Homological stability for configuration spaces on closed manifolds IV

GeMAT seminar
IMAR
16 May 2025

Reminder

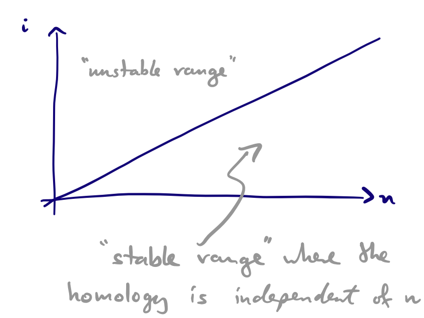
M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open — $M \cong \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$

then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.



compact $\partial M = \phi$

This is false for closed manifolds M.

E.g. $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/_{(2n-2)}$.

What we have seen so far:

Theorem (Combining [Randal-Williams '13] & [Bendersky-Miller '14])

Let $m, n \geq 2i$. Then

$$H_i(C_m(M); R) \cong H_i(C_n(M); R)$$

as long as:

- R is a field and either $\dim(M)$ is odd or $\mathrm{char}(R) \in \{0, 2\}$;
- $R = \mathbb{Z}_{(p)}$ with $p \geq \frac{1}{2}(\dim(M) + 3)$ and either $\dim(M)$ is odd or $\frac{2m - \chi(M)}{2n - \chi(M)} \in \mathbb{Z}_{(p)}^{\times}$

$\iff \nu_p(2m - \chi(M)) = \nu_p(2n - \chi(M))$

Aim for today:

Thm B • Remove the $p \geq \frac{1}{2}(\dim(M)+3)$ hypothesis

$\hookrightarrow$ obstruction theory + more geometric construction.

Thm A • Introduce "replication maps"

$\hookrightarrow$ stability with respect to replication maps instead of stabilisation maps.

$\Downarrow$

Thm C • Replication maps + puncturing trick of [Randal-Williams '13]

$\Rightarrow$ "replication stability" with field coeffs

Coro D • Combining Thm B + Thm C $\longrightarrow$ homological periodicity for $H_*(C_n(M); \mathbb{F}_p)$.

① Replication maps

Theorem A [Cantero–P. '15]

If M admits a non-vanishing vector field, $\Longleftarrow$ • M is open OR • M is closed + $\chi(M)=0$

then $\forall\, r \geq 2$ $\exists$ "replication map" $C_n(M) \xrightarrow{s_r} C_{rn}(M)$

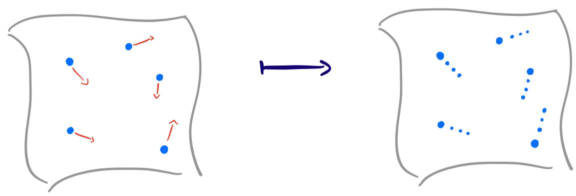
inducing isomorphisms on $H_i(-;\mathbb{Z}_{(p)})$

for all $n \geq 2i$

and $p \nmid r$.

Def Let $v \circlearrowleft \begin{array}{c} TM \\ \downarrow \\ M \end{array}$ be a non-vanishing vector field.

Then s_r :



(picture for $r = 4$, $n = 5$)

Corollary If $\chi(M) = 0$, then for fixed i and p,

$$H_i(C_n(M);\mathbb{Z}_{(p)})$$

depends only on $\nu_p(n)$ in the range $n \geq 2i$.

Proof:

If $m,n \geq 2i$ and $m = p^a\bar{m}$, $n = p^a \bar{n}$ with $\bar{m},\bar{n}$ coprime to p (i.e. $\nu_p(m) = \nu_p(n)$)

then
$$\begin{tikzcd} C_m(M) \arrow[dr, "s_{\bar{n}}"] & \\ & C_{p^a\bar{m}\bar{n}}(M) \\ C_n(M) \arrow[ur, "s_{\bar{m}}"'] & \end{tikzcd}$$
induce $\cong$ on $H_i(-;\mathbb{Z}_{(p)})$. $\square$

In general, for any $\chi(M)$:

Theorem B [Cantero - P. '15]

- If $\dim(M)$ is odd, then
$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}])$$
$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$
in the range $n \geq 2i$.

