Homological stability for configuration spaces on closed manifolds V

GeMAT seminar
IMAR
23 May 2025

Recap

M — connected manifold (without boundary) of $\dim(M) = d \geqslant 2$

$C_n(M) :=$ space of n-point subsets of M

When M is open — $M \cong \mathrm{int}(\bar{M})$, $\partial \bar{M} \neq \emptyset$

$C_n(M)$ is homologically stable with $\mathbb{Z}$ coefficients (and hence with any coeffs)

$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+1}(M);\mathbb{Z})$ when $n \geqslant 2i$ [McDuff, Segal, '70s]

When M is closed — compact, $\partial M = \emptyset$

So far we have seen:

- $H_i(C_n(M);\mathbb{F}) \cong H_i(C_{n+1}(M);\mathbb{F})$ when $n \geqslant 2i$ [Randal-Williams '13]
  if $\dim(M)$ is odd, $\mathbb{F}$ field
  or $\dim(M)$ is even, $\mathbb{F}$ field of char $= 0$ or 2

Note: char $= 0$ result first proven by [Church '12] — we will look at this next time....

- When $\dim(M)$ is odd, for $n \geq 2i$:

$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$
$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}])$$

- When $\dim(M)$ is even, for $n \geq 2i$:

$H_i(C_n(M);\mathbb{Z}_{(p)})$ depends only on $\nu_p(2n-\chi(M))$
(for fixed i, p)
(assume $p \geq 3$ if $\chi(M)$ odd)

$H_i(C_n(M);\mathbb{F}_p) \cong H_i(C_{rn}(M);\mathbb{F}_p)$
for any $r \geq 2$
$r \equiv 1 \pmod p$

$H_i(C_n(M);\mathbb{F}_p) \cong H_i(C_{n+q}(M);\mathbb{F}_p)$
as long as $\chi(M) \neq 0$
$p \geq 3$

where $q = p^{1+\nu_p(\chi(M))}$

[Cantero-P. '15]

Aim for today

Theorem (Kupers-Miller '16)

- If $\dim(M)$ is odd, then for $n \geq 2i$:

$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+1}(M);\mathbb{Z})$$

- If $\dim(M)$ is even, then for $n \geq 2i$:

$$H_i(C_n(M);\mathbb{Z}/k) \cong H_i(C_{n+k}(M);\mathbb{Z}/k) \quad \text{for } k \geq 3 \text{ odd}$$
$$H_i(C_n(M);\mathbb{Z}/k) \cong H_i(C_{n+\frac{k}{2}}(M);\mathbb{Z}/k) \quad \text{for } k \geq 2 \text{ even}$$

In particular, this improves $p^{1+\nu_p(\chi(M))}$-periodicity of $H_i(C_n(M);\mathbb{F}_p)$ [Cantero-P. '15] to p-periodicity of $H_i(C_n(M);\mathbb{F}_p)$ in the stable range $n \geq 2i$.

Rmk: [Nagpal '15] proved that $H_i(C_n(M);\mathbb{F}_p)$ is $p^{(i+3)(2i+2)}$-periodic in an (unspecified) stable range.

Corollary:

If $\dim(M)$ is even, then for fixed $i \geq 1$, $p \geq 3$,

for $m, n \geq 2i$:

$$H_i(C_m(M); \mathbb{F}_p) \cong H_i(C_n(M); \mathbb{F}_p)$$

if either $2m \equiv \chi(M) \equiv 2n \pmod{p}$

or $2m \not\equiv \chi(M) \not\equiv 2n \pmod{p}$

i.e. $H_i(C_n(M); \mathbb{F}_p)$ depends only on whether $2n \equiv \chi(M)$ or not

In particular $H_i(C_n(M); \mathbb{F}_p)$ takes on ≤ 2 different values in the stable range $n \geq 2i$.

