $p \geq \frac{1}{2}(\dim(M)+3)$ and either $\dim(M)$ is odd or $\frac{2m-\chi(M)}{2n-\chi(M)} \in \mathbb{Z}_{(p)}^{\times}$

$\Longleftrightarrow \nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$

Aim for today:

Thm B • Remove the $p \geq \frac{1}{2}(\dim(M)+3)$ hypothesis

↳ obstruction theory + more geometric construction.

Thm A • Introduce "replication maps"

↳ stability with respect to replication maps instead of stabilisation maps.

$\Downarrow$

Thm C • Replication maps + puncturing trick of [Randal-Williams '13]

$\Rightarrow$ "replication stability" with field coeffs

Coro D • Combining Thm B + Thm C $\longrightarrow$ homological periodicity for $H_*(C_n(M); \mathbb{F}_p)$.

① Replication maps

Theorem A [Cantero - P. '15]

$\Longleftarrow$ • M is open

OR • M is closed + $\chi(M)=0$

If M admits a non-vanishing vector field,

then $\forall r \geq 2$ $\exists$ "replication map" $C_n(M) \xrightarrow{\rho_r} C_{rn}(M)$

inducing isomorphisms on $H_i(-;\mathbb{Z}_{(p)})$

for all $n \geq 2i$

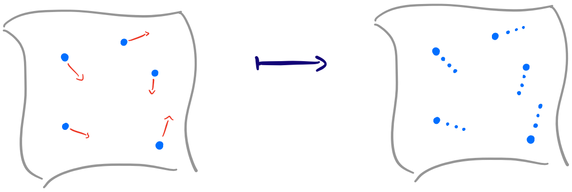
and $p \nmid r$.

Def Let

$$\begin{tikzcd} TM \arrow[d] \\ M \arrow[u, bend left, "v"] \end{tikzcd}$$

be a non-vanishing vector field.

Then ρ_r :



(picture for $r=4$, $n=5$)

Corollary If $\chi(M)=0$, then for fixed i and p,

$H_i(C_n(M);\mathbb{Z}_{(p)})$

depends only on $\nu_p(n)$ in the range $n \geq 2i$.

Proof:

If $m,n \geq 2i$ and $m = p^a \bar{m}$ with $\bar{m}, \bar{n}$ coprime to p (i.e. $\nu_p(m)=\nu_p(n)$)

$n = p^a \bar{n}$

then

$$\begin{tikzcd} C_m(M) \arrow[dr, "\rho_{\bar{n}}"] & \\ & C_{p^a \bar{m}\bar{n}}(M) \\ C_n(M) \arrow[ur, "\rho_{\bar{m}}"'] & \end{tikzcd}$$

induce $\cong$ on $H_i(-;\mathbb{Z}_{(p)})$. □

In general, for any $\chi(M)$:

Theorem B [Cantero - P. '15]

- If $\dim(M)$ is odd, then

$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}])$$
$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$

in the range $n \geq 2i$.

- If $\dim(M)$ is even, then for fixed i and p, (assume $\chi(M)$ even if $p=2$)

$$H_i(C_n(M);\mathbb{Z}_{(p)})$$

depends only on $\nu_p(2n-\chi(M))$ in the range $n \geq 2i$.

For $p \geq \frac{1}{2}(\dim(M)+3)$ this was [Bendersky - Miller '14].

Idea of proof

Recall that

$$\begin{tikzcd} C_n(M) \arrow[r, "\text{scan}"] & \Gamma^c_n(\dot{T}M) \arrow[r] & \Gamma^c_n(\dot{T}M)_{(p)} \arrow[r, phantom, "\simeq"] & \Gamma^c_n(\dot{T}M_{(p)}) \end{tikzcd}$$

H_* isom. in degrees $\leq n/2$
[McDuff, Segal]

localisation [Sullivan '70]
isom. on $H_*(-;\mathbb{Z}_{(p)})$

[Møller '87]

n^{th} path-component
of $\Gamma^c(\dot{T}M_{(p)})$

$$\begin{tikzcd} \mathrm{Map}_r(S^d_{(p)}, S^d_{(p)}) \arrow[d] & & & \\ \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \arrow[r, leftrightarrow] & \dot{T}M_{(p)} \arrow[rr, "\varphi"', "\text{fibiewise degree } r"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \arrow[r] & \Gamma^c(\dot{T}M_{(p)}) \arrow[r, "\varphi \circ -"] & \Gamma^c(\dot{T}M_{(p)}) \\ M \arrow[u, bend left, "\sigma"] & & M & & & \end{tikzcd}$$

<u>Thm</u> [Dold '63]

If $r \in \mathbb{Z}_{(p)}$ is <u>invertible</u>

then φ is a fibiewise self-htz-equivalence

and so $\varphi \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ is a self-htz-equivalence.

<u>Strategy</u>:

- Construct sections $\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$ for $r \in \mathbb{Z}^{\times}_{(p)}$

- Understand how the corresponding $\varphi \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ acts on π_0.

[Berdesky - Miller '14] : $\left.\begin{array}{l} \text{obstruction theory} \\ p \geq \frac{1}{2}(d+3) \end{array}\right\} \longrightarrow \exists$ of σ

New idea : geometric construction
(+ different obstruction theory)

Def

$V_2(\mathbb{R}^{d+1}) :=$ space of orthonormal 2-frames in $\mathbb{R}^{d+1}$

(Stiefel manifold)

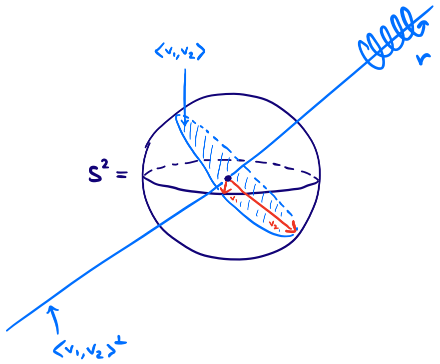
$r \in \mathbb{Z}$

$$\phi_r : V_2(\mathbb{R}^{d+1}) \longrightarrow \mathrm{Map}_r(S^d, S^d)$$

$$(v_1,v_2) \longmapsto \mathbb{R}^{d+1} = \underbrace{\langle v_1,v_2\rangle}_{\substack{\cong \\ \mathbb{C} \\ \circlearrowleft \\ z \mapsto z^r}} \oplus \underset{\substack{\circlearrowleft \\ \mathrm{id}}}{\langle v_1,v_2\rangle^\perp}$$

restricted to S^d = unit sphere in $\mathbb{R}^{d+1}$

Picture ($d=2$):



"Neutron star construction"

$(\phi_r(v_1,v_2)(v_1) = v_1)$

This works fibrewise:

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$ real inner product bundle of rank d

$$\begin{tikzcd} V_2(E \oplus \mathbb{R}) \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{E}) \arrow[dl] \\ & M & \end{tikzcd}$$

1-dim trivial bundle

fibrewise 1-pt compact^n of E

Note: $\dot{E} \cong$ unit sphere bundle of $E \oplus \mathbb{R}$

Applying to $E = TM$ we get

$$\begin{tikzcd}
V_2(TM \oplus \mathbb{R}) \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{T}M) \arrow[dl] \\
& M &
\end{tikzcd}$$

and we can fibrewise localise to get

$$\begin{tikzcd}
V_2(TM \oplus \mathbb{R})_{(p)} \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[dl] \\
& M &
\end{tikzcd}$$

Idea: A section $\begin{tikzcd} V_2(TM \oplus \mathbb{R})_{(p)} \arrow[d, shift left] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$

induces a section $\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d, shift left] \\ M \arrow[u, bend left, "\phi_r \circ \sigma"] \end{tikzcd}$

and hence a self-map

$$(\phi_r \circ \sigma)_* \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$$

for every $r \in \mathbb{Z}$.

Small print: to ensure the target is Γ^c and not just Γ, we have to assume that σ_0 (see below) is compactly-supported.

Notation: By forgetting the second component of a 2-frame,

$$\begin{tikzcd} V_2(TM\oplus\mathbb{R})_{(p)} \arrow[d, shift left] \arrow[r, squiggly] & V_1(TM\oplus\mathbb{R})_{(p)} = \dot{T}M_{(p)} \arrow[d, shift left] \\ M \arrow[u, bend left, "\sigma"] & M \arrow[u, bend left] \end{tikzcd}$$

Call this σ_0 and assume it is compactly-supported

$\rightsquigarrow \deg(\sigma_0) \in \mathbb{Z}_{(p)}$

Lemma (degree formula)

For any σ such that σ_0 is compactly-supported, the self-map $(\phi_r \circ \sigma)_* \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ sends sections of degree k to sections of degree $r(k-\deg(\sigma_0)) + \deg(\sigma_0)$.

Idea of proof

By passing to orientation double covers, assume M orientable.

If M is non-compact, interpret $H_*(\dot{T}M_{(p)})$ as horizontally locally finite homology $H_*^{hlf}(\dot{T}M_{(p)})$, so that inclusions of fibres and sections induce $H_*(S^d_{(p)}) \to H_*^{hlf}(\dot{T}M_{(p)})$ and $H_*^{BM}(M) \to H_*^{hlf}(\dot{T}M_{(p)})$.

$$\begin{tikzcd}[column sep=small] \text{Then} \quad H_d(\dot{T}M_{(p)}) \cong & \mathbb{Z} & \oplus & \mathbb{Z}_{(p)} & \\ \text{zero section} \arrow[r, squiggly] & 0_*[M] \arrow[u] & & & {[F] = [S^d_{(p)}]} \arrow[ul] \end{tikzcd}$$

$$\text{intersection form} = \begin{pmatrix} \chi(M) & (-1)^d \\ 1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d, shift left] & & \\ M \arrow[u, bend left, "\alpha"] \arrow[r, phantom, ""] & \arrow[r, squiggly] & \alpha_*[M] = (1, \deg(\alpha) - \chi(M)) \end{tikzcd}$$

By construction,

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "(\phi_r \circ \sigma)_*"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

- preserves σ_0
- has fibrewise degree r

So $H_d(\dot{T}M_{(p)}) \longrightarrow H_d(\dot{T}M_{(p)})$

- preserves $(\sigma_0)_*[M] = (1, \deg(\sigma_0) - \chi(M))$
- multiplies $[F] = (0,1)$ by r

$\Rightarrow$ it is given by $\begin{pmatrix} 1 & 0 \\ (1-r)(\deg(\sigma_0) - \chi(M)) & r \end{pmatrix} =: A$

Set $k = \deg(\alpha)$

$\ell = \deg((\phi_r \circ \sigma)_* \circ \alpha)$

$$\longrightarrow \quad \begin{array}{l} \alpha_*[M] = (1, k - \chi(M)) \\ (\phi_r \circ \sigma)_*(\alpha_*(M)) = (1, \boxed{\ell - \chi(M)}) \\ \quad \| \\ A.(1, k - \chi(M)) \\ \quad \| \\ (1, \boxed{r(k - \chi(M)) + (1-r)(\deg(\sigma_0) - \chi(M))}) \end{array}$$

$$\boxed{\ell - \chi(M)} = \boxed{r(k - \chi(M)) + (1-r)(\deg(\sigma_0) - \chi(M))}$$

Rearranging $\Longrightarrow$ $\ell = r(k - \deg(\sigma_0)) + \deg(\sigma_0)$.

$\square$

Summary :

$$\begin{tikzcd}
V_2(TM\oplus\mathbb{R})_{(p)} \arrow[dr] \arrow[dd] & \\
& V_1(TM\oplus\mathbb{R})_{(p)} = \dot{T}M_{(p)} \arrow[dl] \\
M \arrow[uu, dashed, bend left, "\sigma"] \arrow[ur, bend right, "\sigma_0"'] &
\end{tikzcd}$$

degree $= \delta \in \mathbb{Z}_{(p)}$

If we can lift σ_0 to σ

then $\forall r \in \mathbb{Z}$ we have $\Gamma^c(\dot{T}M_{(p)}) \xrightarrow{(*)} \Gamma^c(\dot{T}M_{(p)})$

acting on path-components (degrees of sections) by

$$k \longmapsto r(k-\delta)+\delta$$

and if $r \in \mathbb{Z} \setminus p\mathbb{Z}$ then $(*)$ is a htpy equivalence.

