# Homological stability for configuration spaces on closed manifolds III

GeMAT seminar
IMAR
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## Reminder of last week's talk

$M$ —— connected manifold of $\dim(M) = d \geq 2$ with $\partial M = \phi$

$C_n(M)$ = space of $n$-point subsets of $M$

**Theorem** [Bendersky – Miller '14]

For any $M$ as above, we have:

$$H_i(C_m(M);\mathbb{Q}) \cong H_i(C_n(M);\mathbb{Q}) \quad \text{for } m,n \geq 2i$$

if $m,n \neq \frac{1}{2}\chi(M)$ if $\dim(M)$ and $\chi(M)$ are even;

$$H_i(C_m(M);\mathbb{Z}_{(p)}) \cong H_i(C_n(M);\mathbb{Z}_{(p)}) \quad \text{for } m,n \geq 2i$$

if $p \geq \frac{1}{2}(\dim(M)+3)$

& $\nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$ if $\dim(M)$ is even.

i.e. each subsequence

$$\mathcal{C}_k = \{C_n(M) \mid n = \tfrac{1}{2}(\chi(M)+p^k a) \text{, } a \text{ coprime to } p\}$$

is homologically stable with $\mathbb{Z}_{(p)}$ coefficients.

## Steps of the proof

(1) Scanning maps

(2) Localisations of spaces

(3) Bundle maps

(4) Obstruction theory

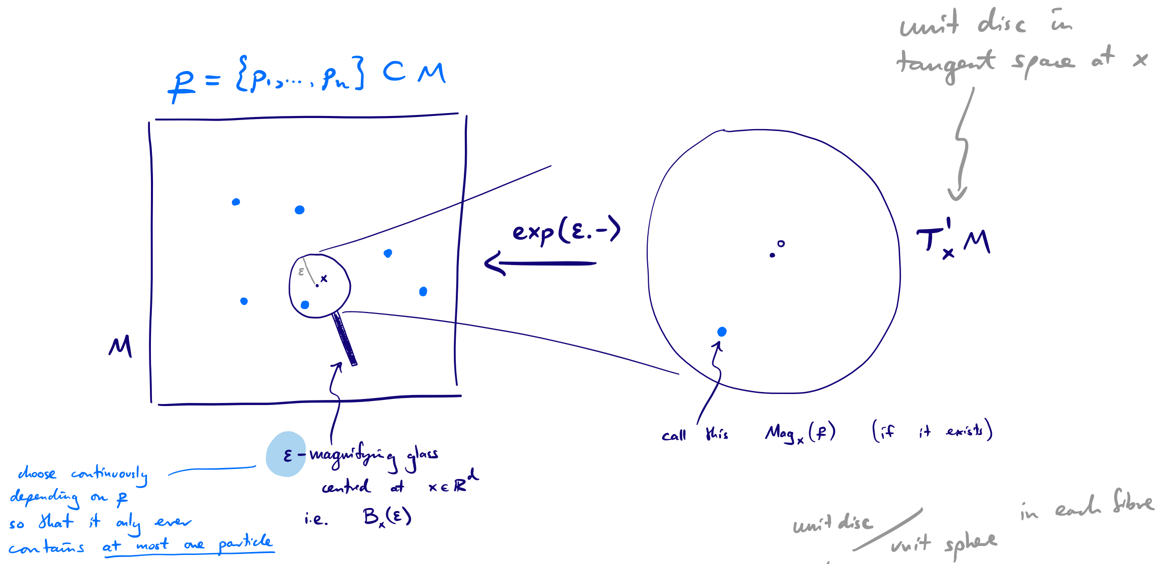
(5) Degree formula

$\left.\right\}$ (1)–(4) last week (+ reminder today)

$\left.\right\}$ (5) today

## (1) Scanning maps

**Construction** (Segal '73, McDuff '75)



unit disc / unit sphere in each fibre

$$\underline{p} \rightsquigarrow \quad M \longrightarrow \dot{T}M \cong T'M / \partial T'M$$

$$x \longmapsto \begin{cases} \mathrm{Mag}_x(\underline{p}) & \text{if } \quad |B_x(\varepsilon) \cap \underline{p}| = 1 \\ \infty & \text{if } \quad |\quad \cdots \quad| = 0 \end{cases}$$

