On the homology of asymptotic monopole moduli spaces

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Outline

1. Background — magnetic monopole moduli spaces $\mathcal{M}_k$ (p. 2)
2. Gibbons–Manton torus bundles (p. 6)
3. Asymptotic monopole moduli spaces $\mathcal{M}_\lambda$ (λ partition of k) (p. 9)
   [Kottke–Singer]
4. Homological stability (p. 13)

   Thm [P.–Tillmann '22]

   Fix $c \geq 1$ and $\lambda = \{k_1, \dots, k_r\}$

   Let $\lambda[n] = \{k_1, \dots, k_r, \underbrace{c, \dots, c}_{n}\}$

   Then $H_i(\mathcal{M}_{\lambda[n]}) \cong H_i(\mathcal{M}_{\lambda[n+1]})$ if $n \geq 2i$.

   Proof – "nonlocal configuration spaces."
5. Towards stable homology (p. 16)

① Background

Maxwell's extended equations of electromagnetism

↳ E = electric field

B = magnetic field

$\rho_{e/m}$ = electric / magnetic charge density

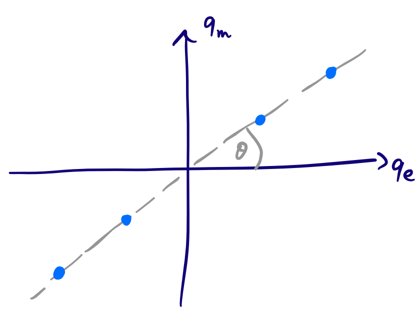
$q_{e/m}$ = electric / magnetic point charge of particle

⋮

$$\nabla \cdot E = 4\pi \rho_e$$
$$\nabla \cdot B = \cancel{0}\ 4\pi \rho_m$$
$$\vdots$$

Symmetric under S^1-action that rotates $\begin{pmatrix} E \\ B \end{pmatrix}$, $\begin{pmatrix} q_e \\ q_m \end{pmatrix}$, etc.

All charged particles observed so far lie on a line:

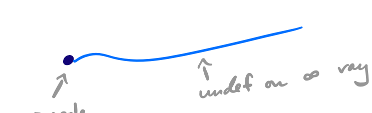


Convention: $\theta = 0$

Dirac (1931): If $\exists$ magnetic monopole $\left(\text{charge} = \begin{pmatrix} 0 \\ q_m \end{pmatrix}\right)$ then all electric charges $\begin{pmatrix} q_e \\ 0 \end{pmatrix}$ are quantised. [QM $\Rightarrow$ $\underbrace{q_m q_e}_{\text{angular momentum of the EM field}}$ is quantised]

Solutions to M.E. with magnetic monopoles — but not defined on all of $\mathbb{R}^3$.

"Dirac monopoles"



't Hooft – Polyakov (70s)
+ Bogomolny – Prasad – Sommerfield

"BPS monopoles"

Solns to a different set of equations (Bogomolny eqⁿs)

Defined on $\mathbb{R}^3$

Behave like Dirac monopoles at large distances

$$\mathcal{M} := \text{moduli space of (BPS) monopoles}$$
$$= \{\text{solns to Bogomolny eqns}\}/\text{gauge}$$

gauge transformations that $\to$ id as you go to ∞ in a fixed direction

$\pi_0(\mathcal{M}) \cong \mathbb{Z}_+$

- Part of the data is $\phi : \mathbb{R}^3 \longrightarrow su(2) \cong \mathbb{R}^3$

  Higgs field

- ∂-cond's at ∞ $\Rightarrow$ $\mathbb{R}^3 \setminus B_R(0) \xrightarrow{\phi} \mathbb{R}^3 \setminus \{0\}$
- The (magnetic) charge of the soln = degree of ϕ (always positive)

$$\mathcal{M}_k := \text{component of total charge} = k$$

4k-dim hyperKähler manifold

admits free action of $S^1 \times \mathbb{R}^3$

total phase — translation

(and $N_k^0 = \mathcal{M}_k / (S^1 \times \mathbb{R}^3)$ is hyperKähler of dim $4k-4$.)