Lemma

The complete obstruction to lifting σ_0 to σ is

$$\begin{cases} 0 & \text{if } \dim(M) \text{ odd} \\ 2\delta - \chi(M) & \text{if } \dim(M) \text{ even} \end{cases}$$

Euler class of pullback of $\begin{array}{c} V_2(TM\oplus\mathbb{R})_{(p)} \\ \downarrow \\ \dot{T}M_{(p)} \end{array}$ along $M \xrightarrow{\sigma_0} \dot{T}M_{(p)}$

Upshot:

$\dim(M)$ odd $\longrightarrow$ choose $\left.\begin{array}{l}\delta \in \mathbb{Z}_{(p)} \\ r \in \mathbb{Z}\setminus p\mathbb{Z}\end{array}\right\}$ freely

$\dim(M)$ even $\longrightarrow$ must have $\delta = \frac{1}{2}\chi(M)$

but $r \in \mathbb{Z}\setminus p\mathbb{Z}$ can be chosen freely

Obs: if $p=2$ and $\chi(M)$ is odd then this does not exist!

$$(*) \quad \Gamma^c_n(\dot{T}M_{(p)}) \simeq \Gamma^c_{r(n-\delta)+\delta}(\dot{T}M_{(p)})$$

Proof of Theorem B

$\dim(M)$ odd

p odd $\longrightarrow$ choose $r = 2$, $\delta = n-1$

$$\Rightarrow \quad r(n-\delta)+\delta = n+1 \quad \text{in } (*)$$

$\forall p \longrightarrow$ choose $r = -1$, $\delta = n+1$

$$\Rightarrow \quad r(n-\delta)+\delta = n+2 \quad \text{in } (*)$$

$\dim(M)$ even

If $\nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$

$$2m - \chi(M) = p^a k \qquad\qquad k,l \in \mathbb{Z}\setminus p\mathbb{Z}$$
$$2n - \chi(M) = p^a l$$

$$\Gamma^c_m(\dot{T}M_{(p)}) \underset{(*)\ r=l}{\simeq} \Gamma^c_{\frac{1}{2}(p^a kl+\chi(M))}(\dot{T}M_{(p)}) \underset{(*)\ r=k}{\simeq} \Gamma^c_n(\dot{T}M_{(p)})$$

$\square$

Proof of Theorem A

Non-vanishing vector field

$$\begin{tikzcd} TM \arrow[d] \\ M \arrow[u, bend left, "v"] \end{tikzcd}$$

$\longrightarrow$ section

$$\begin{tikzcd} V_2(TM \oplus \mathbb{R}) \arrow[d] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd} \qquad \sigma(x) = \begin{pmatrix} (v(x), 0) \\ (0,1) \end{pmatrix}$$

Note: - No need for obstruction theory

- Exists <u>before</u> fibrewise localisation

$\longrightarrow$ σ_0 = ∞-section of $\dot{T}M$

$\Rightarrow$ $\deg(\sigma_0) = 0$

Degree formula $\longrightarrow$ induced $\Gamma^c(\dot{T}M) \longrightarrow \Gamma^c(\dot{T}M)$ acts on π_0 by

$$k \longmapsto r(k - \deg(\sigma_0)) + \deg(\sigma_0) = rk$$

Localising, if $p \nmid r$ we have:

$$\begin{tikzcd} C_n(M) \arrow[r, "\text{scan}"] \arrow[d, "\rho_r"'] & \Gamma^c_n(\dot{T}M) \arrow[r, "\text{loc.}"] \arrow[d] & \Gamma^c_n(\dot{T}M)_{(p)} \arrow[r, phantom, "\simeq"] \arrow[d] & \Gamma^c_n(\dot{T}M_{(p)}) \arrow[d, "\simeq"] \\ C_{rn}(M) \arrow[r, "\text{scan}"] & \Gamma^c_{rn}(\dot{T}M) \arrow[r, "\text{loc.}"] & \Gamma^c_{rn}(\dot{T}M)_{(p)} \arrow[r, phantom, "\simeq"] & \Gamma^c_{rn}(\dot{T}M_{(p)}) \end{tikzcd}$$

$\square$

② Puncturing trick & homological periodicity

Theorem C [Cantero-P. '15]

M — closed, even-dimensional manifold
$\mathbb{F}$ — field of characteristic $p > 0$
$r \geq 2$ —— $p \nmid r$
$p \mid (\chi(M)-1)(r-1)$ } in particular whenever $r \equiv 1 \pmod p$

Then

$$H_i(C_n(M);\mathbb{F}) \cong H_i(C_{rn}(M);\mathbb{F})$$

for $n \geq 2i$.

Note: There are no direct replication maps $C_n(M) \longrightarrow C_{rn}(M)$ unless $\chi(M) = 0$.

When $\chi(M) \neq 0$ the "dissonance" between Thm B + Thm C implies:

Corollary D [Cantero-P. '15] (Homological periodicity)

M — closed, even-dimensional manifold with $\chi(M) \neq 0$
$\mathbb{F}$ — field of characteristic $p \geq 3$

Then

$$H_i(C_n(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})$$

for $n \geq 2i$, where

$$q = p^{a+1}$$
$$a = \nu_p(\chi(M))$$

Note: If $\chi(M) = 0$ then $\nu_p(0) = \infty$, so $q = \infty$.
If $\mathrm{char}(\mathbb{F}) \in \{0,2\}$ then we already know homological stability, so $q = 1$.

Proof of Conj D

Case 1 $\nu_p(2n-x) \leqslant \nu_p(x) = a$

Then $\nu_p(2(n+q)-x)$

$$= \nu_p(\underbrace{2n-x}_{\leqslant a} + \underbrace{2p^{a+1}}_{a+1}) = \nu_p(2n-x)$$

Theorem B $\longrightarrow$ $H_i(C_n(M);\mathbb{Z}_{(p)}) \cong H_i(C_{n+q}(M);\mathbb{Z}_{(p)})$

$$\Rightarrow \quad H_i(C_n(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})$$

(UCT, since $\mathbb{F}$ is a $\mathbb{Z}_{(p)}$-module)

Case 2 $\nu_p(2n-x) > \nu_p(x) = a$

Then $\nu_p(n) = \nu_p(\underbrace{2n-x}_{>a} + \underbrace{x}_{a}) = a$

$$\Rightarrow \quad \frac{n}{p^a} \not\equiv 0 \pmod p$$

$$\Rightarrow \quad \exists\, \ell \geqslant 2 \ : \ \frac{\ell n}{p^a} \equiv 1 \pmod p$$

$$\ell\left(\frac{n+q}{p^a}\right) = \frac{\ell n}{p^a} + \ell p \equiv 1 \pmod p$$

$\Rightarrow$ we may take $r = \frac{\ell n}{p^a}$, $r = \ell\left(\frac{n+q}{p^a}\right)$ in Theorem C

$$H_i(C_n(M);\mathbb{F}) \cong H_i(C_{\frac{\ell n(n+q)}{p^a}}(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})$$

□

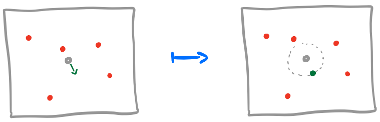
Proof of Theorem C

Recall from talk 1 (following [Randal-Williams '13]) that for any manifold M and basepoint $0 \in M$ there is a LES

$$\begin{tikzcd} \cdots \arrow[r] & H_i C_n(M) \arrow[r] & H_{i-d}(C_{n-1}(M\setminus 0)) \arrow[r, "s_n^i"'] & H_{i-1} C_n(M\setminus 0) \arrow[r] & H_{i-1} C_n(M) \arrow[r] & \cdots \end{tikzcd}$$

induced by

$$S^{d-1} \times C_{n-1}(M\setminus 0) \xrightarrow{t_n} C_n(M\setminus 0)$$



expand radially away from $0 \in M$

add a point specified by the S^{d-1} parameter

So if $H_i = H_i(-;\mathbb{F})$

then $\dim H_i C_n(M) = \dim \ker(s_n^i) + \dim \operatorname{coker}(s_n^{i+1})$

So it's enough to show that the maps

$$s_n^i : H_{i-d}(C_{n-1}(M\setminus 0)) \longrightarrow H_{i-1} C_n(M\setminus 0)$$

stabilise for $n \geq 2i$.

Aim: Construct maps

$$\begin{tikzcd} S^{d-1} \times C_{n-1}(M\setminus 0) \arrow[r, "t_n"] \arrow[d, "f"'] & C_n(M\setminus 0) \arrow[d, "g"] \\ S^{d-1} \times C_{rn-1}(M\setminus 0) \arrow[r, "t_{rn}"'] & C_{rn}(M\setminus 0) \end{tikzcd} \qquad (*)$$

Such that

- The induced vertical maps are isom.'s on H_i for $n \geq 2i$ (after passing to one summand in the Künneth decomposition on the LHS)

- (*) commutes on $H_*(-;\mathbb{F})$

Construction

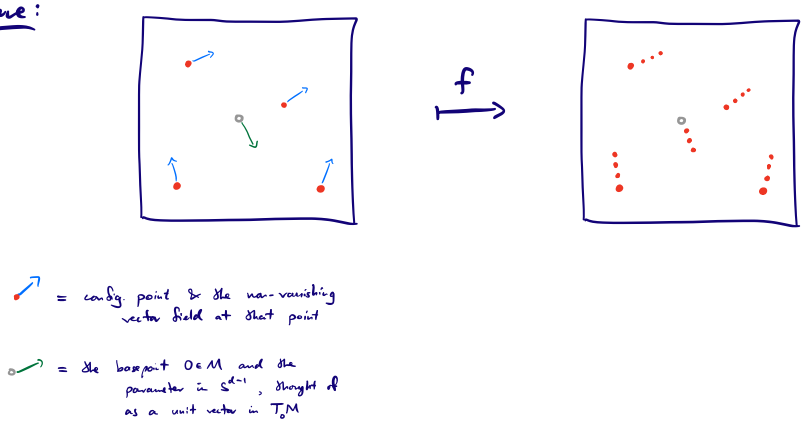
Choose a vector field on M with a unique zero at $0 \in M$.