Agrees with $\infty$ outside $\bigcup_i B_\varepsilon(p_i)$ (compact)

Intersection with 0 section is $p_1, \dots, p_n$ $\Rightarrow$ degree $n$.

$$C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$$

**Reminder** (Degree)

$$\Gamma^c(\dot{T}M) = \text{space of } \underbrace{\text{compactly-supported}}_{\substack{\uparrow \\ \text{equal to } \infty \text{ section} \\ \text{outside cpt subset}}} \text{ sections of } \begin{array}{c} \dot{T}M \\ \downarrow \\ M \end{array}$$

$$= \coprod_{n \in \mathbb{Z}} \Gamma^c_n(\dot{T}M)$$

degree = alg. intersection # of $s_*[M]$ and $\overset{\text{zero-section}}{0}_*[M]$

**Theorem** (McDuff '75 + Segal '79)

The map $C_n(M) \xrightarrow{s} \Gamma^c_n(\dot{T}M)$ induces isomorphisms on $H_i(-;\mathbb{Z})$ for $n \geq 2i$.

and hence with coefficients in any abelian group

$\longrightarrow$ Can study $H_*(\Gamma^c_n(\dot{T}M))$ instead of $H_*(C_n(M))$.

From now on, assume that $M$ is compact, so $\Gamma^c(\dot{T}M) = \Gamma(\dot{T}M)$.

## (2) Localisations

$X$ simply-connected space

$p$ prime or $0$

$\longrightarrow$ $\mathbb{Z}_{(0)} = \mathbb{Q}$

$(p \geq 2)$ $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ coprime to } p \right\}$

$\Big\downarrow$

$p$-localisation of $X$

$$X \longrightarrow X_{(p)}$$

induces isom.'s on $H_i(-;\mathbb{Z}_{(p)})$

simply-connected & integral homology is a $\mathbb{Z}_{(p)}$-module

$\Updownarrow$

$(p=0)$ $\forall$ element is uniquely $q$-divisible, $\forall$ primes $q$

$(p \geq 2)$ $\forall$ element is uniquely $q$-divisible, $\forall$ primes $q \neq p$

Can also be done fibrewise:

$$\begin{tikzcd}
\dot{T}M \arrow[d] \arrow[r] & \dot{T}M_{(p)} \arrow[d] \\
M & M
\end{tikzcd}$$

fibres $= S^d$ 　　 fibres $= S^d_{(p)}$

**Theorem** (Møller '87)

$\forall n \in \mathbb{Z}$, $\quad \Gamma_n(\dot{T}M_{(p)}) \simeq \Gamma_n(\dot{T}M)_{(p)}$

Rmk: Path-components of $\Gamma(\dot{T}M_{(p)})$ are indexed by $n \in \mathbb{Z}_{(p)}$.

**Corollary**

$\forall n \geq 2i$, $\quad H_i(C_n(M);\mathbb{Z}_{(p)}) \cong H_i(\Gamma_n(\dot{T}M_{(p)});\mathbb{Z}_{(p)})$

## (3) Bundle maps

**Lemma:**

$$\begin{tikzcd} \text{Bundle self-maps} & & & \text{Sections} \\ \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] & \mathrm{end}^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ & M \arrow[ul, dashed, bend left, "\sigma"] \arrow[ur, dashed, bend right, "\tau"'] & & M \arrow[u, bend right] \end{tikzcd}$$

such that $\phi \circ \sigma = \tau$ $\longleftrightarrow$

**Def:**

$$\begin{tikzcd} \mathrm{end}_r^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

:= bundle whose fibre over $x \in M$ is the space of self-maps $\dot{T}M_{(p)}|_x \longrightarrow \dot{T}M_{(p)}|_x$

$$\begin{tikzcd} & \simeq S^d_{(p)} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \dot{T}M_{(p)}|_x \arrow[rr] & & \dot{T}M_{(p)}|_x \end{tikzcd}$$

- of degree $r \in \mathbb{Z}_{(p)}$
- sending $\sigma(x)$ to $\tau(x)$

**Theorem** (Dold '63)

If $\phi$ restricts to a hty equivalence on each fibre, then $\phi$ is a fibrewise hty equivalence.