- If $\dim(M)$ is even, then for fixed i and p, (assume $\chi(M)$ even if $p=2$)
$$H_i(C_n(M);\mathbb{Z}_{(p)})$$
depends only on $\nu_p(2n-\chi(M))$ in the range $n \geq 2i$.

For $p \geq \frac{1}{2}(\dim(M)+3)$ this was [Bendersky - Miller '14].

Idea of proof

Recall that

$$C_n(M) \xrightarrow{\text{scan}} \Gamma^c_n(\dot{T}M) \longrightarrow \Gamma^c_n(\dot{T}M)_{(p)} \simeq \Gamma^c_n(\dot{T}M_{(p)})$$

H_* isom. in degrees $\leq n/2$ [McDuff, Segal]

localisation [Sullivan '70] isom. on $H_*(-;\mathbb{Z}_{(p)})$

[Møller '87]

n^{th} path-component of $\Gamma^c(\dot{T}M_{(p)})$

$$
\begin{tikzcd}
\mathrm{Map}_r(S^d_{(p)}, S^d_{(p)}) \arrow[d] \\
\mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\
M \arrow[u, bend left, "\sigma"]
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\dot{T}M_{(p)} \arrow[rr, "\varphi"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\
& M &
\end{tikzcd}
\quad \longrightarrow \quad
\Gamma^c(\dot{T}M_{(p)}) \xrightarrow{\varphi \circ -} \Gamma^c(\dot{T}M_{(p)})
$$

fibrewise degree r

__Thm__ [Dold '63]

If $r \in \mathbb{Z}_{(p)}$ is __invertible__

then φ is a fibrewise self-htz-equivalence

and so $\varphi \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ is a self-htz-equivalence.

__Strategy__:

- Construct sections $\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$ for $r \in \mathbb{Z}_{(p)}^\times$

- Understand how the corresponding $\varphi \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ acts on π_0.

[Bendersky - Miller '14] : $\left.\begin{array}{l} \text{obstruction theory} \\ p \geq \frac{1}{2}(d+3) \end{array}\right\} \longrightarrow \exists$ of σ

New idea : geometric construction

(+ different obstruction theory)

Def

$V_2(\mathbb{R}^{d+1}) :=$ space of orthonormal 2-frames in $\mathbb{R}^{d+1}$

(Stiefel manifold)

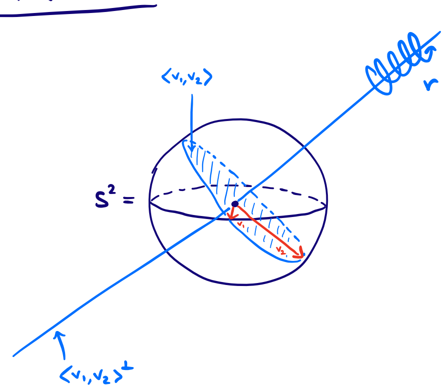
$r \in \mathbb{Z}$

$$\phi_r : V_2(\mathbb{R}^{d+1}) \longrightarrow \mathrm{Map}_r(S^d, S^d)$$

$$(v_1, v_2) \longmapsto \mathbb{R}^{d+1} = \underbrace{\langle v_1, v_2 \rangle}_{\substack{\cong \\ \mathbb{C} \\ \circlearrowleft \\ z \mapsto z^r}} \oplus \underset{\substack{\circlearrowleft \\ \mathrm{id}}}{\langle v_1, v_2 \rangle^\perp}$$

restricted to S^d = unit sphere in $\mathbb{R}^{d+1}$

Picture ($d=2$):



"Neutron star construction"

$\left(\phi_r(v_1,v_2)(v_1) = v_1 \right)$

This works fibrewise:

$E \downarrow M$ real inner product bundle of rank d

1-dim trivial bundle

fibrewise 1-pt compact^n of E

Note: $\dot{E} \cong$ unit sphere bundle of $E \oplus \mathbb{R}$

$$\rightsquigarrow \quad \begin{tikzcd} V_2(E \oplus \mathbb{R}) \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{E}) \arrow[dl] \\ & M & \end{tikzcd}$$