$\Rightarrow$ non-vanishing on $M \setminus 0$

$g := \rho_r$ (replication map)

$$\begin{tikzcd} f := & S^{d-1} \times C_{n-1}(M \setminus 0) \arrow[r, "\mathrm{id} \times \rho_r"] & S^{d-1} \times C_{rn-r}(M \setminus 0) \arrow[d, "\text{``apply } t_k\text{'' } r-1 \text{ times}"] \\ & & S^{d-1} \times C_{rn-1}(M \setminus 0) \end{tikzcd}$$

Picture:



Isomorphisms on H_*

$g = g_r$ induces $\cong$ on H_i for $n \geq 2i$ by Theorem A

f induces $\cong$ on H_i for $n \geq 2i$ by • Theorem A

& • classical hom. stability
+ Serre spectral
sequence argument

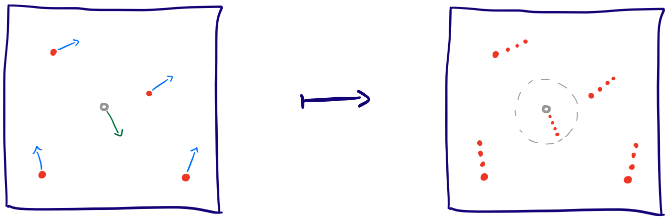
Last step:

Commutativity of the diagram on $\mathbb{F}$-homology

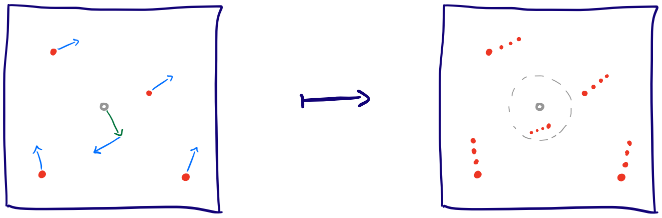
$$\begin{tikzcd}
S^{d-1} \times C_{n-1}(M \setminus 0) \arrow[r, "t_n"] \arrow[d, "f"'] & C_n(M \setminus 0) \arrow[d, "g"] \\
S^{d-1} \times C_{rn-1}(M \setminus 0) \arrow[r, "t_{rn}"'] & C_{rn}(M \setminus 0)
\end{tikzcd}$$

Pictures: $\begin{pmatrix} d=2 \\ n=5 \\ r=4 \end{pmatrix}$

↳ :



↴ :



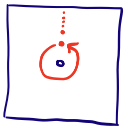
These are determined by maps $h_1, h_2 : S^{d-1} \longrightarrow C_r(\mathbb{R}^d \setminus 0)$

$\rightsquigarrow$ enough to check that

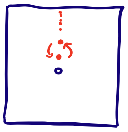
$$(h_1)_* [S^{d-1}] = (h_2)_* [S^{d-1}] \in H_{d-1}(C_r(\mathbb{R}^d \setminus 0); \mathbb{F})$$

Def Elements of $H_{d-1}(C_r(\mathbb{R}^d \setminus 0))$

$\Delta_0 :=$

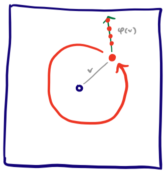


$\pi :=$



For any self-map $S^{d-1} \xrightarrow{\varphi} S^{d-1}$

$\sigma_\varphi :=$



"comet orbiting the sun"

$=$ image of $[S^{d-1}]$ under

$$S^{d-1} \longrightarrow C_r(\mathbb{R}^d \setminus 0)$$

$$v \longmapsto \{v, \underbrace{v + \tfrac{1}{r}\varphi(v), v + \tfrac{2}{r}\varphi(v), \dots, v + \tfrac{r-1}{r}\varphi(v)}_{\text{"tail of the comet"}}\}$$

Example: $\sigma_{id} = \Delta_0$

Note: $(h_1)_*[S^{d-1}] = \sigma_{id} = \Delta_0$

$(h_2)_*[S^{d-1}] = \sigma_\varphi$ for $\varphi : S^{d-1} \longrightarrow S^{d-1}$ the restriction of the non-vanishing vector field.

[Poincaré-Hopf] $\Rightarrow$ $\deg(\varphi) = \chi(M)$.

Lemma (Lemma 5.5 of [Cantero - P. '15])

For any $\varphi : S^{d-1} \longrightarrow S^{d-1}$,

$$\sigma_\varphi = r\Delta_0 + \deg(\varphi)\, r(r-1)\pi$$

Proof: Construct explicit chains 1 dimension higher in order to deduce relations between σ_φ, Δ_0 and π.

Hence $(h_1)_*[S^{d-1}] = (h_2)_*[S^{d-1}]$

$$\Updownarrow \text{ — Lemma}$$

$$r\Delta_0 + r(r-1)\pi = r\Delta_0 + \chi(M)\, r(r-1)\pi$$

$$\Updownarrow \text{ — rearranging}$$

$\operatorname{order}(\pi)$ divides $r(r-1)(\chi(M)-1)$

$$\Uparrow \text{ — } \operatorname{char}(\mathbb{F}) = p$$

p divides $(r-1)(\chi(M)-1)$.

□

Homological stability for configuration spaces on closed manifolds V

GeMAT seminar
IMAR
23 May 2025

Recap

M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M

When M is open

$M \cong \mathrm{int}(\bar{M})$, $\partial \bar{M} \neq \emptyset$

$C_n(M)$ is homologically stable with $\mathbb{Z}$ coefficients (and hence with any coeffs)

$H_i(C_n(M); \mathbb{Z}) \cong H_i(C_{n+1}(M); \mathbb{Z})$ when $n \geq 2i$ [McDuff, Segal, '70s]

When M is closed

compact, $\partial M = \emptyset$

So far we have seen:

- $H_i(C_n(M); \mathbb{F}) \cong H_i(C_{n+1}(M); \mathbb{F})$ when $n \geq 2i$ [Randal-Williams '13]
  if $\dim(M)$ is odd, $\mathbb{F}$ field
  or $\dim(M)$ is even, $\mathbb{F}$ field of $\mathrm{char} = 0$ or 2

Note: char = 0 result **first** proven by [Church '12] — we will look at this next time....

- When $\dim(M)$ is odd, for $n \geqslant 2i$:

$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$

$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}])$$

- When $\dim(M)$ is even, for $n \geqslant 2i$:

$H_i(C_n(M);\mathbb{Z}_{(p)})$ depends only on $\nu_p(2n-\chi(M))$

(for fixed i, p)

(assume $p \geqslant 3$ if $\chi(M)$ odd)

$$H_i(C_n(M);\mathbb{F}_p) \cong H_i(C_{rn}(M);\mathbb{F}_p)$$

for any $r \geqslant 2$
$r \equiv 1 \pmod p$

$$H_i(C_n(M);\mathbb{F}_p) \cong H_i(C_{n+q}(M);\mathbb{F}_p)$$

as long as $\chi(M) \neq 0$
$p \geqslant 3$

where $q = p^{1+\nu_p(\chi(M))}$

[Cantero-P. '15]

Aim for today

Theorem (Kupers-Miller '16)

- If $\dim(M)$ is odd, then for $n \geqslant 2i$:

$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+1}(M);\mathbb{Z})$$

- If $\dim(M)$ is even, then for $n \geqslant 2i$:

$$H_i(C_n(M);\mathbb{Z}/k) \cong H_i(C_{n+k}(M);\mathbb{Z}/k) \quad \text{for } k \geqslant 3 \text{ odd}$$

$$H_i(C_n(M);\mathbb{Z}/k) \cong H_i(C_{n+\frac{k}{2}}(M);\mathbb{Z}/k) \quad \text{for } k \geqslant 2 \text{ even}$$

In particular, this improves $p^{1+\nu_p(\chi(M))}$-periodicity of $H_i(C_n(M);\mathbb{F}_p)$ [Cantero-P. '15] to p-periodicity of $H_i(C_n(M);\mathbb{F}_p)$ in the stable range $n \geqslant 2i$.

Rmk: [Nagpal '15] proved that $H_i(C_n(M);\mathbb{F}_p)$ is $p^{(i+3)(2i+2)}$-periodic in an (unspecified) stable range.

Corollary:

If $\dim(M)$ is even, then for fixed $i \geqslant 1$, $p \geqslant 3$,

for $m, n \geqslant 2i$:

$$H_i(C_m(M); \mathbb{F}_p) \cong H_i(C_n(M); \mathbb{F}_p)$$

if either $2m \equiv \chi(M) \equiv 2n \pmod{p}$

or $2m \not\equiv \chi(M) \not\equiv 2n \pmod{p}$

i.e. $H_i(C_n(M); \mathbb{F}_p)$ depends only on whether $2n \equiv \chi(M)$ or not

In particular $H_i(C_n(M); \mathbb{F}_p)$ takes on $\leqslant 2$ different values in the stable range $n \geqslant 2i$.

Proof

If $2m \not\equiv \chi(M) \not\equiv 2n \pmod{p}$

$\hookrightarrow$ then $\nu_p(2m - \chi(M)) = 0 = \nu_p(2n - \chi(M))$

$\longrightarrow$ result follows from [Cantero - P. '15]

(+ universal coefficient theorem to pass from $\mathbb{Z}_{(p)}$ coeffs to $\mathbb{F}_p$ coeffs)

If $2m \equiv \chi(M) \equiv 2n \pmod{p}$

$\hookrightarrow$ since $p \geqslant 3$, so 2 is invertible mod p,

$m \equiv n \pmod{p}$

$\longrightarrow$ result follows from [Kupers - Miller '16] (p-periodicity).

$\square$

Plan of talk / strategy of proof

from talk # 1

① Recollection of the "puncturing trick" used by Randal-Williams in the case of coeffs in $\mathbb{F}_2$.

② Some homology operations on E_d-algebras.

③ Chain complex lemma.

④ Proof of the theorem.

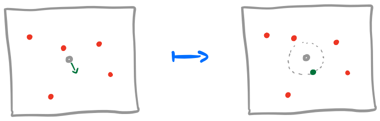
① Puncturing trick

From talk #1 we recall that, for any manifold M with basepoint $0 \in M$ there is an exact sequence:

$$\cdots \to H_i C_n(M) \to H_{i-d}(C_{n-1}(M\setminus 0)) \xrightarrow[S_n^i]{} H_{i-1}(C_n(M\setminus 0)) \to H_{i-1} C_n(M) \to \cdots$$

induced by

$$S^{d-1}\times C_{n-1}(M\setminus 0) \xrightarrow{t_s} C_n(M\setminus 0)$$



expand radially away from $0 \in M$

add a point specified by the S^{d-1} parameter

Obs (1) If the ring of coefficients is a field we have

$$\dim H_i C_n(M) = \dim \ker(S_n^i) + \dim \operatorname{coker}(S_n^{i+1})$$

(and M is of finite type, e.g. closed)

Obs (2) If the ring of coefficients is finite, we have

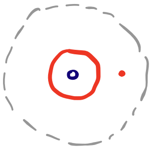
$$|H_i C_n(M)| = |\ker(S_n^i)| \cdot |\operatorname{coker}(S_n^{i+1})|$$

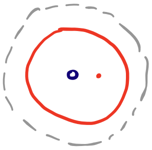
Argument of [Randal-Williams '13]:

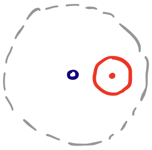
$$\begin{tikzcd}
S^{d-1}\times C_{n-1}(M\setminus 0) \arrow[r, "t_s"] \arrow[d, "\mathrm{id}\times t_p"'] & C_n(M\setminus 0) \arrow[d, "t_p"] \\
S^{d-1}\times C_n(M\setminus 0) \arrow[r, "t_s"'] & C_{n+1}(M\setminus 0)
\end{tikzcd}$$

(non-commutative square)

On H_* this acts by

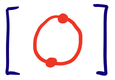
↳ : adding  near the puncture

↴ : adding  near the puncture

The difference is homologous to the operation that adds 

This is the image of

$$\begin{array}{ccc} \left[\odot\right] & \in & H_{d-1}(C_2(\mathbb{R}^d)) \\ \parallel & & \cong \\ 2 \cdot \underbrace{\text{generator}} & \in & H_{d-1}(\mathbb{RP}^{d-1}) \end{array}$$



$\Rightarrow$ with $\mathbb{F}_2$ coefficients it's zero

$\Rightarrow$ commutative square on $H_*(- ; \mathbb{F}_2) = H_*$:

$$\begin{tikzcd}
H_{i-d}(C_{n-1}(M \setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "(t_p)_*"'] & H_{i-1}(C_n(M \setminus 0)) \arrow[d, "(t_p)_*"] \\
H_{i-d}(C_n(M \setminus 0)) \arrow[r, "S_{n+1}^i"] & H_{i-1}(C_{n+1}(M \setminus 0))
\end{tikzcd}$$

$\cong$ in stable range

$\Rightarrow$ $\dim \ker(S_n^i) = \dim \ker(S_{n+1}^i)$

$\dim \operatorname{coker}(S_n^{i+1}) = \dim \operatorname{coker}(S_{n+1}^{i+1})$

in the stable range

$\Rightarrow$ $\square$ by Obs (1).