( $\Rightarrow$ induces a hty equivalence $\Gamma(\dot{T}M_{(p)}) \to \Gamma(\dot{T}M_{(p)})$. )
$\sigma \longmapsto \phi \circ \sigma$

**Upshot**

$$\begin{tikzcd} \mathrm{end}_r^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \arrow[rr, squiggly, "\text{induces}"] & & \Gamma(\dot{T}M_{(p)}) \arrow[r, "\simeq"] & \Gamma(\dot{T}M_{(p)}) \\ M \arrow[u, bend right] & & \sigma \arrow[u, phantom, "\in" rotate=90] & \tau \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$r \in \mathbb{Z}_{(p)}^{\times}$ ⟵ invertible, i.e.

$$\begin{cases} (p=0) & \text{non-}0 \\ (p \geq 2) & \frac{a}{b} \in \mathbb{Q} \text{ with } a,b \text{ coprime to } p \end{cases}$$

## (4) Obstruction theory

In the talk last week we proved:

**Proposition**

If $\dim(M) = d$ is odd and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

then $\forall \sigma, \tau \in \Gamma(\dot{T}M_{(p)})$, $\exists$ section

$$\begin{tikzcd} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$

( Idea of proof — Obstruction theory

+ Theorem (Serre '51) If $d$ is odd,

$$\pi_*(S^d)_{(0)} = 0 \quad \forall * \geq d+1$$

$$\pi_*(S^d)_{(p)} = 0 \quad \forall\, d+1 \leq * \leq d+2p-4$$ )

**Corollary**

Under these conditions on $d$ and $p$,

all path-components of $\Gamma(\dot{T}M_{(p)})$ are homotopy equivalent.

Hence if

- $\dim(M) = d$ is odd,
- $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

we have, for all $m, n \geq 2i$ :

scanning [McDuff-Segal]
+ localisation [Møller]

$$\begin{tikzcd} H_i(C_m(M); \mathbb{Z}_{(p)}) \arrow[r, "\cong"] & H_i(\Gamma_m(\dot{T}M_{(p)}); \mathbb{Z}_{(p)}) \arrow[d, "\cong"] \\ H_i(C_n(M); \mathbb{Z}_{(p)}) \arrow[r, "\cong"] & H_i(\Gamma_n(\dot{T}M_{(p)}); \mathbb{Z}_{(p)}) \end{tikzcd}$$

$\longrightarrow$ complete the proof when $d = \dim(M)$ is odd.

## When $\dim(M) = d$ is even:

- The vanishing results of Serre for $\pi_*(S^d)_{(p)}$ do not hold for even $d$.

$\longrightarrow$ can't prove that $\exists$

$$\begin{tikzcd} \mathrm{end}_1^{\sigma,\tau}(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$

for every $\sigma, \tau \in \Gamma(\dot{T}M_{(p)})$

- Instead:

## Proposition:

If $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

then $\forall\, r \in \mathbb{Z}_{(p)}^{\times}$, $\exists$ section

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend right] \end{tikzcd}$$

When $p = 0$, it is unique up to homotopy.

**Proof**

unbased mapping space

- The fibre of

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$$

is

$$\begin{tikzcd} \mathrm{Map}_r(S^d_{(p)}, S^d_{(p)}) \arrow[d, "\simeq"', "\substack{\text{compose with self-map} \\ \text{of degree } 1/r}"] \\ \mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}) \end{tikzcd}$$

- Obstructions to existence of a section lie in

$$H^i(M; \pi_{i-1}(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))) \qquad (*)$$

But [Serre '51] + some calculations imply that

$\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected.

Hence $(*)$ vanishes for all $i$, so $\exists$ section.

- Obstructions to uniqueness (up to $\simeq$) of a section lie in

$$H^i(M; \pi_i(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))) \qquad (**)$$

But [Møller-Raussen '85] prove (using minimal models) that

$$\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)}) \simeq S^{2d-1}_{(0)}$$

so in the case $p=0$, $\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)})$ is $d$-connected.

(because $2d-2 \geq d$)

Hence $(**)$ vanishes for all $i$, so the section is unique up to $\simeq$.

$\square$

**Upshot:**

In the even-dimensional case, if $p=0$ or $p \geq \frac{1}{2}(d+3)$,

$\forall$ invertible $r \in \mathbb{Z}_{(p)}$,

$\exists$ bundle self-hty-equivalence

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

of fibrewise degree $r$.

(Unique when $p=0$.)