<u>Other versions</u>:

$$\begin{array}{l} \mathcal{X}_k = \widetilde{\mathcal{M}}_k \\ \quad\Big\downarrow \\ \mathcal{M}_k \longrightarrow N_k = \mathcal{M}_k / S^1 \end{array}$$

Eg $\mathcal{M}_1 = S^1 \times \mathbb{R}^3$ [Prasad - Sommerfield]

$\mathcal{N}_1^\circ = *$

$\mathcal{N}_2^\circ =$ "Atiyah - Hitchin manifold" (4-dim)

Thm

[Donaldson '84] + [Boyer - Mann '88]

$$\mathcal{M}_k \cong \{ \mathbb{CP}^1 \xrightarrow{f} \mathbb{CP}^1 \mid \text{rational, degree} = k, \text{ based} \}$$

- based: $f(\infty) = 0$

  S^1- action is the obvious one

  (but $\mathbb{R}^3$ - action is not obvious to see)

- based: $f(\infty) = 1$

  (S^1- action no longer obvious)

  $f \longmapsto$ pair of monic poly's of degree k with no roots in common

Cor:

$$\mathcal{M}_k \cong \{ (Z, P) \in \underbrace{Sp^k(\mathbb{C})}_{= \mathbb{C}^k / \Sigma_k} \times Sp^k(\mathbb{C}) \mid Z \cap P = \phi \}.$$

Homology / homotopy:

$\pi_1(\mathcal{M}_k) \cong \mathbb{Z}$.

Thm [Segal '79]

$\exists$ maps $\mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}$ inducing $\cong$ on π_i for $i \leq k$

(& hence also on H_i)

and $\displaystyle\lim_k H_*(\mathcal{M}_k) \cong \lim_k H_*(B_k)$.

Thm [Cohen-Cohen-Mann-Milgram '91]

$$H_*(\mathcal{M}_k) \cong H_*(B_{2k}).$$

Note:

$$\begin{tikzcd}
 & C_{k,k}(\mathbb{C}) \arrow[dl] \arrow[dr, hook] & \\
C_{2k}(\mathbb{C}) \arrow[rr, dotted, dash, bend right] & & \mathcal{M}_k
\end{tikzcd}$$

$\nexists$ H_*-isom that makes this commute.

$$\mathcal{M}_k \underset{s}{\simeq} \bigvee_{j=1}^{k} D_j(S^1) \underset{s}{\simeq} B(B_{2k})$$

[CCMM '91] [Brown-Peterson '78]

② Gibbons - Manton torus bundles

Warm-up: winding #s.

$F_n(\mathbb{R}^2)$ = ordered config. space on $\mathbb{R}^2$

$$H^1(\cdots) \cong \mathbb{Z}^{\binom{n}{2}} = \mathbb{Z}\{w_{ij} \mid 1 \le i < j \le n\}$$

(UCT) $\cong$

$$\mathrm{Hom}(PB_n, \mathbb{Z})$$

$w_{ij} : PB_n \twoheadrightarrow \mathbb{Z}$ winding # of i^{th} strand around j^{th} strand.

induced by: $F_n(\mathbb{R}^2) \xrightarrow{\alpha_{ij}} S^1$

- forget all pts except p_i and p_j
- translate so that $p_i = 0$
- normalise (rescale) so that $|p_j| = 1$.

$\boxed{\mathbb{Z}\text{-linear combination of } w_{ij}}$ $\longrightarrow$ $\boxed{\text{regular covering of } F_n(\mathbb{R}^2) \text{ with Deck} \cong \mathbb{Z}}$

$\Updownarrow$

$\boxed{\text{principal } \mathbb{Z}\text{-bundle over } F_n(\mathbb{R}^2)}$

Higher dimensions

$$H^{d-1}(F_n(\mathbb{R}^d)) \cong \mathbb{Z}\{w_{ij} \mid 1 \leq i < j \leq n\}.$$

$$F_n(\mathbb{R}^d) \xrightarrow{\alpha_{ij}} S^{d-1}$$

$$w_{ij} := \alpha_{ij}^*([S^{d-1}]).$$

Dimension 3

Recall that $\quad H^2(X;\mathbb{Z}) \xleftrightarrow{1:1} \{\text{principal } S^1\text{-bundles}\}$

$\boxed{\mathbb{Z}\text{-linear combination of } w_{ij}} \longrightarrow \boxed{\text{principal } S^1\text{-bundle over } F_n(\mathbb{R}^3)}$

Def

- Fix $\quad \lambda = \{k_1, \dots, k_n\} \qquad k_i \in \mathbb{Z}$
- For $1 \leq j \leq n$ let $\begin{array}{c} S^j_\lambda \\ \downarrow \\ F_n(\mathbb{R}^3) \end{array}$ = prin. S^1-bundle class. by $\sum_{\substack{i=1 \\ i\neq j}}^{n} k_i \cdot w_{ij}$
- Gibbons-Manton torus bundle:

$$T_\lambda := \begin{array}{c} \bigoplus_{j=1}^{n} S^j_\lambda \\ \downarrow \\ F_n(\mathbb{R}^3). \end{array}$$

The j-th S^1 parameter encodes interaction of the j-th particle with each of the other particles, weighted by λ.