② Homology operations

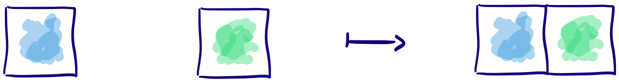
$$C(\mathbb{R}^d) = \coprod_{n\geq 0} C_n(\mathbb{R}^d)$$

$H_* C(\mathbb{R}^d)$ has two key operations:

Pontrjagin product

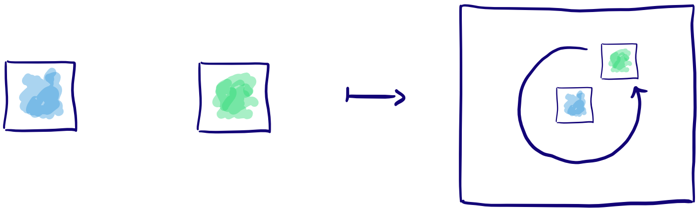
$$H_i C_m(\mathbb{R}^d) \otimes H_j C_n(\mathbb{R}^d) \longrightarrow H_{i+j} C_{m+n}(\mathbb{R}^d)$$

commutative when $d \geq 2$



Browder bracket

$$H_i C_m(\mathbb{R}^d) \otimes H_j C_n(\mathbb{R}^d) \xrightarrow{\phi} H_{i+j+d-1} C_{m+n}(\mathbb{R}^d)$$



Lemma For $\alpha \in H_j C_m(\mathbb{R}^d)$, the difference between the two ways around the square

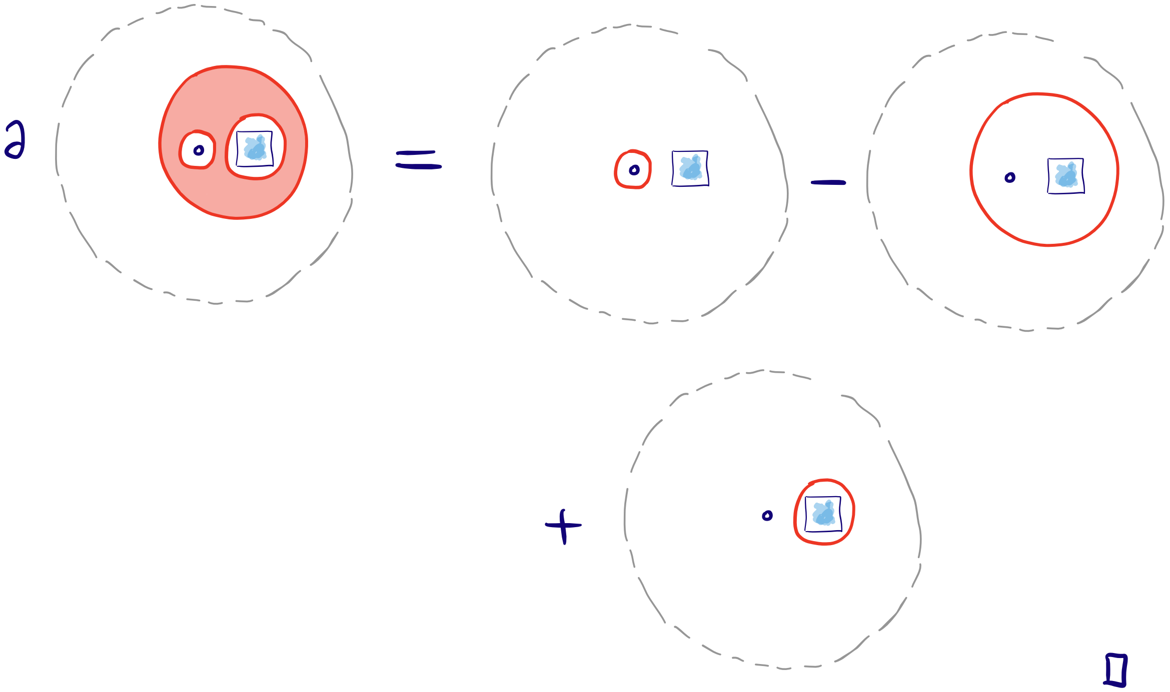
$$\begin{tikzcd}
H_{i-d}(C_n(M \setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "t_\alpha"'] & H_{i-1}(C_{n+1}(M \setminus 0)) \arrow[d, "t_\alpha"] \\
H_{i-d+j}(C_{n+m}(M \setminus 0)) \arrow[r, "S_{n+m}^{i+j}"] & H_{i-1+j}(C_{n+1+m}(M \setminus 0))
\end{tikzcd}$$

add a copy of α near the puncture

is equal to $t_{\phi(\alpha, \bullet)}$.

Proof :

$\alpha =$ 



□

Lemma

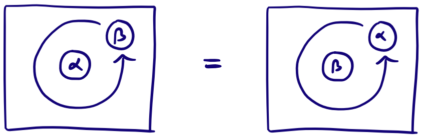
(a) For d odd, $\phi(\cdot,\cdot) = 0$ with $\mathbb{Z}$ coefficients.

(b) For d even, $\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet)$ is divisible by $2k$.

Proof

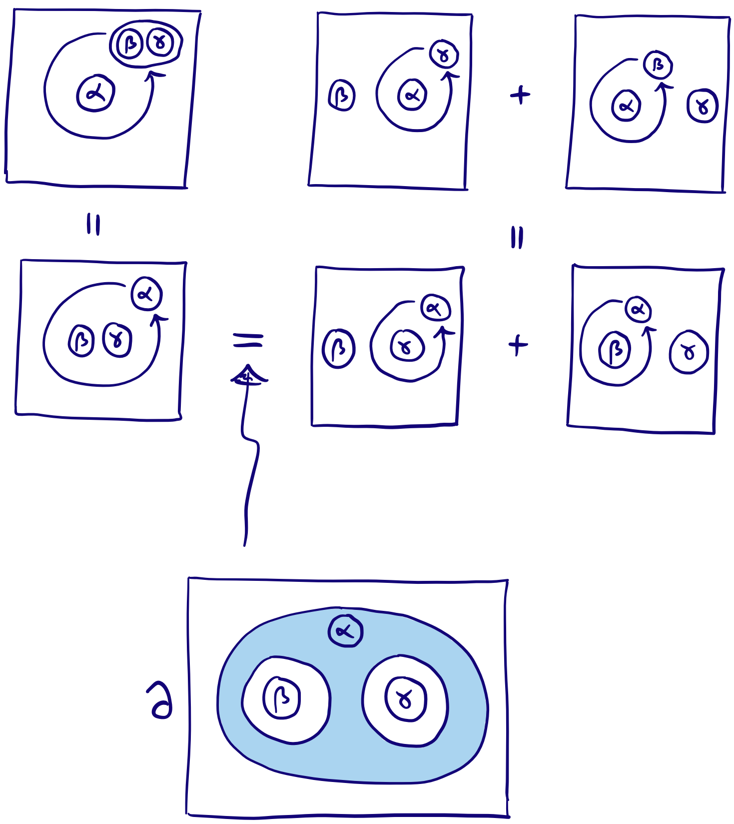
(a) $\phi(\cdot,\cdot) =$  $= 2\cdot\text{generator} \in H_{d-1}(\mathbb{RP}^{d-1}) \cong \mathbb{Z}/2$.

(b) **Fact 1** : ϕ is symmetric



Fact 2 : ϕ is a derivation, ie.

$$\phi(\alpha,\beta\gamma) = \beta\,\phi(\alpha,\gamma) + \phi(\alpha,\beta)\,\gamma$$



Iterating Fact 2 (& Fact 1) we get:

$$\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet) = \underbrace{\bullet\bullet\cdots\bullet}_{k-1}\,\phi(\bullet,\bullet) + \bullet\,\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k-1},\bullet)$$

$$\cdots = k\,.\,\underbrace{\bullet\bullet\cdots\bullet}_{k-1}\,\phi(\bullet,\bullet)$$

$\phi(\bullet,\bullet)$ divisible by 2 $\longrightarrow$ $\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet)$ divisible by $2k$

□

Corollary The square

$$\begin{tikzcd}
H_{i-d}(C_n(M\setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "(t_p)^m"'] & H_{i-1}(C_{n+1}(M\setminus 0)) \arrow[d, "(t_p)^m"] \\
H_{i-d}(C_{n+m}(M\setminus 0)) \arrow[r, "S_{n+m}^i"] & H_{i-1}(C_{n+1+m}(M\setminus 0))
\end{tikzcd}$$

m-fold iterated stabilization map

commutes if
- we take coeffs in a ring R : char(R) divides $2m$,
- or we take coeffs in any ring and dim(M) is odd.

no condition on m in the second case.

Corollary^2

Under these conditions,

$$\ker(S_n^i) \cong \ker(S_{n+m}^i)$$

$$\& \quad \text{coker}(S_n^i) \cong \text{coker}(S_{n+m}^i)$$

in the stable range.

Obs(2)

$\Rightarrow$ If $R = \mathbb{Z}/k$

$$\text{then} \quad |H_i(C_n(M);\mathbb{Z}/k)| = |\ker(S_n^i)| \cdot |\text{coker}(S_n^{i+1})|$$

$$\|$$

$$|H_i(C_{n+m}(M);\mathbb{Z}/k)| = |\ker(S_{n+m}^i)| \cdot |\text{coker}(S_{n+m}^{i+1})|$$

$$\text{for} \quad m = \begin{cases} k & k \text{ odd} \\ k/2 & k \text{ even} \\ 1 & \dim(M) \text{ odd} \end{cases}$$

Upshot : $|H_i(C_n(M);\mathbb{Z}/k)|$ is m-periodic in the stable range (for fixed i, k)

③ Chain complex lemma

C_* chain complex over $\mathbb{Z}$
bounded above and below
finitely generated & free in each degree

(E.g. cellular chain complex of a finite CW-complex)

Lemma

finite set $\qquad\qquad$ abelian group

$$Ⓐ \quad \left.\begin{array}{l} \left|H_j(C_* \otimes \mathbb{Z}/p^s)\right| \\ \text{for all } \; j \leq i \\ \qquad\quad\; 1 \leq s \leq r \end{array}\right\} \xrightarrow{\text{determines}} H_i(C_* \otimes \mathbb{Z}/p^r)$$

$$Ⓑ \quad \left.\begin{array}{l} \left|H_j(C_* \otimes \mathbb{Z}/p^s)\right| \\ \text{for all } \; j \leq i \\ \qquad\quad\; 1 \leq s \\ \qquad\quad\; \text{primes } p \end{array}\right\} \xrightarrow{\text{determines}} H_i(C_*)$$

In particular, if X is a finite CW-complex,
$H_*(X;\mathbb{Z})$ is determined by $\left|H_*(X;\mathbb{Z}/p^s)\right|$ $\forall p, s$.

Idea of proof

Ⓐ

$$C_* \otimes \mathbb{Z}/p^r \underset{\text{quasi-isomorphic}}{\simeq} \text{finite } \oplus \text{ of copies of } \quad \cdots \to 0 \to \mathbb{Z}/p^r \to 0 \to \cdots$$

$$\text{and} \quad \cdots \to 0 \to \mathbb{Z}/p^r \xrightarrow{\cdot p^s} \mathbb{Z}/p^r \to 0 \to \cdots$$

for $1 \leqslant s \leqslant r-1$.

$\rightsquigarrow$ enough to determine the <u>multiplicities</u> of these pieces.

Write down a formula for $|H_j(C_* \otimes \mathbb{Z}/p^s)|$ in terms of these multiplicities:

$$\begin{tikzcd} \{\text{multiplicities}\} \arrow[r, "\text{determine via system}", "\text{of linear equations}"'] & \left\{ |H_j(C_* \otimes \mathbb{Z}/p^s)| \quad \begin{matrix} j \leqslant i \\ 1 \leqslant s \leqslant r \end{matrix} \right\} \arrow[l, bend left] \end{tikzcd}$$

Matrix is <u>invertible</u>, hence

(Actually to determine multiplicities of the pieces shifted by i, need to know multiplicities of the pieces shifted by $i-1$.