Proof

If $2m \not\equiv \chi(M) \not\equiv 2n \pmod{p}$

$\hookrightarrow$ then $\nu_p(2m - \chi(M)) = 0 = \nu_p(2n - \chi(M))$

$\longrightarrow$ result follows from [Cantero-P. '15]

(+ universal coefficient theorem to pass from $\mathbb{Z}_{(p)}$ coeffs to $\mathbb{F}_p$ coeffs)

If $2m \equiv \chi(M) \equiv 2n \pmod{p}$

$\hookrightarrow$ since $p \geq 3$, so 2 is invertible mod p,

$m \equiv n \pmod{p}$

$\longrightarrow$ result follows from [Kupers-Miller '16] (p-periodicity).

$\square$

Plan of talk / strategy of proof

from talk # 1

1. Recollection of the "puncturing trick" used by Randal-Williams in the case of coeffs in $\mathbb{F}_2$.

2. Some homology operations on E_d-algebras.

3. Chain complex lemma.

4. Proof of the theorem.

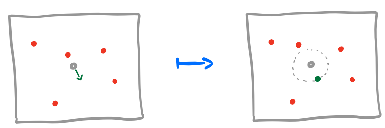
① Puncturing trick

From talk #1 we recall that, for any manifold M with basepoint $0 \in M$ there is an exact sequence:

$$\cdots \to H_i C_n(M) \to H_{i-d}(C_{n-1}(M\setminus 0)) \xrightarrow[S_n^i]{} H_{i-1}(C_n(M\setminus 0)) \to H_{i-1} C_n(M) \to \cdots$$

induced by

$$S^{d-1}\times C_{n-1}(M\setminus 0) \xrightarrow{t_s} C_n(M\setminus 0)$$



expand radially away from $0 \in M$
add a point specified by the S^{d-1} parameter

Obs (1) If the ring of coefficients is a field we have

$$\dim H_i C_n(M) = \dim \ker(S_n^i) + \dim \operatorname{coker}(S_n^{i+1})$$

(and M is of finite type, e.g. closed)

Obs (2) If the ring of coefficients is finite, we have

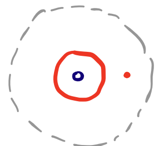
$$|H_i C_n(M)| = |\ker(S_n^i)| \cdot |\operatorname{coker}(S_n^{i+1})|$$

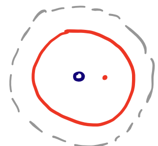
Argument of [Randal-Williams '13]:

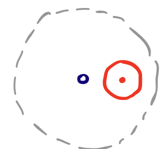
$$\begin{tikzcd}
S^{d-1}\times C_{n-1}(M\setminus 0) \arrow[r, "t_s"] \arrow[d, "\mathrm{id}\times t_p"'] & C_n(M\setminus 0) \arrow[d, "t_p"] \\
S^{d-1}\times C_n(M\setminus 0) \arrow[r, "t_s"'] & C_{n+1}(M\setminus 0)
\end{tikzcd}$$

(non-commutative square)

On H_* this acts by

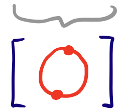
$\llcorner_{>}$: adding  near the puncture

$\urcorner_{\downarrow}$: adding  near the puncture

The difference is homologous to the operation that adds 

This is the image of $[$  $] \in H_{d-1}(C_2(\mathbb{R}^d))$

$$\begin{array}{ccc} \| & & \cong \\ 2 \cdot \underbrace{\text{generator}}_{[\bigcirc]} & \in & H_{d-1}(\mathbb{RP}^{d-1}) \end{array}$$



$\Rightarrow$ with $\mathbb{F}_2$ coefficients it's zero

$\Rightarrow$ commutative square on $H_*(- ; \mathbb{F}_2) = H_*$:

$$\begin{tikzcd} H_{i-d}(C_{n-1}(M \setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "(t_p)_*"'] & H_{i-1}(C_n(M \setminus 0)) \arrow[d, "(t_p)_*"] \\ H_{i-d}(C_n(M \setminus 0)) \arrow[r, "S_{n+1}^i"] & H_{i-1}(C_{n+1}(M \setminus 0)) \end{tikzcd}$$

$\cong$ in stable range

$\Rightarrow$ $\dim \ker(S_n^i) = \dim \ker(S_{n+1}^i)$
$\dim \operatorname{coker}(S_n^{i+1}) = \dim \operatorname{coker}(S_{n+1}^{i+1})$

in the stable range

$\Rightarrow$ $\square$ by Obs (1).