Applying to $E = TM$ we get

$$\begin{tikzcd} V_2(TM \oplus \mathbb{R}) \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{T}M) \arrow[dl] \\ & M & \end{tikzcd}$$

and we can fibrewise localise to get

$$\begin{tikzcd} V_2(TM \oplus \mathbb{R})_{(p)} \arrow[rr, "\phi_r"] \arrow[dr] & & \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[dl] \\ & M & \end{tikzcd}$$

Idea: A section

$$\begin{tikzcd} V_2(TM \oplus \mathbb{R})_{(p)} \arrow[d, shift left] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

induces a section

$$\begin{tikzcd} \mathrm{end}_r(\dot{T}M_{(p)}) \arrow[d, shift left] \\ M \arrow[u, bend left, "\phi_r \circ \sigma"] \end{tikzcd}$$

and hence a self-map

$$(\phi_r \circ \sigma)_* \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$$

for every $r \in \mathbb{Z}$.

Small print: to ensure the target is Γ^c and not just Γ, we have to assume that σ_0 (see below) is compactly-supported.

Notation: By forgetting the second component of a 2-frame,

$$\begin{tikzcd} V_2(TM \oplus \mathbb{R})_{(p)} \arrow[d] & \rightsquigarrow & V_1(TM \oplus \mathbb{R})_{(p)} = \dot{T}M_{(p)} \arrow[d] \\ M \arrow[u, bend left, "\sigma"] & & M \arrow[u, bend left] \end{tikzcd}$$

Call this σ_0 and assume it is compactly-supported

$\rightsquigarrow \deg(\sigma_0) \in \mathbb{Z}_{(p)}$

Lemma (degree formula)

For any σ such that σ_0 is compactly-supported, the self-map $(\phi_r \circ \sigma)_* \circ - : \Gamma^c(\dot{T}M_{(p)}) \longrightarrow \Gamma^c(\dot{T}M_{(p)})$ sends sections of degree k to sections of degree $r(k - \deg(\sigma_0)) + \deg(\sigma_0)$.

__Idea of proof__

By passing to orientation double covers, assume M orientable.
If M is non-compact, interpret $H_*(\dot{T}M_{(p)})$ as horizontally locally finite homology $H_*^{hlf}(\dot{T}M_{(p)})$, so that inclusions of fibres and sections induce $H_*(S^d_{(p)}) \to H_*^{hlf}(\dot{T}M_{(p)})$ and $H_*^{BM}(M) \to H_*^{hlf}(\dot{T}M_{(p)})$.

Then $H_d(\dot{T}M_{(p)}) \cong \mathbb{Z} \oplus \mathbb{Z}_{(p)}$

zero section $\rightsquigarrow 0_*[M] \mapsto \mathbb{Z}$, $\quad [F] = [S^d_{(p)}] \mapsto \mathbb{Z}_{(p)}$

intersection form $= \begin{pmatrix} \chi(M) & (-1)^d \\ 1 & 0 \end{pmatrix}$

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[d] \\ M \arrow[u, bend left, "\alpha"] \end{tikzcd} \rightsquigarrow \alpha_*[M] = (1, \deg(\alpha) - \chi(M))$$

By construction,

$$\begin{tikzcd} \dot{T}M_{(p)} \arrow[rr, "(\phi_r\circ\sigma)_*"] \arrow[dr] & & \dot{T}M_{(p)} \arrow[dl] \\ & M & \end{tikzcd}$$

- preserves σ_0
- has fibrewise degree r

So $H_d(\dot{T}M_{(p)}) \longrightarrow H_d(\dot{T}M_{(p)})$

- preserves $(\sigma_0)_*[M] = (1, \deg(\sigma_0) - \chi(M))$
- multiplies $[F] = (0,1)$ by r

$\Rightarrow$ it is given by $\begin{pmatrix} 1 & 0 \\ (1-r)(\deg(\sigma_0) - \chi(M)) & r \end{pmatrix} =: A$