⟶ Do this recursively.

⟶ Base case: <u>zero</u>, since C_* bounded below.)

Ⓑ

A more careful version of the argument in Ⓐ implies that

$$\left| H_i(C_* \otimes \mathbb{Z}/p^r) \right| \xrightarrow{\text{determines}} H_i(C_* \otimes \mathbb{Z}/p^r)$$

for all i and r $\qquad$ for all i and r

<u>and</u> the projections

$$\begin{tikzcd} H_i(C_* \otimes \mathbb{Z}/p^r) \arrow[d] \\ H_i(C_* \otimes \mathbb{Z}/p^s) \end{tikzcd} \quad \text{for } s \leq r$$

This determines

$$H_i(C_* \otimes \hat{\mathbb{Z}}_p) \cong \lim_{\infty \leftarrow r} H_i(C_* \otimes \mathbb{Z}/p^r)$$

p-adic integers

$\hat{\mathbb{Z}}_p$ is torsion-free $\Rightarrow$ flat $\mathbb{Z}$-module

$\Rightarrow H_*(C_* \otimes \hat{\mathbb{Z}}_p) \cong H_*(C_*) \otimes \hat{\mathbb{Z}}_p$

Each $H_i(C_*)$ is fin. generated, so it is determined by knowing $H_i(C_*) \otimes \hat{\mathbb{Z}}_p$ for all primes p.

□

④ Finishing the proof of the theorem

We already know that, in the stable range $n \geq 2i$:

$$\left.\begin{aligned} &|H_i(C_n(M);\mathbb{Z}/k)| \cong |H_i(C_{n+k}(M);\mathbb{Z}/k)| \,, \\ \text{and}\quad &|H_i(C_n(M);\mathbb{Z}/k)| \cong |H_i(C_{n+\frac{k}{2}}(M);\mathbb{Z}/k)| \quad \text{if } k \text{ is even,} \end{aligned}\right\} (*)$$

$$\text{and}\quad |H_i(C_n(M);\mathbb{Z}/k)| \cong |H_i(C_{n+1}(M);\mathbb{Z}/k)| \quad \text{if } \dim(M) \text{ is odd.} \quad (**)$$

dim(M) even

- $(*)$ holds in particular for any $k = p^r$.
- chain complex lemma Ⓐ

M closed

$\Rightarrow C_n(M)$ finite CW-complex

$$\rightsquigarrow \quad \begin{aligned} &H_i(C_n(M);\mathbb{Z}/p^r) \cong H_i(C_{n+p^r}(M);\mathbb{Z}/p^r) \quad p \geq 3 \\ &H_i(C_n(M);\mathbb{Z}/2^r) \cong H_i(C_{n+2^{r-1}}(M);\mathbb{Z}/2^r) \end{aligned}$$

- $\Rightarrow$ periodicity with general $\mathbb{Z}/k$ coeffs by decomposing $\mathbb{Z}/k \cong \bigoplus$ of $\mathbb{Z}/p^r$.

dim(M) odd

- $(**)$ holds in particular for any $k = p^r$.
- chain complex lemma Ⓑ

$$\rightsquigarrow \quad H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+1}(M);\mathbb{Z}) .$$

□

Homological stability for configuration spaces on closed manifolds VI

GeMAT seminar
IMAR
11 July 2025

Reminder

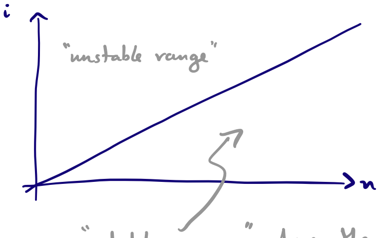
M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open — $M \cong \operatorname{int}(\bar{M})$, $\partial\bar{M} \neq \phi$

then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.



"stable range" where the homology is independent of n

This is false for closed manifolds M.

compact
$\partial M = \phi$

E.g. $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2n-2)$.

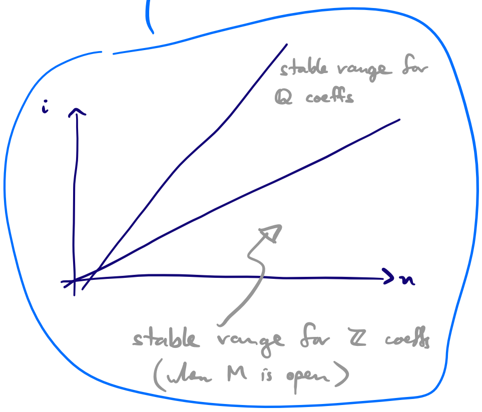
Theorem 1 (Church '12)

If M is connected
orientable
finite-type

$H^*(M;\mathbb{Q})$ is finite-dim.

e.g. (interiors of) compact manifolds

then $H_i(C_n(M);\mathbb{Q}) \cong H_i(C_{n+1}(M);\mathbb{Q})$ for $n \geq i+1$



This is a corollary of:

Theorem 2 (Church '12)

Under these conditions, for each fixed $i \geq 0$,

the sequence of Σ_n-representations $H^i(F_n(M);\mathbb{Q})$

ordered configuration spaces

is
- uniformly representation stable
- & monotone

for $n \geq 4i$

can improve to $n \geq 2i$ if $\dim(M) \geq 3$.

First task: define what this means....

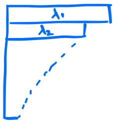
Representation Theory of symmetric groups (over $\mathbb{Q}$)

(cf. Fulton-Harris)

V_λ irreducible reps of Σ_n

$\updownarrow$ $\quad\updownarrow$ 1:1

λ partitions of n $\longleftrightarrow$ $\lambda = (\lambda_1, \dots, \lambda_r)$: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$, $\lambda_1 + \cdots + \lambda_r = n$



Eg

$\lambda = (n)$ trivial repr. on $\mathbb{Q}$

$\lambda = (n-1,1)$ standard repr. on $\mathbb{Q}^n/\mathbb{Q}$

$\lambda = (n-3,1,1,1)$ $\Lambda^3(\mathbb{Q}^n/\mathbb{Q})$

Def. $\left.\begin{array}{l}\lambda \text{ partition of } k \\ n \geqslant k + \lambda_1\end{array}\right\} \rightsquigarrow \lambda[n] = (n-k, \lambda_1, \dots, \lambda_r)$ partition of n

$$V(\lambda)_n := V_{\lambda[n]}$$

For a Σ_n-representation W, $c_\lambda(W)$:= multiplicity of $V(\lambda)_n$ in W.

Representation stability (Church-Farb '10)

$$\begin{tikzcd} V_1 \arrow[r, "\phi_1"] & V_2 \arrow[r, "\phi_2"] & \cdots \arrow[r] & V_n \arrow[r, "\phi_n"] & V_{n+1} \arrow[r] & \cdots \\ \circlearrowleft & \circlearrowleft & & \circlearrowleft & \circlearrowleft & \\ \Sigma_1 \arrow[r, hook] & \Sigma_2 \arrow[r, hook] & \cdots \arrow[r, hook] & \Sigma_n \arrow[r, hook] & \Sigma_{n+1} \arrow[r, hook] & \cdots \end{tikzcd}$$

finite-dimensional
$\mathbb{Q}$-vector spaces
+ compatible actions

Def

This is uniformly representation stable for $n \geq N$ if:

$\forall n \geq N$, (1) ϕ_n is injective

(2) the $\mathbb{Q}[\Sigma_{n+1}]$-span of $\phi_n(V_n)$ is V_{n+1}

(3) $\forall \lambda$, $c_\lambda(V_n) = c_\lambda(V_{n+1})$.

Rmk If $\underset{\underset{V(\lambda)_n}{\cong}}{W} \subseteq V_n$, then (3) does not imply that $\phi_n(W) \cong V(\lambda)_{n+1}$,

or even that $\phi_n(W)$ contains $V(\lambda)_{n+1}$.

↳ not a Σ_{n+1}-repr.!
should consider its Σ_{n+1}-span.

Def (Church '12)

$\{V_n, \phi_n\}$ is monotone for $n \geq N$ if:

$\forall n \geq N$, (4) if $\underset{\underset{(V(\lambda)_n)^{\oplus k}}{\cong}}{W} \subseteq V_n$

then the $\mathbb{Q}[\Sigma_{n+1}]$-span of $\phi_n(W)$ contains an isomorphic copy of $(V(\lambda)_{n+1})^{\oplus k}$.

Restating Theorem 2:

Theorem 2 (Church '12)

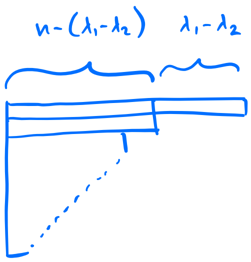
If M is connected, orientable, finite-type then for each fixed $i \geq 0$,

the sequence $\cdots \longrightarrow H^i(F_n(M);\mathbb{Q}) \longrightarrow H^i(F_{n+1}(M);\mathbb{Q}) \longrightarrow \cdots$

induced by $F_n(M) \xleftarrow{\text{forget}} F_{n+1}(M)$

satisfies (1)–(4) above for $n \geq 4i$.

Rmk Suppose that λ is a partition of n and $n-(\lambda_1-\lambda_2)-1 \geq 4i$



Thm 2 $\Rightarrow$ mult of V_λ in $H^i(F_n(M);\mathbb{Q})$

$= $ mult of $V_{(\lambda_2-1,\lambda_2,\lambda_3,\dots)}$ in $H^i(F_{n-(\lambda_1-\lambda_2)-1};\mathbb{Q})$

$= 0$

How this implies Theorem 1

For $n \geq 4i$ Can be improved to the range stated in Thm 1 by more careful analysis.

$$\dim H_i(C_n(M);\mathbb{Q})$$

$$= \dim H^i(C_n(M);\mathbb{Q})$$

$$= \dim \left(H^i(F_n(M);\mathbb{Q})\right)^{\Sigma_n}$$

$= c_\lambda\left(H^i(F_n(M);\mathbb{Q})\right)$ for $\lambda = (n) =$ 

independent of n by Thm 2.

Ideas of proof of Theorem 2

Leray spectral sequence

Cf. [Dimca, Sheaves in topology]

$$X \xrightarrow{f} Y \quad \rightsquigarrow \quad f_* : Sh(X) \longrightarrow Sh(Y)$$

$$\mathcal{F} \mapsto \left(U \longmapsto \mathcal{F}(f^{-1}(U)) \right)$$

$\rightsquigarrow$ derived functors $R^i f_* : Sh(X) \longrightarrow Sh(Y)$

$$\left(\begin{array}{l} \text{presheaf } U \longmapsto H^i(f^{-1}(U); \mathcal{F}) \\ R^i f_*(\mathcal{F}) = \text{sheaffification of this} \end{array} \right)$$

$\mathcal{F}$ sheaf on X

$$\boxed{E_2^{p,q} = H^p(Y; R^q f_*(\mathcal{F})) \Rightarrow H^*(X; \mathcal{F})} \qquad (*)$$

$$\begin{array}{c} E_2^{p,q}(n) \\ /\!/ \end{array}$$

[Totaro '96] Explicit description of $E_2^{p,q}$ of $(*)$ for

$F_n(M) \hookrightarrow M^n$ and $\mathcal{F} = \mathbb{Q}$

which converges to $H^*(F_n(M); \mathbb{Q})$.

Idea

① Prove uniform repr. stab.^{y} + monotonicity for $E_2^{p,q}(n)$.

② Deduce the same for $E_r^{p,q}(n)$ and hence $E_\infty^{p,q}(n)$.

