Homological stability for configuration spaces on closed manifolds VI

GeMAT seminar
IMAR
11 July 2025

Reminder

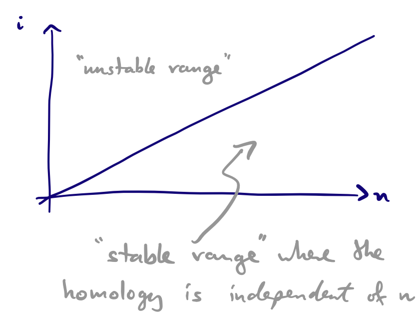
M — connected manifold (without boundary) of $\dim(M) = d \geq 2$

$C_n(M) :=$ space of n-point subsets of M

Theorem [Arnol'd, McDuff, Segal, 70's]

If M is open — $M \cong \mathrm{int}(\bar{M})$, $\partial \bar{M} \neq \phi$

then $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ for $n \geq 2i$.



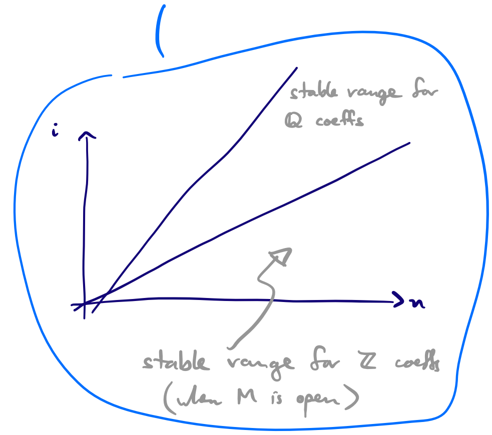
This is false for closed manifolds M. (closed: compact, $\partial M = \phi$)

E.g. $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2n-2)$.

Theorem 1 (Church '12)

If M is connected, orientable, finite-type (e.g. (interiors of) compact manifolds; i.e. $H^*(M;\mathbb{Q})$ is finite-dim.)

then $H_i(C_n(M);\mathbb{Q}) \cong H_i(C_{n+1}(M);\mathbb{Q})$ for $n \geq i+1$



This is a corollary of:

Theorem 2 (Church '12)

Under these conditions, for each fixed $i \geq 0$,

the sequence of Σ_n-representations $H^i(F_n(M);\mathbb{Q})$ (ordered configuration spaces)

is
- uniformly representation stable
- & monotone

for $n \geq 4i$ (can improve to $n \geq 2i$ if $\dim(M) \geq 3$.)

First task: define what this means....

Representation theory of symmetric groups (over $\mathbb{Q}$)

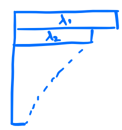
(cf. Fulton-Harris)

V_λ irreducible reps of Σ_n

$\updownarrow$ 1:1

λ partitions of n $\longleftrightarrow$ $\lambda = (\lambda_1, \dots, \lambda_r)$: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$

$\lambda_1 + \cdots + \lambda_r = n$



Eg $\lambda = (n)$ trivial repr. on $\mathbb{Q}$

$\lambda = (n-1, 1)$ standard repr. on $\mathbb{Q}^n / \mathbb{Q}$

$\lambda = (n-3, 1, 1, 1)$ $\Lambda^3 (\mathbb{Q}^n / \mathbb{Q})$

Def. λ partition of k, $n \geq k + \lambda_1$ $\Big\} \rightsquigarrow$ $\lambda[n] = (n-k, \lambda_1, \dots, \lambda_r)$ partition of n

$$V(\lambda)_n := V_{\lambda[n]}$$

For a Σ_n-representation W, $c_\lambda(W) :=$ multiplicity of $V(\lambda)_n$ in W.

Representation stability (Church–Farb '10)

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots \longrightarrow V_n \xrightarrow{\phi_n} V_{n+1} \longrightarrow \cdots$$

$$\circlearrowleft \quad \circlearrowleft \quad \circlearrowleft \quad \circlearrowleft$$

$$\Sigma_1 \hookrightarrow \Sigma_2 \hookrightarrow \cdots \hookrightarrow \Sigma_n \hookrightarrow \Sigma_{n+1} \hookrightarrow \cdots$$

finite-dimensional $\mathbb{Q}$-vector spaces + compatible actions

Def

This is uniformly representation stable for $n \geq N$ if :

$\forall n \geq N$, (1) ϕ_n is injective

(2) the $\mathbb{Q}[\Sigma_{n+1}]$-span of $\phi_n(V_n)$ is V_{n+1}

(3) $\forall \lambda$, $c_\lambda(V_n) = c_\lambda(V_{n+1})$.