② Homology operations

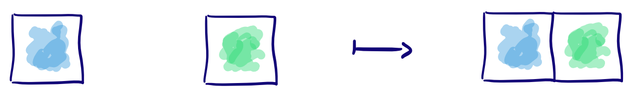
$$C(\mathbb{R}^d) = \coprod_{n \geq 0} C_n(\mathbb{R}^d)$$

$H_* C(\mathbb{R}^d)$ has two key operations:

Pontryagin product

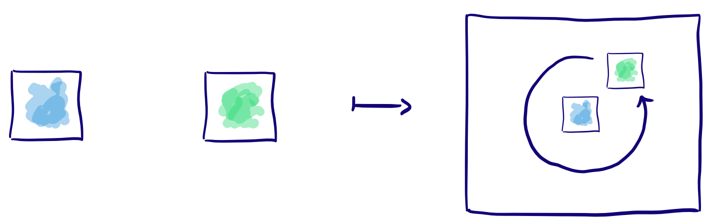
commutative when $d \geq 2$

$$H_i C_m(\mathbb{R}^d) \otimes H_j C_n(\mathbb{R}^d) \longrightarrow H_{i+j} C_{m+n}(\mathbb{R}^d)$$



Browder bracket

$$H_i C_m(\mathbb{R}^d) \otimes H_j C_n(\mathbb{R}^d) \xrightarrow{\phi} H_{i+j+d-1} C_{m+n}(\mathbb{R}^d)$$



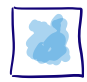
Lemma For $\alpha \in H_j C_m(\mathbb{R}^d)$, the difference between the two ways around the square

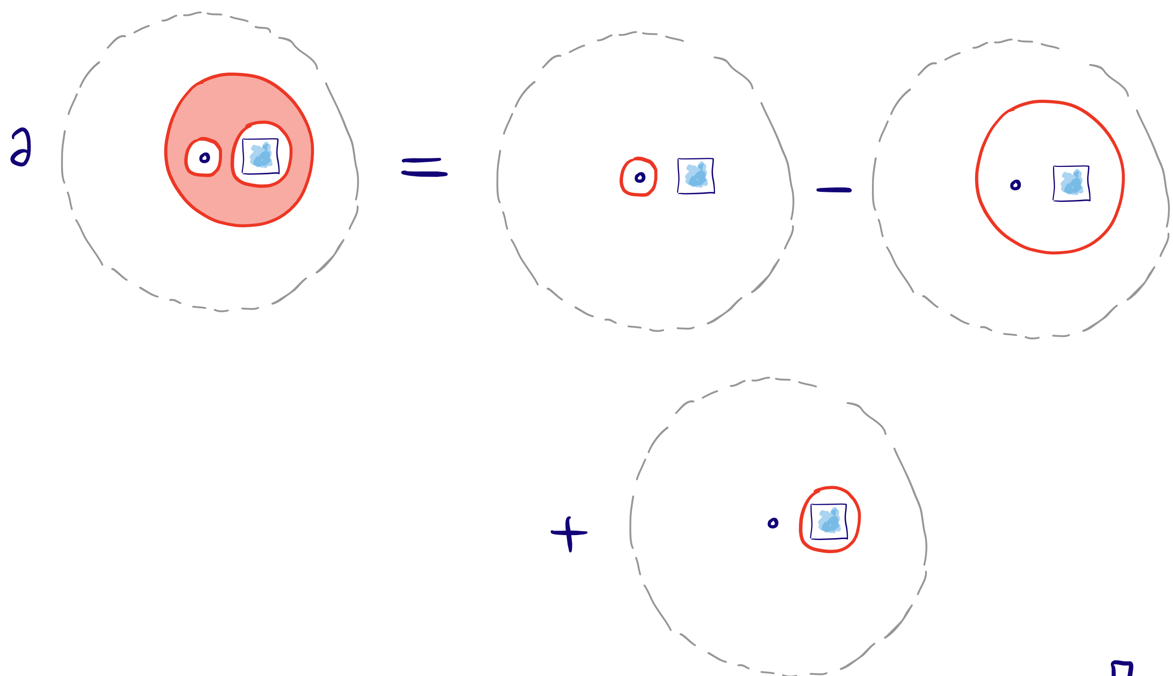
$$\begin{tikzcd}
H_{i-d}(C_n(M \setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "t_\alpha"'] & H_{i-1}(C_{n+1}(M \setminus 0)) \arrow[d, "t_\alpha"] \\
H_{i-d+j}(C_{n+m}(M \setminus 0)) \arrow[r, "S_{n+m}^{i+j}"] & H_{i-1+j}(C_{n+1+m}(M \setminus 0))
\end{tikzcd}$$