③ Deduce the same for the limit, i.e. $H^i(F_n(M); \mathbb{Q})$.

Rmks

- Steps ② and ③ are relatively straightforward.
- However, they only work for the combined property of uniform repr. stab.^{y} + monotonicity.

(Uniform repn stab.^{y} by itself does not "propagate" through a spectral sequence.)

Sub-configuration spaces

Eg

$\Lambda = \{\{1,2\},\{3,5\},\{4\}\}$

$|\Lambda| = 3$

$\bar{\Lambda} = (2,2,1)$

Λ partition of $\{1,\dots,n\}$

Notation $|\Lambda| :=$ # of blocks of the partition.

$\bar{\Lambda} :=$ the induced partition of n.

Def

Λ-subconfiguration space

$$F_\Lambda(M) = \left\{ (p_1,\dots,p_n) \in M^n \;\middle|\; \begin{array}{l}\text{if } i,j \text{ lie in the same block of } \Lambda \\ \text{then } p_i \neq p_j\end{array}\right\}$$

Λ-diagonal

$$M^\Lambda = \left\{ (p_1,\dots,p_n) \in M^n \;\middle|\; \begin{array}{l}\text{if } i,j \text{ lie in the same block of } \Lambda \\ \text{then } p_i = p_j\end{array}\right\}$$

Eg $\Lambda =$ trivial partition $\longrightarrow$ $F_\Lambda(M) = F_n(M)$

$M^\Lambda =$ diagonal copy of M

$\Lambda =$ discrete partition $\longrightarrow$ $F_\Lambda(M) = M^n = M^\Lambda$

Thm (Arnol'd '69 + Brieskorn '73)

$$H^*(F_n(\mathbb{R}^d);\mathbb{Q}) \cong \bigoplus_{\Lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q})$$

(isom. of $\mathbb{Q}[\Sigma_n]$-modules)

Thm (Totaro '96)

(isoms of $\mathbb{Q}[\Sigma_n]$-modules)

$$E_2^{*,*}(n) \cong \bigoplus_{\Lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q}) \otimes H^*(M^\Lambda;\mathbb{Q})$$

Obs.

$$\cong \bigoplus_{\lambda}\left(\bigoplus_{\overline{\Lambda}=\lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q}) \otimes H^*(M^\Lambda;\mathbb{Q})\right)$$

preserved by Σ_n-action

.... decompose $H^*(M^\Lambda;\mathbb{Q})$ in a Σ_n-invariant way via the Künneth formula ....

$\cong \bigoplus$ summands, each of which is isomorphic to $\mathrm{Ind}_{\Sigma_k\times\Sigma_{n-k}}^{\Sigma_n}(W\boxtimes\mathbb{Q})$ for some Σ_k-representation W

Prop. (Church '12)

In (fixed) bidegree (p,q), we have

$$k = q + \mathrm{length}(\lambda) + \max\{i \mid \lambda_i \geq 2\}.$$

Thm (Church '12) (**)

For any fixed Σ_k-representation W, the sequence

$$\mathrm{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} (W \boxtimes \mathbb{Q})$$

is uniformly representation stable for $n \geq 2k$
& monotone for $n \geq k$.

Rmk
- This is pure representation theory.
- [Church] gives an elementary (but intricate) proof using the branching rule for $\mathrm{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} (V_\lambda \boxtimes \mathbb{Q})$.
- [Sam-Weyman] give an alternative proof, using Schur-Weyl duality to work instead with $GL_n(\mathbb{Q})$-representations.

Finishing the proof

(recap)

$$\left.\begin{array}{l}\text{Theorem } (**) \\ \quad + \text{ fact that } k \text{ is fixed}\end{array}\right) \Longrightarrow \quad \text{URS} + \text{M} \quad \text{for} \quad E_2^{p,q}(n)$$

$$\overset{\text{propagation}}{\Longrightarrow} \quad \text{URS} + \text{M} \quad \text{for} \quad H^i(F_n(M); \mathbb{Q})$$

$$\underset{\lambda = \square\square\square\square\square\square}{\Longrightarrow} \quad \text{stability} \quad \text{for} \quad H^i(C_n(M); \mathbb{Q})$$

Rmk Can prove that stability is induced by $C_{n+1}(M) \overset{\text{transfer}}{\dashrightarrow} C_n(M)$.

Homological stability for configuration spaces on closed manifolds VII

GeMAT seminar
IMAR
15 July 2025

Context

- If M is a connected, open manifold, then
$$H_i(C_n(M)) \cong H_i(C_{n+1}(M)) \quad \text{for } n \geqslant 2i.$$
[McDuff, Segal]

- When M is closed this is generally false, but:

Thm If M is a connected manifold, then
$$H_i(C_n(M);\mathbb{Q}) \cong H_i(C_{n+1}(M);\mathbb{Q}) \quad \text{for } n \geqslant i+1.$$

- So far we have seen proofs by

Church	(talk V)	via representation stability
Randal-Williams	(talk I)	via transfer maps
Bendersky – Miller	(talks II/III)	via scanning maps & obstruction theory

- Today: it will be a corollary of:

Thm [Knudsen '17] For any d-manifold $(d \geqslant 2)$:
$$\bigoplus_{n \geqslant 0} H_*(C_n(M);\mathbb{Q}) \cong H(\mathfrak{g}_M) \quad \text{as bigraded } \mathbb{Q}\text{-vspaces,}$$
Lie algebra homology

where $\mathfrak{g}_M$ is the following bigraded Lie algebra:

$$\mathfrak{g}_M = H_c^{\overset{\text{degree}}{-*}}\big(M ; \mathcal{L}(\underbrace{\mathbb{Q}^{\omega}[d-1]}_{\text{weight } 1})\big)$$

$\mathbb{Q}^{\omega}$ = orientation local system on M

$\mathbb{Q}^{\omega}[d-1]$ = consider it to be concentrated in degree $d-1$

$\mathcal{L}(-)$ = free graded Lie algebra.

Rmk This can be simplified at the cost of splitting into two cases:

d odd $$\mathcal{L}(\mathbb{Q}^{\omega}[d-1]) = \underset{\text{weight } 1}{\mathbb{Q}^{\omega}[d-1]}$$ with trivial bracket.

$$\mathfrak{g}_M \cong H_*(M;\mathbb{Q})[-1]$$ with trivial bracket.

d even $$\mathcal{L}(\mathbb{Q}^{\omega}[d-1]) = \underset{\substack{x \\ \text{weight } 1}}{\mathbb{Q}^{\omega}[d-1]} \oplus \underset{\substack{[x,x] \\ \text{weight } 2}}{\mathbb{Q}[2d-2]}$$

$$\mathfrak{g}_M \cong H_*(M;\mathbb{Q})[-1] \oplus H_*(M;\mathbb{Q}^{\omega})[d-2]$$

as a bigraded vector space — the Lie bracket turns out to be determined by the cup product structure on M.

Corollary

As a graded $\mathbb{Q}$-vspace, $H_*(C_n(M);\mathbb{Q})$ depends only on d and the graded $\mathbb{Q}$-vspace $H_*(M;\mathbb{Q})$, plus the cup product structure if d is even.

[Bödigheimer-Cohen-Taylor '89] [Félix-Thomas '00]
d odd d even

Lie algebra homology

$\mathfrak{g}$ Lie algebra over $\mathbb{Q}$

Def $H_*(\mathfrak{g}) := \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{Q},\mathbb{Q})$ $\qquad U\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n} \Big/ [x,y] = xy-yx$

Thm (Chevalley-Eilenberg)

$$H_*(\mathfrak{g}) \cong H_*\big(\underbrace{\Lambda^*\mathfrak{g}}_{\text{differential induced by } [-,-]\,:\,\Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}}\big)$$

Can use this to generalise to graded Lie algebras:

Def $\mathfrak{g}$ graded Lie algebra

$$H_*(\mathfrak{g}) := H_*\big(\overbrace{\underbrace{\mathrm{Sym}^*(\mathfrak{g}[1])}_{\text{differential induced by } [-,-]}}^{\text{Chevalley-Eilenberg complex } C_*^{CE}(\mathfrak{g})}\big)$$

Note: $\mathfrak{g}$ ungraded $\longleftrightarrow$ concentrated in degree zero

$\Rightarrow$ $\mathfrak{g}[1]$ is concentrated in degree 1

$\Rightarrow$ $\mathrm{Sym}^*(\mathfrak{g}[1]) = \Lambda^*\mathfrak{g}$

Plan

- Some ideas of the proof
- How to deduce stability
- Example calculations

Idea of proof

Uses factorisation homology

Def · $\mathrm{Disk}_d = \left\{ \begin{array}{l} \text{manifolds diffeo to } \coprod_n \mathbb{R}^d \\ \text{smooth embeddings} \end{array} \right.$

· n-disk algebra A = symmetric monoidal functor

$$(\mathrm{Disk}_d, \sqcup) \xrightarrow{A} (\mathrm{Ch}_{\mathbb{Q}}, \otimes)$$

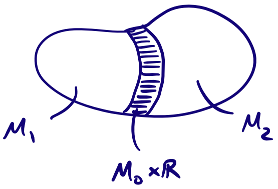
· $\int_{-} A : (\mathrm{Mfld}_d, \sqcup) \longrightarrow (\mathrm{Ch}_{\mathbb{Q}}, \otimes)$

is the unique symmetric monoidal functor s.t.

∃ and uniqueness
Thm (Ayala-Francis'15)

· $\int_{\mathbb{R}^d} A \simeq A$

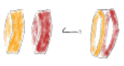
· if



= M

then $\int_M A \simeq \int_{M_1} A \underset{\int_{M_0\times\mathbb{R}} A}{\otimes} \int_{M_2} A$

$\int_{M_0\times\mathbb{R}} A$ has a ring structure and $\int_{M_i} A$ is a module over it via applying $\int_{-} A$ to the inclusions



⟷ $(M_0\times\mathbb{R}) \sqcup (M_0\times\mathbb{R}) \hookrightarrow M_0\times\mathbb{R}$

and $(M_0\times\mathbb{R}) \sqcup M_i \hookrightarrow M_i$

· $\int_M A$ =: factorisation homology of M with coefficients in A.

From d-disk algebras to Lie algebras

Part of the structure of a d-disk algebra is

$$C_*\left(\mathrm{Emb}\left(\coprod_2 \mathbb{R}^d, \mathbb{R}^d\right)\right) \otimes A \otimes A \longrightarrow A$$

by abuse of notation, A denotes $A(\mathbb{R}^d)$

Note that
$$\begin{aligned} &H_*\left(\mathrm{Emb}\left(\coprod_2 \mathbb{R}^d, \mathbb{R}^d\right)\right) \\ &\cong H_*(F_2(\mathbb{R}^d);\mathbb{Q}) \\ &\cong H_*(S^{d-1};\mathbb{Q}) \\ &\cong \mathbb{Q}[0] \oplus \mathbb{Q}[d-1] \end{aligned}$$

Restricting to (a cycle representing a generator of) $\mathbb{Q}[d-1]$, we get:

$$\begin{tikzcd} A[d-1] \otimes A \arrow[r] \arrow[d] & A \\ m_{d-1} : A[d-1] \otimes A[d-1] \arrow[r] & A[d-1] \end{tikzcd}$$

This is a Lie bracket on $A[d-1]$.

(Jacobi identity ⟷ Yang-Baxter relation in $H_*(F_3(\mathbb{R}^d))$.)

$$\begin{tikzcd} F : \{d\text{-disk algebras}\} \arrow[r, shift left] & \{\overset{\text{graded}}{\text{Lie}} \text{ algebras}\} \arrow[l, bend left, "U_d"] \end{tikzcd}$$

left adjoint
"d-enveloping algebra"

Main steps of the proof

$$(*) = \int_M \overset{\text{d-enveloping algebra}}{\mathcal{U}_d} \Big(\overset{\text{free Lie algebra — bigraded}}{\mathcal{L}} \Big(\underbrace{\widetilde{C}_*((\mathbb{R}^d)^+)[-1]}_{\text{graded vector space (weight 1)}} \Big)\Big)$$

bigraded chain cx.