## (5) Degree formula

Any bundle self-htpy equivalence

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\phi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

of fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$

induces a self-htpy equivalence $\Gamma(\dot{T}M_{(p)}) \xrightarrow[\phi\circ -]{\simeq} \Gamma(\dot{T}M_{(p)})$.

Its effect on degrees of sections is

$$\begin{tikzcd} \pi_0(\Gamma(\dot{T}M_{(p)})) \arrow[r, "(\phi\circ-)_*", "\cong"'] \arrow[d, "\deg"', "\cong"] & \pi_0(\Gamma(\dot{T}M_{(p)})) \arrow[d, "\deg"', "\cong"] \\ \mathbb{Z}_{(p)} \arrow[r, "\phi_\#"'] & \mathbb{Z}_{(p)} \end{tikzcd}$$

**Proposition** (degree formula)

If $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

and $\phi: \dot{T}M_{(p)} \longrightarrow \dot{T}M_{(p)}$ has fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$,

then

$$\phi_\#(k) = rk + \tfrac{1}{2}(1-r)\chi(M)$$

Rmk

This is false for $\dim(M) = d$ odd.

In that case we can find $\phi$ of fibrewise degree $r = 1$

such that $\phi_\#(k) = \ell$ for any two specified $k, \ell \in \mathbb{Z}_{(p)}$.

# How this finishes the proof

Assuming that $\dim(M) = d$ is even and $p = 0$ or $p \geqslant \frac{1}{2}(d+3)$,

$$\left.\begin{array}{c}\text{obstruction theory}\\ +\\ \text{degree formula}\end{array}\right\} \Rightarrow \boxed{\begin{array}{ll}\forall r \in \mathbb{Z}_{(p)}^{\times} & \Gamma_k(\dot{T}M_{(p)}) \simeq \Gamma_\ell(\dot{T}M_{(p)})\\ \forall k, \ell \in \mathbb{Z}_{(p)} & \quad \text{if } \ell = rk + \frac{1}{2}(1-r)\chi(M)\end{array}} \quad (*)$$

Now let $m, n \in \mathbb{Z}$, $m, n \geqslant 2i$

and assume that $\nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$.

when $p = 0$ this is interpreted as $2m - \chi(M) \neq 0 \neq 2n - \chi(M)$.

Then $r := \dfrac{2n - \chi(M)}{2m - \chi(M)} \in \mathbb{Z}_{(p)}^{\times}$

and
$$rm + \frac{1}{2}(1-r)\chi(M) = \frac{(2n - \cancel{\chi(M)})m + \frac{1}{2}(\cancel{2m} - 2n)\chi(M)}{2m - \chi(M)}$$
$$= \frac{2nm - n\chi(M)}{2m - \chi(M)} = n$$

Hence:

scanning [McDuff-Segal]
+ localisation [Møller]

$$\begin{tikzcd}
H_i(C_m(M);\mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_m(\dot{T}M_{(p)});\mathbb{Z}_{(p)}) \arrow[d, phantom, "\cong", "(*) \text{ with } r = \frac{2n - \chi(M)}{2m - \chi(M)}"'] \\
H_i(C_n(M);\mathbb{Z}_{(p)}) \arrow[r, phantom, "\cong"] & H_i(\Gamma_n(\dot{T}M_{(p)});\mathbb{Z}_{(p)})
\end{tikzcd}$$

$\square$

# Proof of the degree formula

If $\dim(M) = d$ is even and $p = 0$ or $p \geq \frac{1}{2}(d+3)$,

and $\phi : \dot{T}M_{(p)} \longrightarrow \dot{T}M_{(p)}$ has fibrewise degree $r \in \mathbb{Z}_{(p)}^{\times}$,

then $$\phi_{\#}(k) = rk + \tfrac{1}{2}(1-r)\chi(M) \qquad (*)$$

## Steps

(1) $M$ orientable $\Rightarrow$ $\forall r \in \mathbb{Z}_{(p)}^{\times}$ $\exists \phi$ of fibrewise degree $r$ satisfying $(*)$

(2) $M$ orientable $\Rightarrow$ $\phi_{\#}$ depends only on the fibrewise degree of $\phi$

$\rightsquigarrow$ degree formula when $M$ orientable

(3) Degree formula when $M$ non-orientable

## Rmks

- (3) is easy — just pass to orientation double covers.
- If we just care about orientable manifolds, (1) is sufficient.
- If we care about non-orientable manifolds, we need (2) to deduce (3).
- (2) is obvious when $p = 0$, because in that case $\phi$ is determined up to $\simeq$ by its fibrewise degree (uniqueness).