Rmk If $W \subseteq V_n$, $W \cong V(\lambda)_n$, then (3) does not imply that $\phi_n(W) \cong V(\lambda)_{n+1}$, or even that $\phi_n(W)$ contains $V(\lambda)_{n+1}$.

↳ not a Σ_{n+1}-repr.! should consider its Σ_{n+1}-span.

Def (Church '12)

$\{V_n, \phi_n\}$ is monotone for $n \geq N$ if :

$\forall n \geq N$, (4) if $W \subseteq V_n$, $W \cong (V(\lambda)_n)^{\oplus k}$

then the $\mathbb{Q}[\Sigma_{n+1}]$-span of $\phi_n(W)$ contains an isomorphic copy of $(V(\lambda)_{n+1})^{\oplus k}$.

Restating Theorem 2:

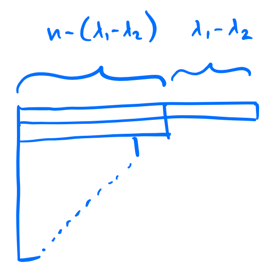
Theorem 2 (Church '12)

If M is connected, orientable, finite-type, then for each fixed $i \geq 0$,

the sequence $\cdots \to H^i(F_n(M);\mathbb{Q}) \to H^i(F_{n+1}(M);\mathbb{Q}) \to \cdots$

(induced by $F_n(M) \xleftarrow{\text{forget}} F_{n+1}(M)$)

satisfies (1)–(4) above for $n \geq 4i$.



Rmk Suppose that λ is a partition of n and $n-(\lambda_1-\lambda_2)-1 \geq 4i$

Thm 2 $\Rightarrow$ mult of V_λ in $H^i(F_n(M);\mathbb{Q})$

$= $ mult of $V_{(\lambda_2-1,\lambda_2,\lambda_3,\dots)}$ in $H^i(F_{n-(\lambda_1-\lambda_2)-1};\mathbb{Q})$

$= 0$

How this implies Theorem 1

For $n \geq 4i$ — Can be improved to the range stated in Thm 1 by more careful analysis.

$$\dim H_i(C_n(M);\mathbb{Q})$$
$$= \dim H^i(C_n(M);\mathbb{Q})$$
$$= \dim \left(H^i(F_n(M);\mathbb{Q})\right)^{\Sigma_n}$$
$$= c_\lambda\left(H^i(F_n(M);\mathbb{Q})\right)$$

for $\lambda = (n) =$ 

↑ independent of n by Thm 2.

Ideas of proof of Theorem 2

Leray spectral sequence

Cf. [Dimca, Sheaves in topology]

$X \xrightarrow{f} Y \rightsquigarrow f_* : Sh(X) \longrightarrow Sh(Y)$

$\mathcal{F} \mapsto (U \mapsto \mathcal{F}(f^{-1}(U)))$

$\rightsquigarrow$ derived functors $R^i f_* : Sh(X) \longrightarrow Sh(Y)$

(presheaf $U \mapsto H^i(f^{-1}(U); \mathcal{F})$
$R^i f_*(\mathcal{F})$ = sheafification of this)

$\mathcal{F}$ sheaf on X

$$E_2^{p,q} = H^p(Y; R^q f_*(\mathcal{F})) \Rightarrow H^*(X; \mathcal{F}) \qquad (*)$$

[Totaro '96] Explicit description of $E_2^{p,q}$ (= $E_2^{p,q}(n)$) of $(*)$ for

$F_n(M) \hookrightarrow M^n$ and $\mathcal{F} = \mathbb{Q}$

which converges to $H^*(F_n(M); \mathbb{Q})$.