Set $k = \deg(\alpha)$
$\ell = \deg((\phi_r\circ\sigma)_*\circ\alpha)$

$\longrightarrow$ $\alpha_*[M] = (1, k-\chi(M))$

$(\phi_r\circ\sigma)_*(\alpha_*(M)) = (1, \ell - \chi(M))$

$\quad\| $

$A.(1, k-\chi(M))$

$\quad\|$

$(1, r(k-\chi(M)) + (1-r)(\deg(\sigma_0) - \chi(M)))$

$\ell - \chi(M) = r(k-\chi(M)) + (1-r)(\deg(\sigma_0) - \chi(M))$

Rearranging $\Rightarrow$ $\ell = r(k - \deg(\sigma_0)) + \deg(\sigma_0)$. $\square$

Summary :

$$\begin{tikzcd}
V_2(TM \oplus \mathbb{R})_{(p)} \arrow[d] \arrow[dr] & \\
M \arrow[u, dashed, bend left, "\sigma"] \arrow[r, bend right, "\sigma_0"'] & V_1(TM \oplus \mathbb{R})_{(p)} = \dot{T}M_{(p)} \arrow[l]
\end{tikzcd}$$

degree $= \delta \in \mathbb{Z}_{(p)}$

If we can lift σ_0 to σ

then $\forall r \in \mathbb{Z}$ we have $\Gamma^c(\dot{T}M_{(p)}) \xrightarrow{(*)} \Gamma^c(\dot{T}M_{(p)})$

acting on path-components (degrees of sections) by

$$k \longmapsto r(k-\delta)+\delta$$

and if $r \in \mathbb{Z} \setminus p\mathbb{Z}$ then $(*)$ is a hty equivalence.

Lemma

The complete obstruction to lifting σ_0 to σ is

$$\begin{cases} 0 & \text{if } \dim(M) \text{ odd} \\ 2\delta - \chi(M) & \text{if } \dim(M) \text{ even} \end{cases}$$

Euler class of pullback of $\begin{matrix} V_2(TM \oplus \mathbb{R})_{(p)} \\ \downarrow \\ \dot{T}M_{(p)} \end{matrix}$ along $M \xrightarrow{\sigma_0} \dot{T}M_{(p)}$

Upshot:

$\dim(M)$ odd $\longrightarrow$ choose $\left.\begin{array}{l} \delta \in \mathbb{Z}_{(p)} \\ r \in \mathbb{Z} \setminus p\mathbb{Z} \end{array}\right\}$ freely

$\dim(M)$ even $\longrightarrow$ must have $\delta = \frac{1}{2}\chi(M)$

but $r \in \mathbb{Z} \setminus p\mathbb{Z}$ can be chosen freely

Obs: if $p=2$ and $\chi(M)$ is odd then this does not exist!

$$(*) \qquad \Gamma^c_n(\dot{T}M_{(p)}) \simeq \Gamma^c_{r(n-\delta)+\delta}(\dot{T}M_{(p)})$$

Proof of Theorem B

$\dim(M)$ odd

p odd $\longrightarrow$ choose $r = 2$, $\delta = n-1$

$$\Rightarrow \quad r(n-\delta)+\delta = n+1 \quad \text{in } (*)$$

$\forall p \longrightarrow$ choose $r = -1$, $\delta = n+1$

$$\Rightarrow \quad r(n-\delta)+\delta = n+2 \quad \text{in } (*)$$

$\dim(M)$ even

If $\nu_p(2m-\chi(M)) = \nu_p(2n-\chi(M))$

$$2m - \chi(M) = p^a k$$
$$2n - \chi(M) = p^a \ell \qquad k,\ell \in \mathbb{Z}\setminus p\mathbb{Z}$$

$$\Gamma^c_m(\dot{T}M_{(p)}) \underset{(*)\ r=\ell}{\simeq} \Gamma^c_{\frac{1}{2}(p^a k\ell + \chi(M))}(\dot{T}M_{(p)}) \underset{(*)\ r=k}{\simeq} \Gamma^c_n(\dot{T}M_{(p)})$$

$\square$

Proof of Theorem A

Non-vanishing vector field $\begin{array}{c} TM \\ v \uparrow\downarrow \\ M \end{array}$

$\longrightarrow$ section $\begin{array}{c} V_2(TM \oplus \mathbb{R}) \\ \sigma \uparrow\downarrow \\ M \end{array}$ $\qquad \sigma(x) = \begin{pmatrix} (v(x), 0) \\ (0, 1) \end{pmatrix}$