Interpretation:

A point in $\tilde{T}_\lambda$ consists of

1. An ordered configuration in $\mathbb{R}^3$
2. For each config. point, an S^1-parameter that encodes its interaction with all the other points. ("weighted by λ")

One dimension lower, the analogue of 2 is a $\mathbb{Z}$-parameter, i.e. an integer, that records the total winding of one pt around the others.
(weighted by λ)

3 Asymptotic monopole moduli spaces

Recall: Borel construction

$$\begin{array}{c} P \\ \downarrow \\ X \end{array} \quad \text{principal } G\text{-bundle} \quad \rightsquigarrow \quad G \text{ acts on } P$$

F any space with action of G

diagonal action.

$$\rightsquigarrow \quad \begin{array}{c} P \times_G F := (P \times F)/G \\ \downarrow \\ X \end{array}$$

bundle over X with fibre F (and str. group G).

Eg: principal $\mathbb{Z}/2$-bundle $\xrightarrow[\mathbb{Z}/2 \curvearrowright \mathbb{R}]{}$ associated line bundle.
$\|$
double covering

Recall that $\mathcal{M}_k$ has an action of S^1.

Def $\lambda = \{k_1, \ldots, k_n\}$ $\quad k_i \geq 1$

$$\begin{array}{c} \widetilde{\mathcal{M}}_\lambda := T_\lambda \times_{(S^1)^n} (\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_n}) \\ \big\downarrow \\ F_n(\mathbb{R}^3) \end{array}$$

(Borel construction)

Σ_n acts on $\mathbb{Z}^n$

$\Sigma_\lambda :=$ stabiliser of $(k_1, \ldots, k_n)$

$$\begin{array}{c} \mathcal{M}_\lambda := \widetilde{\mathcal{M}}_\lambda / \Sigma_\lambda \\ \big\downarrow \\ F_n(\mathbb{R}^3)/\Sigma_\lambda =: C_\lambda(\mathbb{R}^3) \end{array}$$

fibre = product of symmetric powers of $\mathcal{M}_k$ for different k.

Note: Actually use normalised config. spaces (centre of mass $= 0$, sum of squares $= 1$) and centred monopole spaces $\mathcal{M}_k / \mathbb{R}^3$.

Eg $\mathcal{M}_1 / \mathbb{R}^3 \cong S^1$

So if $k_1 = k_2 = \cdots = k_n = 1$, the Borel construction does nothing and $\widetilde{\mathcal{M}}_\lambda = \Upsilon_\lambda$

$$\mathcal{M}_\lambda = \Upsilon_\lambda / \Sigma_n .$$

Note: if $\lambda = (k)$

then $\Upsilon_{(k)}$ = trivial S^1-bundle over $C_1(\mathbb{R}^3) = *$

$$= S^1$$

$$\mathcal{M}_{(k)} = \left(S^1 \times_{S^1} \mathcal{M}_k \right) / \Sigma_1$$

$$= \mathcal{M}_k .$$

Intuition:

Looking at a monopole in $\mathbb{R}^3$ from "far away", we see n "clusters" of different magnetic charges, concentrated at n points in $\mathbb{R}^3$. Locally, each one looks like a classical (non-asymptotic) monopole. The structure of the Gibbons-Manton torus bundles encodes how the different "clusters" interact with each other.

Thm [Kottke-Singer, Memo AMS, 2022]

$\exists$ partial compactification of $\mathcal{M}_k$

(the codim-1 faces of a full compactification)

given by: $\displaystyle\bigcup_{\lambda} \mathcal{M}_\lambda$

where $\lambda = \{k_1, \ldots, k_n\}$, $k_i \geq 1$, $\sum_{i=1}^{n} k_i = k$.

- $\mathcal{M}_{(k)} = \mathcal{M}_k$ = interior
- $\mathcal{M}_\lambda$ for λ non-trivial $\rightsquigarrow$ ∂-faces.

Note:

- Unpublished work of Fritzsch-Kottke-Singer extends this to a full compact^n.
- Strata $\longleftrightarrow$ rooted trees with leaves labelled by +ve integers summing to k.

of codimension d

of height d

④ Homological stability

Thm [P.–Tillmann, '22/'23]

arXiv published

Fix $\lambda = \{k_1, \dots, k_r\}$

$c \geq 1$

Set $\lambda[n] = \{k_1, \dots, k_r, \underbrace{c, \dots, c}_{n}\}$

$\exists\ \mathcal{M}_{\lambda[n]} \longrightarrow \mathcal{M}_{\lambda[n+1]}$

inducing isom's on $H_i(-)$ for $n \geq 2i$.