add a copy of α near the puncture

is equal to $t_{\phi(\alpha, \cdot)}$.

Proof:

$\alpha =$ 



□

Lemma

(a) For d odd, $\phi(\bullet,\bullet) = 0$ with $\mathbb{Z}$ coefficients.

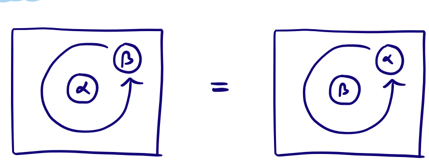
(b) For d even, $\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet)$ is divisible by $2k$.

Proof

(a) $\phi(\bullet,\bullet) = \odot = 2.\,\text{generator} \in H_{d-1}(\mathbb{R}P^{d-1}) \cong \mathbb{Z}/2$.

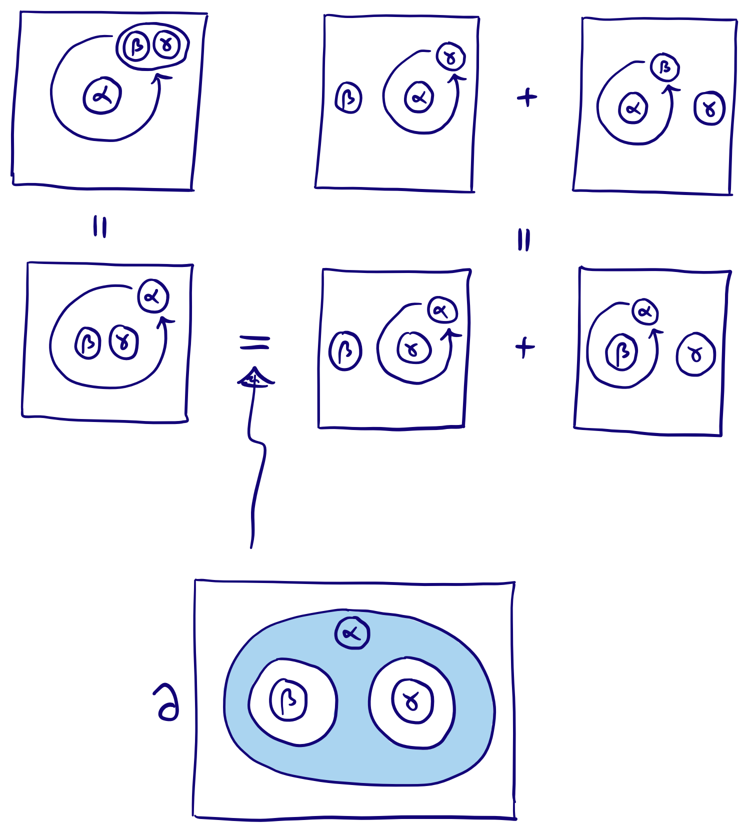
(b)