Note: - No need for obstruction theory
- Exists <u>before</u> fibrewise localisation

$\longrightarrow$ $\sigma_0 = \infty$-section of $\dot{T}M$

$\Rightarrow$ $\deg(\sigma_0) = 0$

Degree formula $\longrightarrow$ induced $\Gamma^c(\dot{T}M) \longrightarrow \Gamma^c(\dot{T}M)$ acts on π_0 by

$$k \longmapsto r(k - \deg(\sigma_0)) + \deg(\sigma_0)$$
$$= rk$$

Localising, if $p \nmid r$ we have:

$$\begin{tikzcd} C_n(M) \arrow[r, "\text{scan}"] \arrow[d, "\rho_r"'] & \Gamma^c_n(\dot{T}M) \arrow[r, "\text{loc.}"] \arrow[d] & \Gamma^c_n(\dot{T}M)_{(p)} \arrow[r, phantom, "\simeq"] \arrow[d] & \Gamma^c_n(\dot{T}M_{(p)}) \arrow[d, "\simeq"] \\ C_{rn}(M) \arrow[r, "\text{scan}"] & \Gamma^c_{rn}(\dot{T}M) \arrow[r, "\text{loc.}"] & \Gamma^c_{rn}(\dot{T}M)_{(p)} \arrow[r, phantom, "\simeq"] & \Gamma^c_{rn}(\dot{T}M_{(p)}) \end{tikzcd}$$

$\square$

② Puncturing trick & homological periodicity

Theorem C [Cantero–P. '15]

M — closed, even-dimensional manifold

$\mathbb{F}$ — field of characteristic $p > 0$

$r \geq 2$ — $p \nmid r$, $p \mid (\chi(M)-1)(r-1)$ $\Big\}$ in particular whenever $r \equiv 1 \pmod p$

Then

$$H_i(C_n(M);\mathbb{F}) \cong H_i(C_{rn}(M);\mathbb{F})$$

for $n \geq 2i$.

Note: There are no direct replication maps $C_n(M) \to C_{rn}(M)$ unless $\chi(M) = 0$.

When $\chi(M) \neq 0$ the "dissonance" between Thm B + Thm C implies:

Corollary D [Cantero–P. '15] (Homological periodicity)

M — closed, even-dimensional manifold with $\chi(M) \neq 0$

$\mathbb{F}$ — field of characteristic $p \geq 3$

Then

$$\boxed{H_i(C_n(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})}$$

for $n \geq 2i$, where

$$q = p^{a+1}$$
$$a = \nu_p(\chi(M))$$

Note: If $\chi(M) = 0$ then $\nu_p(0) = \infty$, so $q = \infty$.
If $\mathrm{char}(\mathbb{F}) \in \{0,2\}$ then we already know homological stability, so $q = 1$.

Proof of Conj D

Case 1 $\nu_p(2n-x) \leqslant \nu_p(x) = a$

Then $\nu_p(2(n+q)-x)$

$$= \nu_p(\underbrace{2n-x}_{\leqslant a} + \underbrace{2p^{a+1}}_{a+1}) = \nu_p(2n-x)$$

Theorem B $\longrightarrow$ $H_i(C_n(M);\mathbb{Z}_{(p)}) \cong H_i(C_{n+q}(M);\mathbb{Z}_{(p)})$

$\Rightarrow$ $H_i(C_n(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})$

(UCT, since $\mathbb{F}$ is a $\mathbb{Z}_{(p)}$-module)

Case 2 $\nu_p(2n-x) > \nu_p(x) = a$

Then $\nu_p(n) = \nu_p(\underbrace{2n-x}_{>a} + \underbrace{x}_{a}) = a$

$\Rightarrow$ $\frac{n}{p^a} \not\equiv 0 \pmod p$

$\Rightarrow$ $\exists\, \ell \geqslant 2$: $\frac{\ell n}{p^a} \equiv 1 \pmod p$

$$\ell\left(\frac{n+q}{p^a}\right) = \frac{\ell n}{p^a} + \ell p \equiv 1 \pmod p$$