Rmk: An easy corollary of Segal's result is that

$\mathcal{M}_{(k_1, \dots, k_n)} \longrightarrow \mathcal{M}_{(k_1, \dots, k_j + 1, \dots, k_n)}$ induces isom's

on $H_i(-)$ for $i \leq k_j$.

Instead of increasing the charge of a fixed "cluster", our result studies the effect of increasing the number of "clusters" of a fixed charge.

Idea of proof :

Prove a more general result about configuration spaces with "non-local labels".

Thm M — connected, non-compact manifold

Z — based, path-connected space

$$\begin{tikzcd} E_n \arrow[r, "s_n"] \arrow[d, two heads, "\pi_n"'] & E_{n+1} \arrow[d, two heads, "\pi_{n+1}"] \\ C_n(M) \arrow[r] & C_{n+1}(M) \end{tikzcd}$$

fibrations

$c_n \in C_n(M)$

classical stabilisation map

choose basepoints

Assume that (a) $\pi_n^{-1}(c_n) \cong Z^n$

(b) monodromy of π_n : $\begin{tikzcd} \pi_1 C_n(M) \arrow[r] \arrow[d, two heads] & hAut(Z^n) \\ \Sigma_n \arrow[ur] & \end{tikzcd}$

(c) $s_n|_{Z^n}$: $\begin{tikzcd} Z^n \arrow[r] \arrow[d, "\cong"'] & Z^{n+1} \\ \{*\} \times Z^n \arrow[ur] & \end{tikzcd}$

Then $E_n \to E_{n+1} \to \cdots$ is homologically stable.

Proof sketch :

Serre spectral sequence argument

$\longrightarrow$ reduce to proving twisted hom. stab. for $C_n(M)$ w.r.t. a certain "polynomial" coeff. system.

$\longrightarrow$ this was proven in [P. '18].

Examples of E_n

$$\begin{array}{c} E_n \\ \downarrow \\ C_n(M) \end{array}$$

① $E_n = C_n(M;Z)$ config. points labelled by elements of Z ("Local labels")

② <u>Lemma:</u>

E_n = Gibbons-Manton torus bundles

+ anything obtained from them by Borel construction.

↳ <u>Key point:</u> constructing the lifted stabilisation maps.

(Need to understand pullbacks of Gibbons-Manton circle factors along the (classical) stabilisation maps.)

□.

<u>Non-example</u>

③ Hurwitz spaces ⟶ the monodromy action is more complicated, and does <u>not</u> factor through $\underset{\substack{\shortparallel \\ B_n}}{\pi_1 C_n(D^2)} \twoheadrightarrow \Sigma_n$

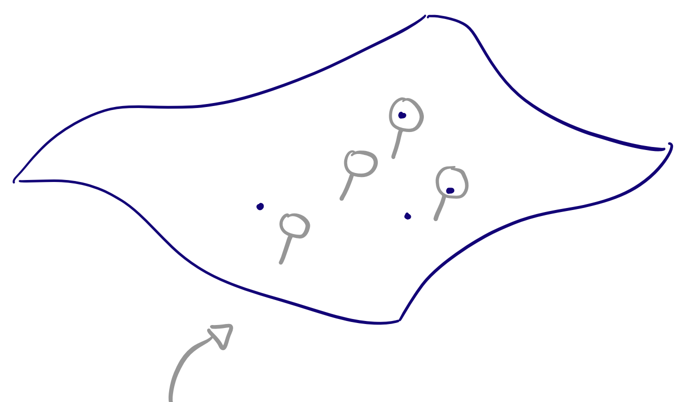
<u>Open Q:</u> What is the stable homology?

E.g. $(c=1)$ $\displaystyle \lim_{k\to\infty} H_*(\mathcal{M}_{\underbrace{(1,1,\dots,1)}_{k}}) \cong \ ?$

5 Towards stable homology

Idea:

- $\mathcal{M}_\lambda$ is a configuration space where configs are equipped with pairwise non-local data.
- Calculation of stable H_* of ordinary configuration spaces (no extra data) uses scanning maps.



"magnifying glasses" record what the config. looks like locally near each point

- ⇝ cannot work for non-local data.
- Idea — scan with many magnifying glasses.
- First step: Theorem-in-progress ('25+)

  Explicit model for stable H_* of ordinary configuration spaces via multi-scanning maps.

- Next: Try to lift to T_λ and $\mathcal{M}_\lambda$ ....

Using more concrete description of T_λ via pullbacks of Hopf bundles.