Fact 1: ϕ is symmetric



Fact 2 : ϕ is a derivation, i.e.

$$\phi(\alpha, \beta\gamma) = \beta\,\phi(\alpha,\gamma) + \phi(\alpha,\beta)\,\gamma$$



Iterating Fact 2 (& Fact 1) we get:

$$\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet) = \underbrace{\bullet\bullet\cdots\bullet}_{k-1}\,\phi(\bullet,\bullet) + \bullet\,\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k-1},\bullet)$$

$$\cdots = k\,.\,\underbrace{\bullet\bullet\cdots\bullet}_{k-1}\,\phi(\bullet,\bullet)$$

$\phi(\bullet,\bullet)$ divisible by 2 $\longrightarrow$ $\phi(\underbrace{\bullet\bullet\cdots\bullet}_{k},\bullet)$ divisible by $2k$

$\square$

Corollary The square

$$\begin{tikzcd} H_{i-d}(C_n(M\setminus 0)) \arrow[r, "S_n^i"] \arrow[d, "(t_p)^m"'] & H_{i-1}(C_{n+1}(M\setminus 0)) \arrow[d, "(t_p)^m"] \\ H_{i-d}(C_{n+m}(M\setminus 0)) \arrow[r, "S_{n+m}^i"] & H_{i-1}(C_{n+1+m}(M\setminus 0)) \end{tikzcd}$$

m-fold iterated stabilization map

commutes if
- we take coeffs in a ring R : char (R) divides $2m$,
- or we take coeffs in any ring and $\dim(M)$ is odd.

no condition on m in the second case.

Corollary 2

Under these conditions,

$$\ker(S_n^i) \cong \ker(S_{n+m}^i)$$

& $$\operatorname{coker}(S_n^i) \cong \operatorname{coker}(S_{n+m}^i)$$

in the stable range.

Obs(2)

$\Rightarrow$ If $R = \mathbb{Z}/k$

then $$|H_i(C_n(M); \mathbb{Z}/k)| = |\ker(S_n^i)| \cdot |\operatorname{coker}(S_n^{i+1})|$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$|H_i(C_{n+m}(M); \mathbb{Z}/k)| = |\ker(S_{n+m}^i)| \cdot |\operatorname{coker}(S_{n+m}^{i+1})|$$

for $$m = \begin{cases} k & k \text{ odd} \\ k/2 & k \text{ even} \\ 1 & \dim(M) \text{ odd} \end{cases}$$

Upshot: $|H_i(C_n(M); \mathbb{Z}/k)|$ is m-periodic in the stable range

(for fixed i, k)

③ Chain complex lemma

C_* chain complex over $\mathbb{Z}$

bounded above and below

finitely generated & free in each degree

(E.g. cellular chain complex of a finite CW-complex)

Lemma

(A)

finite set

$$\left.\begin{array}{l} \left|H_j(C_* \otimes \mathbb{Z}/p^s)\right| \\ \text{for all } j \leq i \\ \qquad 1 \leq s \leq r \end{array}\right\} \xrightarrow{\text{determines}} H_i(C_* \otimes \mathbb{Z}/p^r)$$

abelian group

(B)

$$\left.\begin{array}{l} \left|H_j(C_* \otimes \mathbb{Z}/p^s)\right| \\ \text{for all } j \leq i \\ \qquad 1 \leq s \\ \qquad \text{primes } p \end{array}\right\} \xrightarrow{\text{determines}} H_i(C_*)$$

In particular, if X is a finite CW-complex,

$H_*(X;\mathbb{Z})$ is determined by $\left|H_*(X;\mathbb{Z}/p^s)\right|$ $\forall p, s$.

Idea of proof

(A)

$C_* \otimes \mathbb{Z}/p^r \simeq$ finite $\oplus$ of copies of $\cdots \to 0 \to \mathbb{Z}/p^r \to 0 \to \cdots$

quasi-isomorphic

and $\cdots \to 0 \to \mathbb{Z}/p^r \xrightarrow{\cdot p^s} \mathbb{Z}/p^r \to 0 \to \cdots$

for $1 \leq s \leq r-1$.