Calculate this in two different ways :

$$(*) \simeq \bigoplus_{n \geq 0} C_*(C_n(M); \mathbb{Q})$$

$$(*) \simeq C_*^{CE}\big(H_c^{-*}(M ; \mathcal{L}(\mathbb{Q}^w[d-1]))\big)$$

Then take homology. $\square$.

Deducing stability

By the main theorem,

$$(*) \qquad \bigoplus_n H_*(C_n(M);\mathbb{Q}) \cong H_*(C^{CE}_*(\mathfrak{g}_M))$$

$$C^{CE}_*(\mathfrak{g}_M) = \mathrm{Sym}^*(\mathfrak{g}_M[1])$$

$$= \mathrm{Sym}^*\Big(\underbrace{H_c^{-*}(M;\underbrace{\mathbb{Q}^w[d-1]}_{\text{weight }1} \oplus \underbrace{\mathbb{Q}[2d-2]}_{\text{weight }2})}_{H_*(M;\mathbb{Q})}[1]\Big)$$

The $\begin{pmatrix}\text{degree-}0\\ \text{weight-}1\end{pmatrix}$ summand is one-dimensional.

Choose a generator p and extend it to a bihomogeneous basis B of $\mathfrak{g}_M[1]$.

Key proposition:

For $x \in C^{CE}_*(\mathfrak{g}_M)$, if $\mathrm{weight}(x) \geq \underset{\text{homological degree}}{|x|} + 1$ then x is divisible by p.

(Algebraic incarnation of hom. stab.^{y}).

(With a bit of work, implies hom. stab^y on LHS of $(*)$.)

Proof Assume $d \geq 3$. ($d=2$ is slightly more complicated)

Write $x = x_1 \cdots x_r$ $\quad x_i \in B$

$\Rightarrow$ $\mathrm{weight}(x_j) \geq |x_j| + 1$ for some j

Since $x_j \in \mathfrak{g}_M[1]$, $\mathrm{weight}(x_j) = 1$ or 2

Suppose $\mathrm{weight}(x_j) = 2$.

$\longrightarrow$ $d-1 \leq |x_j| \leq 2d-1$

$\longrightarrow$ contradiction to $2 \geq |x_j| + 1$ and $d \geq 3$.

Hence $\mathrm{weight}(x_j) = 1$.

$\longrightarrow$ $0 \leq |x_j| \leq d$

$\longrightarrow$ hence $|x_j| = 0$

$\longrightarrow$ $x_j = \lambda p$. $\square$

Rmk Proof just uses the formal structure of the bigrading of $\mathfrak{g}_M[1]$.

Reminder:

$$\bigoplus_n H_*(C_n(M);\mathbb{Q}) \cong H_*\big(\overbrace{\mathrm{Sym}^*(\mathfrak{g}_M[1])}^{\overset{C_*^{CE}(\mathfrak{g}_M)}{\shortparallel}}\big)$$

$$\mathfrak{g}_M[1] \cong \underbrace{H_*(M;\mathbb{Q})}_{wt=1} \left(\oplus \underbrace{H_*(M;\mathbb{Q}^w)[d-1]}_{wt=2} \right) \text{ if } d \text{ is even}$$

Examples

$M = \mathbb{R}^d$

d odd

no differentials b/c odd dim.

$$C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x] \qquad w(x) = 1$$

$$|x| = 0$$

$$\|$$

$$H_*(C_*^{CE}(\mathfrak{g}_M))$$

$$H_*(C_n(\mathbb{R}^d); \mathbb{Q}) \cong \text{weight-}n \text{ part of } \mathbb{Q}[x]$$

$$= \mathbb{Q}\{x^n\} \qquad |x^n| = 0$$

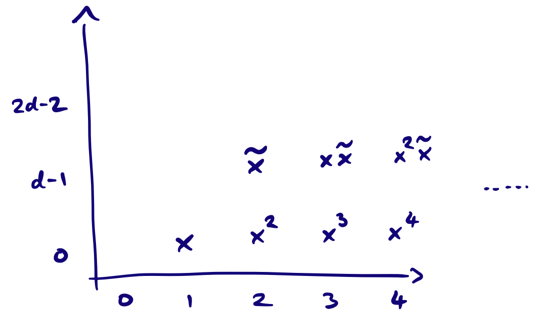
d even

$$H_*(C_*^{CE}(\mathfrak{g}_M)) = C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x] \otimes \Lambda[\tilde{x}]$$

no differentials b/c no non-0 cup products

$$w(x) = 1 \qquad |x| = 0$$

$$w(\tilde{x}) = 2 \qquad |\tilde{x}| = d-1$$



$M = \mathbb{R}^n \setminus \text{point}$

d odd

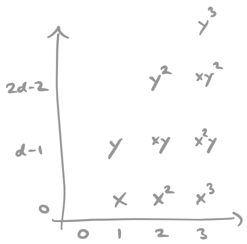
$$H_*(C_*^{CE}(\mathfrak{g}_M)) \cong \mathbb{Q}[x,y]$$

$w(x) = 1 = w(y)$

$|x| = 0$

$|y| = d-1$

$$H_*(C_n(\mathbb{R}^d \setminus \text{point}); \mathbb{Q}) \cong \text{weight-}n \text{ piece of this}$$



$$= \mathbb{Q}\{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$$

degrees $0, d-1, 2d-2, \dots, nd-n$

$$\cong \begin{cases} \mathbb{Q} & * = i(d-1) \quad 0 \leq i \leq n \\ 0 & \text{o/w.} \end{cases}$$

d even

$$H_*(C_*^{CE}(\mathfrak{g}_M)) = C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x, \tilde{y}] \otimes \Lambda[\tilde{x}, y]$$

no differentials b/c no non-0 cup products

$w(x) = 1 = w(y)$

$w(\tilde{x}) = 2 = w(\tilde{y})$

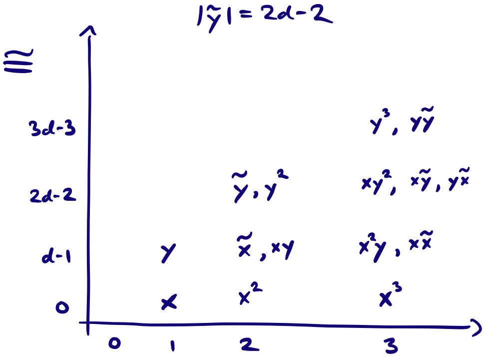
$|x| = 0$

$|y| = d-1$

$|\tilde{x}| = d-1$

$|\tilde{y}| = 2d-2$

$$\bigoplus_n H_*(C_n(\mathbb{R}^d \setminus \text{point}); \mathbb{Q}) \cong$$



[Drummond-Cole–Knudsen] use this Lie algebra model to calculate explicitly $\dim_{\mathbb{Q}} H_i(C_n(\Sigma); \mathbb{Q})$ for $\underline{\text{all}}$ i, n, Σ.

finite-type surface.

Homological stability for configuration spaces on closed manifolds VIII

GeMAT seminar
IMAR
16 July 2025

Reminder

In the last talk (i.e. yesterday) we saw:

Thm [Knudsen '17] For any d-manifold ($d \geq 2$):

$$\bigoplus_{n \geq 0} H_*(C_n(M);\mathbb{Q}) \cong H(\mathfrak{g}_M) \quad \text{as bigraded } \mathbb{Q}\text{-vspaces,}$$

where $$\mathfrak{g}_M = H_c^{\overset{\text{degree}}{-*}}(M; \mathcal{L}(\underbrace{\mathbb{Q}^w[d-1]}_{\text{weight } 1}))$$

$\mathbb{Q}^w[d-1]$ = orientation local system on M in degree $d-1$

$\mathcal{L}(-)$ = free graded Lie algebra.

... which implies $\mathbb{Q}$-homological stability for $C_n(M)$.

Aim for today:

[Randal-Williams '23]

A more direct proof of this:
- without factorisation homology
- as a corollary of a more general result.

Setup:

Work in $\mathrm{Top}_*^{\mathbb{N}}$ $=$ $\underbrace{\mathbb{N}\text{-indexed}}_{}$ sequences of based spaces.

weight grading from last talk

Fix M : connected d-manifold

$\cong$ interior of a compact manifold with boundary.

Definition

$C(M) \in \mathrm{Top}_*^{\mathbb{N}}$ $\qquad C(M)(n) := C_n(M)^+$ one-point compact^n

This is a commutative monoid object:

$$\begin{tikzcd}
C(M)(n_1) \wedge C(M)(n_2) \arrow[r] \arrow[d, equal] & C(M)(n_1+n_2) \arrow[d, equal] \\
C_{n_1}(M)^+ \wedge C_{n_2}(M)^+ \arrow[r] \arrow[d, equal] & C_{n_1+n_2}(M)^+ \\
\left(C_{n_1}(M) \times C_{n_2}(M)\right)^+ &
\end{tikzcd}$$

$$(c_1, c_2) \longmapsto \begin{cases} c_1 \cup c_2 & \text{if } c_1 \cap c_2 = \emptyset \\ \infty & \text{o/w} \end{cases}$$

Definition:

For $Y \in \mathrm{Top}_*^{\mathbb{N}}$, $\quad \mathrm{Com}(Y) :=$ free commutative monoid on Y.

$$\mathrm{Com}(Y)(n) = \text{weight-}n \text{ component of } SP^\infty\Big(\bigvee_{i \geq 0} Y(i)\Big)$$

$$SP^\infty(Z) = \operatorname*{colim}_{j \to \infty} \left(Z \wedge \cdots \wedge Z / \Sigma_j \right)$$

$$= \Big\{ \underbrace{\{y_1, \dots, y_k\}}_{\text{multiset}} : y_i \in Y(n_i) \setminus * \,,\ \Sigma n_i = n \Big\}^+$$

as a set

operation $=$ union of multisets

Notation:

X based space $\longrightarrow$ $X[n] \in Top_*^{\mathbb{N}}$

$$X[n](m) = \begin{cases} * & m \neq n \\ X & m = n \end{cases}$$

Eg

If $Y = M^+[k]$

Then $\quad \mathrm{Com}(M^+[k])(nk) = \left\{ \{p_1, \dots, p_n\} : p_i \in M \right\}^+ = \left(M^n / \Sigma_n \right)^+$

$\quad \mathrm{Com}(M^+[k])(m) = * \quad$ if $k \nmid m$

NB: In particular, $\mathrm{Com}(M^+[k])(0) \cong S^0$.

Observation:

$c(M)$ can be built out of $\mathrm{Com}(-)$:

It is the pushout of:

$$\begin{tikzcd}
\mathrm{Com}(M^+[2]) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & S^0[0] \\
\mathrm{Com}(M^+[1]) &
\end{tikzcd}$$

double every point

(*) diagram of commutative monoids

(In every positive grading, the pushout has the effect of collapsing the "fat diagonal" of $(M^n/\Sigma_n)^+$.)