Plan for the remainder of the talk:

- Proof of (1),
- Proof of (2).

For the remainder of the talk we assume that $M$ is orientable.

## Step 1

$\forall r \in \mathbb{Z}_{(p)}^{\times}$ $\exists \phi$ of fibrewise degree $r$ satisfying

$$\phi_{\#}(k) = rk + \tfrac{1}{2}(1-r)\chi(M) \qquad (*)$$

$$\begin{tikzcd} TM \arrow[d] & & M \arrow[r] & BSO(d) \\ M & \text{is classified by a map} & & \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} \dot{T}M \arrow[d] & & M \arrow[r] \arrow[rr, bend left] & BSO(d) \arrow[r] & BMap_1(S^d, S^d) \\ M & \text{is classified by a map} & & & \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] & & M \arrow[r] \arrow[rr, bend left, "\tau"] & BSO(d) \arrow[r] & BMap_1(S^d_{(p)}, S^d_{(p)}) \\ M & \text{is classified by a map} & & & \end{tikzcd}$$

Recall from earlier that $Map_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected,

hence $BMap_1(S^d_{(p)}, S^d_{(p)})$ is $d$-connected.

$\Rightarrow$ $\tau$ is nullhomotopic

$$\Rightarrow \quad \begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "\psi"] \arrow[dr] & & M \times S^d_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

Given $r \in \mathbb{Z}_{(p)}^{\times}$, let

$$\phi := \psi^{-1} \circ (id \times f_r) \circ \psi$$

where $f_r : S^d_{(p)} \to S^d_{(p)}$ is the (unique up to $\simeq$) map of degree $r$.

Composition with $\psi$ induces

$$\Gamma(\dot{T}M_{(p)}) \cong \Gamma(M \times S^d_{(p)}) \cong \mathrm{Map}(M, S^d_{(p)})$$

## Lemma

This restricts, on each path-component, to

$$\Gamma_k(\dot{T}M_{(p)}) \cong \mathrm{Map}_{k - \frac{1}{2}\chi(M)}(M, S^d_{(p)}).$$

### End of proof of Step 1:

$$\begin{tikzcd}
\Gamma_k(\dot{T}M_{(p)}) \arrow[r, "\psi \circ -"] \arrow[d, "\phi \circ -"'] & \mathrm{Map}_{k - \frac{1}{2}\chi(M)}(M, S^d_{(p)}) \arrow[d, "f_r \circ -"] \\
\Gamma_\ell(\dot{T}M_{(p)}) & \mathrm{Map}_{r(k - \frac{1}{2}\chi(M))}(M, S^d_{(p)}) \arrow[l, "\psi^{-1} \circ -"']
\end{tikzcd}$$

$$\ell = r\left(k - \tfrac{1}{2}\chi(M)\right) + \tfrac{1}{2}\chi(M)$$
$$= rk + \tfrac{1}{2}(1-r)\chi(M).$$

### Proof of Lemma.

$$\psi : \dot{T}M_{(p)} \longrightarrow M \times S^d_{(p)}$$

$$\psi_* : H_d(\dot{T}M_{(p)}) \xrightarrow{\cong} H_d(M \times S^d_{(p)}) \cong \mathbb{Z} \times \mathbb{Z}_{(p)}$$

generated by $[M] \in \mathbb{Z}$
and $[F] \in \mathbb{Z}_{(p)}$

intersection number $\langle -,- \rangle : H_d(\dot{T}M_{(p)}) \times H_d(\dot{T}M_{(p)}) \longrightarrow \mathbb{Z}_{(p)}$

$$\begin{tikzcd}
\dot{T}M_{(p)} \arrow[d] \\
M \arrow[u, bend left, "0"]
\end{tikzcd}
\qquad \psi_*(0_*[M]) = [M] + k[F] \quad \text{for some } k \in \mathbb{Z}_{(p)}.$$

$$\chi(M) = \langle 0_*[M], 0_*[M] \rangle$$
$$= \langle \psi_*^{-1}[M] + k\psi_*^{-1}[F] \,,\, \psi_*^{-1}[M] + k\psi_*^{-1}[F] \rangle$$
$$= 2k$$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