$\Rightarrow$ we may take $r = \frac{\ell n}{p^a}$, $r = \ell\left(\frac{n+q}{p^a}\right)$ in Theorem C

$$H_i(C_n(M);\mathbb{F}) \cong H_i(C_{\frac{\ell n(n+q)}{p^a}}(M);\mathbb{F}) \cong H_i(C_{n+q}(M);\mathbb{F})$$

$\square$

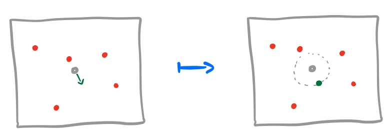
Proof of Theorem C

Recall from talk 1 (following [Randal-Williams '13]) that for any manifold M and basepoint $0 \in M$ there is a LES

$$\cdots \to H_i C_n(M) \to H_{i-d}(C_{n-1}(M \setminus 0)) \xrightarrow{S_n^i} H_{i-1} C_n(M \setminus 0) \to H_{i-1} C_n(M) \to \cdots$$

induced by

$$S^{d-1} \times C_{n-1}(M \setminus 0) \xrightarrow{t_n} C_n(M \setminus 0)$$



expand radially away from $0 \in M$
add a point specified by the S^{d-1} parameter

So if $H_i = H_i(-;\mathbb{F})$

then $\dim H_i C_n(M) = \dim \ker(S_n^i) + \dim \operatorname{coker}(S_n^{i+1})$

So it's enough to show that the maps

$$S_n^i : H_{i-d}(C_{n-1}(M \setminus 0)) \longrightarrow H_{i-1} C_n(M \setminus 0)$$

stabilise for $n \geq 2i$.

Aim: Construct maps

$$\begin{tikzcd} S^{d-1} \times C_{n-1}(M \setminus 0) \arrow[r, "t_n"] \arrow[d, "f"] & C_n(M \setminus 0) \arrow[d, "g"] \\ S^{d-1} \times C_{rn-1}(M \setminus 0) \arrow[r, "t_{rn}"] & C_{rn}(M \setminus 0) \end{tikzcd} \qquad (*)$$

Such that
- The induced vertical maps are isom.'s on H_i for $n \geq 2i$ (after passing to one summand in the Künneth decomposition on the LHS)
- (*) commutes on $H_*(-;\mathbb{F})$

Construction

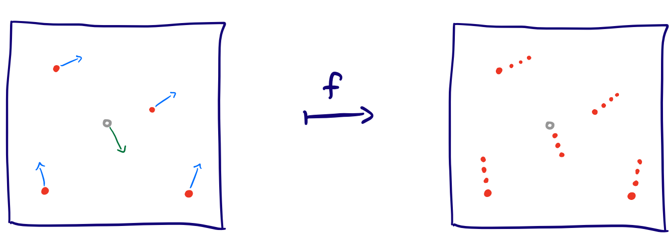
Choose a vector field on M with a unique zero at $0 \in M$.

$\Rightarrow$ non-vanishing on $M \setminus 0$

$g := \rho_r$ (replication map)

$$f := \quad \begin{array}{ccc} S^{d-1} \times C_{n-1}(M \setminus 0) & \xrightarrow{\mathrm{id} \times \rho_r} & S^{d-1} \times C_{rn-r}(M \setminus 0) \\ & & \Big\downarrow \text{“apply } t_k\text{” } r-1 \text{ times} \\ & & S^{d-1} \times C_{rn-1}(M \setminus 0) \end{array}$$

Picture:



 = config. point & the non-vanishing vector field at that point

 = the basepoint $0 \in M$ and the parameter in S^{d-1}, thought of as a unit vector in $T_0 M$

Isomorphisms on H_*

$g = g_r$ induces $\cong$ on H_i for $n \geq 2i$ by Theorem A

f induces $\cong$ on H_i for $n \geq 2i$ by
- Theorem A
- & classical hom. stability + Serre spectral sequence argument