$\rightsquigarrow$ enough to determine the multiplicities of these pieces.

Write down a formula for $|H_j(C_* \otimes \mathbb{Z}/p^s)|$ in terms of these multiplicities:

$$\{\text{multiplicities}\} \xrightarrow[\text{of linear equations}]{\text{determine via system}} \left\{ |H_j(C_* \otimes \mathbb{Z}/p^s)| \quad \begin{matrix} j \leq i \\ 1 \leq s \leq r \end{matrix} \right\}$$

Matrix is invertible, hence

(Actually to determine multiplicities of the pieces shifted by i, need to know multiplicities of the pieces shifted by $i-1$.

$\longrightarrow$ Do this recursively.

$\longrightarrow$ Base case: zero, since C_* bounded below.)

(B)

A more careful version of the argument in (A) implies that

$$\left| H_i(C_* \otimes \mathbb{Z}/p^r) \right| \quad \text{for all } i \text{ and } r \quad \xrightarrow{\text{determines}} \quad H_i(C_* \otimes \mathbb{Z}/p^r) \quad \text{for all } i \text{ and } r$$

and the projections

$$\begin{array}{c} H_i(C_* \otimes \mathbb{Z}/p^r) \\ \downarrow \\ H_i(C_* \otimes \mathbb{Z}/p^s) \end{array} \quad \text{for } s \leq r$$

This determines

$$H_i(C_* \otimes \hat{\mathbb{Z}}_p) \cong \lim_{\infty \leftarrow r} H_i(C_* \otimes \mathbb{Z}/p^r)$$

p-adic integers

$\hat{\mathbb{Z}}_p$ is torsion-free $\Rightarrow$ flat $\mathbb{Z}$-module

$\Rightarrow H_*(C_* \otimes \hat{\mathbb{Z}}_p) \cong H_*(C_*) \otimes \hat{\mathbb{Z}}_p$

Each $H_i(C_*)$ is fin. generated, so it is determined by knowing $H_i(C_*) \otimes \hat{\mathbb{Z}}_p$ for all primes p.

$\square$

④ Finishing the proof of the theorem

We already know that, in the stable range $n \geq 2i$:

$$|H_i(C_n(M); \mathbb{Z}/_k)| \cong |H_i(C_{n+k}(M); \mathbb{Z}/_k)| ,$$

and $$|H_i(C_n(M); \mathbb{Z}/_k)| \cong |H_i(C_{n+\frac{k}{2}}(M); \mathbb{Z}/_k)| \quad \text{if } k \text{ is even,} \qquad (*)$$

and $$|H_i(C_n(M); \mathbb{Z}/_k)| \cong |H_i(C_{n+1}(M); \mathbb{Z}/_k)| \quad \text{if } \dim(M) \text{ is odd.} \qquad (**)$$

dim(M) even

- $(*)$ holds in particular for any $k = p^r$.
- chain complex lemma Ⓐ

  M closed
  $\Rightarrow$ $C_n(M)$ finite CW-complex

  $\Rightarrow$

$$H_i(C_n(M); \mathbb{Z}/_{p^r}) \cong H_i(C_{n+p^r}(M); \mathbb{Z}/_{p^r}) \quad p \geq 3$$
$$H_i(C_n(M); \mathbb{Z}/_{2^r}) \cong H_i(C_{n+2^{r-1}}(M); \mathbb{Z}/_{2^r})$$

- $\Rightarrow$ periodicity with general $\mathbb{Z}/_k$ coeffs by decomposing $\mathbb{Z}/_k \cong \bigoplus$ of $\mathbb{Z}/_{p^r}$.

dim(M) odd

- $(**)$ holds in particular for any $k = p^r$.
- chain complex lemma Ⓑ

  $\Rightarrow$

$$H_i(C_n(M); \mathbb{Z}) \cong H_i(C_{n+1}(M); \mathbb{Z}) . \qquad \square$$