Idea

1. Prove uniform repr. stab.^y + monotonicity for $E_2^{p,q}(n)$.
2. Deduce the same for $E_r^{p,q}(n)$ and hence $E_\infty^{p,q}(n)$.
3. Deduce the same for the limit, i.e. $H^i(F_n(M); \mathbb{Q})$.

Rmks

- Steps ② and ③ are relatively straightforward.
- However, they only work for the <u>combined</u> property of uniform repr. stab.^{y} + monotonicity.

  (Uniform repn stab.^{y} by itself does not "propagate" through a spectral sequence.)

Eg

$\Lambda = \{\{1,2\}, \{3,5\}, \{4\}\}$

$|\Lambda| = 3$

$\bar{\Lambda} = (2,2,1)$

Sub-configuration spaces

Λ partition of $\{1, \ldots, n\}$

Notation $|\Lambda| :=$ # of blocks of the partition.

$\bar{\Lambda} :=$ the induced partition of n.

Def (Λ-subconfiguration space)

$$F_\Lambda(M) = \left\{ (p_1, \ldots, p_n) \in M^n \;\middle|\; \begin{array}{l} \text{if } i,j \text{ lie in the same block of } \Lambda \\ \text{then } p_i \neq p_j \end{array} \right\}$$

(Λ-diagonal)

$$M^\Lambda = \left\{ (p_1, \ldots, p_n) \in M^n \;\middle|\; \begin{array}{l} \text{if } i,j \text{ lie in the same block of } \Lambda \\ \text{then } p_i = p_j \end{array} \right\}$$

Eg $\Lambda =$ trivial partition $\longrightarrow$ $F_\Lambda(M) = F_n(M)$

$M^\Lambda =$ diagonal copy of M

$\Lambda =$ discrete partition $\longrightarrow$ $F_\Lambda(M) = M^n = M^\Lambda$

Thm (Arnol'd '69 + Brieskorn '73)

$$H^*(F_n(\mathbb{R}^d);\mathbb{Q}) \cong \bigoplus_{\Lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q})$$

(isom. of $\mathbb{Q}[\Sigma_n]$-modules)

Thm (Totaro '96)

(isoms of $\mathbb{Q}[\Sigma_n]$-modules)

$$E_2^{*,*}(n) \cong \bigoplus_{\Lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q}) \otimes H^*(M^\Lambda;\mathbb{Q})$$

Obs.

$$\cong \bigoplus_{\lambda}\left(\bigoplus_{\overline{\Lambda}=\lambda} H^{(d-1)(n-|\Lambda|)}(F_\Lambda(\mathbb{R}^d);\mathbb{Q}) \otimes H^*(M^\Lambda;\mathbb{Q})\right)$$

preserved by Σ_n-action

.... decompose $H^*(M^\Lambda;\mathbb{Q})$ in a Σ_n-invariant way via the Künneth formula ....

$\cong \bigoplus$ summands, each of which is isomorphic to $\mathrm{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n}(W \boxtimes \mathbb{Q})$ for some Σ_k-representation W

Prop. (Church '12)

In (fixed) bidegree (p,q), we have

$$k = q + \mathrm{length}(\lambda) + \max\{i \mid \lambda_i \geq 2\}.$$

Thm (Church '12) (**)

For any fixed Σ_k-representation W, the sequence

$$\mathrm{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} (W \boxtimes \mathbb{Q})$$

is uniformly representation stable for $n \geq 2k$
& monotone for $n \geq k$.

Rmk
- This is pure representation theory.
- [Church] gives an elementary (but intricate) proof using the branching rule for $\mathrm{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} (V_\lambda \boxtimes \mathbb{Q})$.
- [Sam–Weyman] give an alternative proof, using Schur–Weyl duality to work instead with $GL_n(\mathbb{Q})$-representations.

Finishing the proof (recap)

Theorem (**) + fact that k is fixed $\Rightarrow$ URS + M for $E_2^{p,q}(n)$

$\overset{\text{propagation}}{\Longrightarrow}$ URS + M for $H^i(F_n(M); \mathbb{Q})$

$\Longrightarrow$ stability for $H^i(C_n(M); \mathbb{Q})$ (λ = single-row Young diagram)

Rmk Can prove that stability is induced by $C_{n+1}(M) \overset{\text{transfer}}{\dashrightarrow} C_n(M)$.