Theorem (Randal-Williams)

$C(M)$ is also the homotopy pushout of (*).

$$\Updownarrow$$

$$C(M) \simeq B\big(\mathrm{Com}(M^+[1]),\ \mathrm{Com}(M^+[2]),\ S^0[0]\big)$$

two-sided bar construction:

$B(X, M, Y)$ M comm. monoid

X, Y modules over M

Notation: $H_{n,i}(Y) := \tilde{H}_i(Y(n); \mathbb{Q})$

Fact: $H_{*,*}(B(X,M,Y)) \cong \mathrm{Tor}_*^{H_{*,*}(M)}\big(H_{*,*}(X), H_{*,*}(Y)\big)$

Mild generalisation:

$$\begin{tikzcd} L \arrow[d, "\pi"'] \\ M \end{tikzcd}$$ real vector bundle

$\rightsquigarrow$ $C(M;L) \in \mathrm{Top}_*^{\mathbb{N}}$

$$n \longmapsto C_n(M;L) = \left\{ (p_1, \dots, p_n) \in L^n \;\middle|\; \underset{\text{if } i \neq j}{\pi(p_i) \neq \pi(p_j)} \right\} \Big/ \Sigma_n$$

$C(M;L)$ is again a commutative monoid, and is the pushout of the following diagram of commutative monoids:

$$\left(\begin{tikzcd} L\oplus L \arrow[d] & & (L\oplus L)_{\Sigma_2} \arrow[d] \arrow[r, phantom, "="] & L^{(2)} \\ M \arrow[rr, phantom, "\rightsquigarrow"] & & M & \end{tikzcd}\right)$$

direct sum of vector bundles

$p \in M$ unordered pair of "labels" in $L|_p$

$$\begin{tikzcd} \mathrm{Com}\left(\left(L^{(2)}\right)^+[2]\right) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"] & S^0[0] \\ \mathrm{Com}\left(L^+[1]\right) & \end{tikzcd} \qquad (**)$$

induced by

$$\begin{tikzcd} (p,\{\ell_1,\ell_2\}) \arrow[d, mapsto] \\ \{(p,\ell_1),(p,\ell_2)\} \end{tikzcd}$$

I.e. Δ doubles each point, and the new points "share" the two labels of the original point between them.

Theorem (Randal-Williams)

$C(M;L)$ is also the homotopy pushout of $(**)$.

Plan

- Why this recovers Knudsen's formula (and hence $\mathbb{Q}$-hom. stab.).
- Proof of Theorem.

Knudsen's formula

Choose L = orientation line bundle of M

$\oplus$ trivial $(d-1)$-dim. bundle

$\rightsquigarrow$ L is orientable and $\underbrace{2d}_{\text{even}}$-dimensional

$$\begin{tikzcd} C_n(M;L) \arrow[d] & \arrow[l, squiggly] \text{orientable } 2dn\text{-dim. manifold} \\ C_n(M) & \arrow[ul, squiggly] \text{vector bundle} \rightsquigarrow \text{hty equivalence} \end{tikzcd}$$

$$\rightsquigarrow \quad H^*(C_n(M);\mathbb{Q}) \cong H^*(C_n(M;L);\mathbb{Q})$$

$$\overset{\text{P.D.}}{\cong} H^{BM}_{2dn-*}(C_n(M;L);\mathbb{Q})$$

$$\cong \tilde{H}_{2dn-*}(C_n(M;L)^+;\mathbb{Q})$$

$$= H_{n,2dn-*}(C(M;L))$$

$\rightsquigarrow$ Modulo regrading, we just need to calculate

$$H_{*,*}(C(M;L))$$

By the theorem,

$$H_{*,*}(C(M;L)) \cong \mathrm{Tor}_*^{H_{**}(\mathrm{Com}((L^{(2)})^+[2]))}\left(H_{**}(\mathrm{Com}(L^+[1])),\ \mathbb{Q}[0]\right)$$

Lemma

$$C_{**}(\mathrm{Com}(L^+[1])) \overset{\text{q.i.}}{\simeq} \mathrm{Sym}^*(\tilde{C}_*(L^+;\mathbb{Q})[1])$$

$$C_{**}(\mathrm{Com}((L^{(2)})^+[2])) \simeq \mathrm{Sym}^*(\tilde{C}_*((L^{(2)})^+;\mathbb{Q})[2])$$

replace with $\tilde{H}_*$ by another quasi-isomorphism

(Every $\mathbb{Q}$-chain cx. is q.i. to its homology.)

Hence

$$\underbrace{H_{*,*}(C(M;L))} \cong \mathrm{Tor}_*^{\mathrm{Sym}^*(\tilde{H}_*((L^{(2)})^+;\mathbb{Q})[2])}\Big(\underbrace{\mathrm{Sym}^*(\tilde{H}_*(L^+;\mathbb{Q})[1])}, \mathbb{Q}[0]\Big)$$

=> Can calculate this using the "Koszul resolution" of this.

Finally, rewrite the result in terms of M instead of L and $L^{(2)}$ using the Thom isomorphism:

$$\tilde{H}_*(L^+;\mathbb{Q}) \cong \Sigma^d \tilde{H}_*(M^+;\mathbb{Q}^w)$$

Thom iso for $L \to M$

$$\Sigma_2 \circlearrowright \tilde{H}_*((L\oplus L)^+;\mathbb{Q}) \cong \Sigma^{2d}\tilde{H}_*(M^+;\mathbb{Q}) \circlearrowleft (-1)^d$$

Thom iso for $L\oplus L \to M$

$$\Downarrow$$

$$\tilde{H}_*((L^{(2)})^+;\mathbb{Q}) \cong \begin{cases} \Sigma^{2d}\tilde{H}_*(M^+;\mathbb{Q}) & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

Rmk This is where the dichotomy between even & odd dimensions arises.

$\rightsquigarrow$ This recovers Knudsen's formula, after re-indexing. $\square$

Proof of Theorem

Write $R := \mathrm{Com}(L^{+}[1])$

$S := \mathrm{Com}((L^{(2)})^{+}[2])$

Then R and $S^{\circ}[0]$ become S-modules via

$\Delta : S \longrightarrow R$ and $\varepsilon : S \longrightarrow S^{\circ}[0]$

(point-doubling) (augmentation)

Recall that we have

$$\begin{tikzcd} S \arrow[r, "\varepsilon"] \arrow[d] & S^{\circ}[0] \arrow[d] \\ R \arrow[r] & C(M;L) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} = R \underset{S}{\otimes} S^{\circ}[0]$$

(categorical pushout square)

There is a natural map:

homotopy pushout / bar construction / derived tensor product

(categorical) pushout / tensor product

$$B(R, S, S^{\circ}[0]) \longrightarrow C(M;L)$$

Aim: This is a weak equivalence.

Key lemma: R is a flat S-module,

i.e. $R \underset{S}{\otimes} - : S\text{-mod} \longrightarrow R\text{-mod}$ preserves weak equivalences.

[NB: sweeping under the carpet technicalities about spaces being "well-based", etc.]

Key lemma ⇒ Theorem

basic properties of the bar construction

$$\begin{tikzcd}
B(S,S,S^\circ[0]) \arrow[r, "\simeq"] \arrow[d, squiggly, "R \underset{S}{\otimes} -"] & S^\circ[0] \\
B(R,S,S^\circ[0]) \arrow[r, "\simeq"] & R \underset{S}{\otimes} S^\circ[0] = C(M;L)
\end{tikzcd}$$

Key lemma

□

Idea of proof of Key lemma

Recall that $R(n) = \left(L^n/\Sigma_n\right)^+$

Def Filtration $F_\bullet R$ of R by

$F_p R(n) :=$ subspace of $\left(L^n/\Sigma_n\right)^+$ of multisets $\{\ell_1,\dots,\ell_n\}$ where $\leq p$ of the points $\pi(\ell_1),\dots,\pi(\ell_n) \in M$ are not duplicated.

(together with the point at ∞)

NB: Duplicated means strictly duplicated, and not (e.g.) triplicated.

Eg For an element $\ell \in L$,

$\{\ell,\ell,\ell\} \in F_1 R(3)$

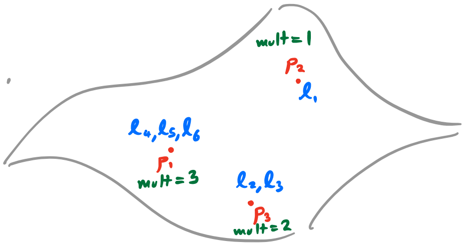
but $\{\ell,\ell,\ell\} \notin F_0 R(3)$

Rmk: This can also be described as follows:

Point of L^n/Σ_n ⟷ configurations in M with multiplicities in $\mathbb{Z}_{\geq 1}$, such that the multiplicities sum to n, and where each point $p \in M$ of the configuration, with multiplicity $i \geq 1$, is equipped with i labels in $\pi^{-1}(p)$.

$$\begin{tikzcd} L \arrow[d, "\pi"] \\ M \end{tikzcd}$$

E.g.



$\in L^6/\Sigma_6$

$$F_p R(n) = \{\leq p \text{ multiplicities are odd}\} \cup \{\infty\} \subseteq \underset{\substack{\| \\ R(n)}}{\left(L^n/\Sigma_n\right)^+}$$

Rmk: When $L = M$, this is the same filtration used by **Arnol'd** $(M = \mathbb{R}^2)$ and **Segal** to prove homological stability of $C_n(M)$.

Note that $F_0 R \cong S$, because being in filtration 0 means that all points come in pairs, which means that the configuration is in the image of $\Delta : S \to R$.

(& note that each $\Delta : S(n) \to R(n)$ is an embedding)

Claim: $\forall p \geq 0$, $F_p R \underset{S}{\otimes} -$ preserves weak equivalences.

This will complete the proof, since $F_\bullet R$ is a *finite* filtration in each weight-grading. $(F_n R(n) = R(n))$

Proof of Claim by induction on p

$p=0$: ✓ since $F_0 R \cong S$

$p \geq 1$: Consider the following square:

graded wedge sum $(X \otimes Y)(n) = \bigvee_{n=i+j} X(i) \wedge Y(j)$

$$\begin{tikzcd} F_{p-2}R(p)[p] \otimes S \arrow[r, "\text{forget}", "\text{colours}"'] \arrow[d] & F_{p-1}R \arrow[d] \\ R(p)[p] \otimes S \arrow[r, "\text{forget}", "\text{colours}"'] & F_p R \end{tikzcd}$$

(⋆)

$\leq p-1$ odd multiplicities

$\leq p$ odd multiplicities

as below, but the red config must have ≥ 1 multiplicity ≥ 1

blue config of weight $n-p$ with only even multiplicities

red config of weight p

● = formal definition

● = interpretation in terms of config's with multiplicities

NB: In weight-grading $n < p$, the LHS is trivial (i.e. a point).

Lemma (can be checked from the description in green)

(⋆) is a pushout square

Hence, for any S-module V, so is:

$$\begin{tikzcd} F_{p-2}R(p)[p] \otimes V \arrow[r] \arrow[d] & F_{p-1}R \underset{S}{\otimes} V \arrow[d] \\ R(p)[p] \otimes V \arrow[r] & F_p R \underset{S}{\otimes} V \end{tikzcd}$$

$(\star) \underset{S}{\otimes} V$

Consider

$$\begin{tikzcd} F_{p-2}R(p) \arrow[r] \arrow[d, phantom, "\parallel"] & R(p) \arrow[d, phantom, "\parallel"] \\ \{\text{subspace where at least 2 points coincide}\}^+ \arrow[r, "(\ast)"] & (L^p/\Sigma_p)^+ \end{tikzcd}$$

Key topological fact

$\exists$ open neighborhood of in $(L^p/\Sigma_p)^+$ that deformation retracts onto it.

NB: Here we use the assumption on M that it is the interior of a compact manifold with boundary.

$\Rightarrow$ Hence the map $(\ast)$ is a cofibration.

$\Rightarrow$ The LH vertical map of $(\star)\otimes_S V$ is also a cofibration.

$\Rightarrow$ $(\star)\otimes_S V$ is a homotopy pushout square

(Lemma)

Now we have:

- The square $(\star)\otimes_S V$ is natural w.r.t the S-module V.
- The LH side (top and bottom) preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$.
  (by general properties of the graded wedge sum $\otimes$)
- The top-right also preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$, by the inductive hypothesis.
- Since the square $(\star)\otimes_S V$ is a homotopy pushout square (by above), this implies that the bottom-right also preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$.

$\square$