$\psi_*(\sigma_*[M]) = [M] + \ell[F]$ for some $\ell \in \mathbb{Z}_{(p)}$.

$$\deg(\psi \circ \sigma) = \ell$$

$$\begin{aligned} \deg(\sigma) &= \langle 0_*[M], \sigma_*[M] \rangle \\ &= \left\langle \psi_*^{-1}[M] + \frac{\chi(M)}{2}\psi_*^{-1}[F] \ , \ \psi_*^{-1}[M] + \ell\psi_*^{-1}[F] \right\rangle \\ &= \frac{\chi(M)}{2} + \ell \end{aligned}$$

Hence $\deg(\psi \circ \sigma) = \deg(\sigma) - \frac{1}{2}\chi(M)$. □

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## Step 2

Any two bundle self-htpy equivalences

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "{\phi, \phi'}"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M \arrow[ul, bend left, "\sigma"] & \end{tikzcd}$$

of the same fibrewise degree $v \in \mathbb{Z}_{(p)}^\times$

act in the same way on degrees of sections:

$$\forall \text{ section } \sigma, \quad \deg(\phi' \circ \sigma) = \deg(\phi \circ \sigma).$$

### Proof

Let $\phi^{-1}$ denote a fibrewise htpy inverse of $\phi$.

⇝ Enough to show that $\underbrace{(\phi' \circ \phi^{-1})}_{\substack{\| \\ \theta}} \circ -$ preserves degrees of sections.

$\theta \longleftarrow$ fibrewise degree $= 1$.

Degree of a section is calculated in $H_d(\dot{T}M_{(p)})$.

$\rightsquigarrow$ Enough to show that $\theta_* : H_d(\dot{T}M_{(p)}) \longrightarrow H_d(\dot{T}M_{(p)})$ is the $\underline{\text{identity}}$.

Use a trivialisation

$$\begin{tikzcd}
\dot{T}M_{(p)} \arrow[r, "\psi"] \arrow[dr] & M \times S^d_{(p)} \arrow[d] \\
& M
\end{tikzcd}$$

to write

$$\begin{tikzcd}
H_d(\dot{T}M_{(p)}) \arrow[r, phantom, "\cong"] \arrow[d, "\theta_*"'] & H_d(M \times S^d_{(p)}) \arrow[r, phantom, "\cong"] & \mathbb{Z} \times \mathbb{Z}_{(p)} \arrow[d, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"] \\
H_d(\dot{T}M_{(p)}) \arrow[r, phantom, "\cong"] & H_d(M \times S^d_{(p)}) \arrow[r, phantom, "\cong"] & \mathbb{Z} \times \mathbb{Z}_{(p)}
\end{tikzcd}$$

Now $\theta$ is a bundle map $\rightsquigarrow$ $\alpha = 1$
$\beta = 0$

Also, $\delta$ = fibrewise degree of $\theta$
$= 1$

Hence $\theta_* = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ for some $\gamma \in \mathbb{Z}_{(p)}$.

**Lemma** $\boxed{\exists N : \theta^N \simeq \text{id}.}$

Hence $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}^N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$//$

$\begin{pmatrix} 1 & 0 \\ N\gamma & 1 \end{pmatrix}$

$\Rightarrow N\gamma = 0$

$\Rightarrow \gamma = 0 \quad \Rightarrow \theta_* = \text{id}.$ $\square$

## Proof of lemma

(a bit more technical...)

$$\begin{tikzcd}
\theta \in \text{space of bundle maps of degree } 1 \arrow[d, phantom, "\shortparallel"] \\
\Gamma(\mathrm{end}_1(\dot{T}M_{(p)})) \\
\text{topological monoid} \arrow[u]
\end{tikzcd}$$

$$[\theta] \in \pi_0(\Gamma(\mathrm{end}_1(\dot{T}M_{(p)}))) \longleftarrow \text{monoid}$$

Aim: $[\theta]$ has finite order.

Actually will prove that the whole monoid is a torsion group.

For short, write $\begin{tikzcd} P = \mathrm{end}_1(\dot{T}M_{(p)}) \arrow[d] \\ M \end{tikzcd}$, so our aim is

to show that the monoid $\pi_0(\Gamma(P))$ is a torsion group.