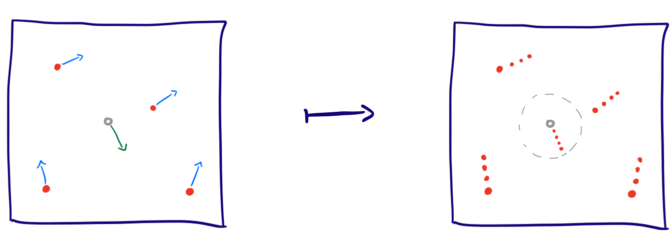
Last step:

Commutativity of the diagram on $\mathbb{F}$-homology

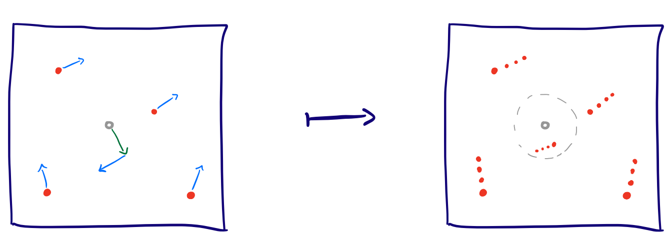
$$\begin{tikzcd}
S^{d-1} \times C_{n-1}(M \setminus 0) \arrow[r, "t_n"] \arrow[d, "f"'] & C_n(M \setminus 0) \arrow[d, "g"] \\
S^{d-1} \times C_{rn-1}(M \setminus 0) \arrow[r, "t_{rn}"'] & C_{rn}(M \setminus 0)
\end{tikzcd}$$

Pictures: $\begin{pmatrix} d=2 \\ n=5 \\ r=4 \end{pmatrix}$

$\hookrightarrow$:



$\lrcorner$:

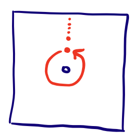


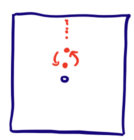
These are determined by maps $h_1, h_2 : S^{d-1} \longrightarrow C_r(\mathbb{R}^d \setminus 0)$

$\rightsquigarrow$ enough to check that

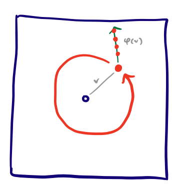
$$(h_1)_*[S^{d-1}] = (h_2)_*[S^{d-1}] \in H_{d-1}(C_r(\mathbb{R}^d \setminus 0); \mathbb{F})$$

Def Elements of $H_{d-1}(C_r(\mathbb{R}^d \setminus 0))$

$\Delta_0 :=$ 

$\pi :=$ 

For any self-map $S^{d-1} \xrightarrow{\varphi} S^{d-1}$

$\sigma_\varphi :=$ 

"comet orbiting the sun"

$=$ image of $[S^{d-1}]$ under

$$S^{d-1} \longrightarrow C_r(\mathbb{R}^d \setminus 0)$$
$$v \longmapsto \{v, \underbrace{v + \tfrac{1}{r}\varphi(v), v + \tfrac{2}{r}\varphi(v), \dots, v + \tfrac{r-1}{r}\varphi(v)}_{\text{"tail of the comet"}}\}$$

Example: $\sigma_{id} = \Delta_0$

Note: $(h_1)_*[S^{d-1}] = \sigma_{id} = \Delta_0$

$(h_2)_*[S^{d-1}] = \sigma_\varphi$ for $\varphi : S^{d-1} \longrightarrow S^{d-1}$ the restriction of the non-vanishing vector field.

[Poincaré–Hopf] $\Rightarrow$ $\deg(\varphi) = \chi(M)$.

Lemma (Lemma 5.5 of [Cantero - P. '15])

For any $\varphi : S^{d-1} \to S^{d-1}$,

$$\sigma_\varphi = r\Delta_0 + \deg(\varphi)\, r(r-1)\pi$$

Proof: Construct explicit chains 1 dimension higher in order to deduce relations between σ_φ, Δ_0 and π.

Hence $(h_1)_*[S^{d-1}] = (h_2)_*[S^{d-1}]$

$\Updownarrow$ Lemma

$$r\Delta_0 + r(r-1)\pi = r\Delta_0 + \chi(M)\, r(r-1)\pi$$

$\Updownarrow$ rearranging

order(π) divides $r(r-1)(\chi(M)-1)$

$\Uparrow$ char$(\mathbb{F}) = p$

p divides $(r-1)(\chi(M)-1)$.

$\square$