① Choose a CW-structure on $M$ with a single $d$-cell.

$M^{d-1} :=$ $(d-1)$-skeleton of this CW-structure.

Denote the restriction of $\begin{tikzcd} P \arrow[d] \\ M \end{tikzcd}$ to $M^{d-1}$ by $\begin{tikzcd} P^{d-1} \arrow[d] \\ M^{d-1} \end{tikzcd}$.

② Obstructions to vanishing of $\pi_0(\Gamma(P^{d-1}))$ lie in

$$H^i(M^{d-1}; \pi_i(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))).$$

Since
- $\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})$ is $(d-1)$-connected,
- $M^{d-1}$ is $(d-1)$-dimensional,

these vanish and hence $\pi_0(\Gamma(P^{d-1})) = *$.

③ $M^{d-1} \hookrightarrow M$ is a cofibration

$\Rightarrow$ restriction : $\Gamma(P) \longrightarrow \Gamma(P^{d-1})$ is a fibration.

Fibre $= \left\{ \text{extensions of a section } \begin{tikzcd} P^{d-1} \arrow[d] \\ M^{d-1} \arrow[u, bend left, "s"] \end{tikzcd} \text{ to } \begin{tikzcd} P \arrow[d] \\ M \end{tikzcd} \right\}$

$\simeq \Omega^d\left(\text{fibre of } \begin{matrix} P \\ \downarrow \\ M \end{matrix}\right)$

$= \Omega^d \mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})$

$$\begin{tikzcd} Q^{d-1} \arrow[r, phantom, "\subset"] \arrow[d] & Q \arrow[d] \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] & P^{d-1} \arrow[d] \\ S^{d-1} \arrow[u, bend left, "s'"] \arrow[r, phantom, "\subset"] & D^d \arrow[r] & M^{d-1} \arrow[u, bend right, "s"'] \end{tikzcd}$$

characteristic map of the d-cell

extension of s to $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$

trivialisable $\cong D^d \times F$

$\longleftrightarrow$ extension of s' to $\begin{matrix} Q \\ \downarrow \\ D^d \end{matrix}$

$\longleftrightarrow$ extension of s' : $S^{d-1} \longrightarrow F$
to $D^d \longrightarrow F$

$\longleftrightarrow$ based maps $D^d/S^{d-1} \longrightarrow F$
$\quad \cong S^d$

Hence we have a fibre sequence:

$$\Omega^d \mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}) \longrightarrow \Gamma(P) \longrightarrow \Gamma(P^{d-1})$$

④ This induces an exact sequence of monoids:

$$\pi_0(\Omega^d \mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})) \xrightarrow{(*)} \pi_0(\Gamma(P)) \longrightarrow \pi_0(\Gamma(P^{d-1}))$$

⑤ This has <u>two</u> monoid structures:

- composition of maps,
- concatenation of (d-fold) loops.

this one makes $(*)$ into a monoid homomorphism

this one allows the identification with $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$

By the Eckmann-Hilton argument, they agree, and hence we have:

$$\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})) \longrightarrow \pi_0(\Gamma(P)) \longrightarrow \pi_0(\Gamma(P^{d-1}))$$

(exact sequence of monoids).

⑥ By above, $\pi_0(\Gamma(P^{d-1})) = *$,

so $\pi_0(\Gamma(P)) =$ quotient of $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$

(in particular it is a group)

so it is enough to show that $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$ is a torsion group.

⑦ Recall from earlier that

$$\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)}) \simeq S^{2d-1}_{(0)}$$

[Møller - Raussen '85]

so it is $d$-connected (since $2d-2 \geq d$),

$$\begin{aligned} \text{so} \quad 0 &= \pi_d(\mathrm{Map}_1(S^d_{(0)}, S^d_{(0)})) \\ &\cong \pi_d(\mathrm{Map}_1(S^d, S^d)_{(0)}) \\ &\cong \pi_d((\mathrm{Map}_1(S^d, S^d)_{(p)})_{(0)}) \\ &\cong \pi_d(\mathrm{Map}_1(S^d, S^d)_{(p)}) \otimes \mathbb{Q} \\ &\cong \pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)})) \otimes \mathbb{Q} \end{aligned}$$

and hence $\pi_d(\mathrm{Map}_1(S^d_{(p)}, S^d_{(p)}))$ is a torsion group. □