Homological stability for configuration spaces on closed manifolds VII

GeMAT seminar
IMAR
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Context

- If M is a connected, open manifold, then
$$H_i(C_n(M)) \cong H_i(C_{n+1}(M)) \quad \text{for } n \geqslant 2i.$$
[McDuff, Segal]

- When M is closed this is generally false, but:

Thm If M is a connected manifold, then
$$H_i(C_n(M);\mathbb{Q}) \cong H_i(C_{n+1}(M);\mathbb{Q}) \quad \text{for } n \geqslant i+1.$$

- So far we have seen proofs by

Church	(talk V)	via	representation stability
Randal-Williams	(talk I)	via	transfer maps
Bendersky – Miller	(talks II/III)	via	scanning maps & obstruction theory

- Today: it will be a corollary of:

Thm [Knudsen '17] For any d-manifold $(d \geqslant 2)$:
$$\bigoplus_{n \geqslant 0} H_*(C_n(M);\mathbb{Q}) \cong H(\mathfrak{g}_M) \quad \text{as bigraded } \mathbb{Q}\text{-vspaces,}$$
Lie algebra homology

where $\mathfrak{g}_M$ is the following bigraded Lie algebra:

$$\mathfrak{g}_M = H_c^{\overset{\text{degree}}{-*}}(M; \mathcal{L}(\underbrace{\mathbb{Q}^\omega[d-1]}_{\text{weight } 1}))$$

$\mathbb{Q}^\omega$ = orientation local system on M

$\mathbb{Q}^\omega[d-1]$ = consider it to be concentrated in degree $d-1$

$\mathcal{L}(-)$ = free graded Lie algebra.

Rmk This can be simplified at the cost of splitting into two cases:

d odd $\quad \mathcal{L}(\mathbb{Q}^\omega[d-1]) = \underset{\text{weight } 1}{\mathbb{Q}^\omega[d-1]}$ with trivial bracket.

$$\mathfrak{g}_M \cong H_*(M;\mathbb{Q})[-1] \quad \text{with trivial bracket.}$$

d even $\quad \mathcal{L}(\mathbb{Q}^\omega[d-1]) = \underset{\text{weight } 1}{\underset{x}{\mathbb{Q}^\omega[d-1]}} \oplus \underset{\text{weight } 2}{\underset{[x,x]}{\mathbb{Q}[2d-2]}}$

$$\mathfrak{g}_M \cong H_*(M;\mathbb{Q})[-1] \oplus H_*(M;\mathbb{Q}^\omega)[d-2]$$

as a bigraded vector space — the Lie bracket turns out to be determined by the cup product structure on M.

Corollary

As a graded $\mathbb{Q}$-vspace, $H_*(C_n(M);\mathbb{Q})$ depends only on d and the graded $\mathbb{Q}$-vspace $H_*(M;\mathbb{Q})$, plus the cup product structure if d is even.

[Bödigheimer-Cohen-Taylor '89] (d odd) [Félix-Thomas '00] (d even)

Lie algebra homology

$\mathfrak{g}$ Lie algebra over $\mathbb{Q}$

Def $H_*(\mathfrak{g}) := \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{Q}, \mathbb{Q})$

$$U\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \Big/ [x,y] = xy - yx$$

Thm (Chevalley–Eilenberg)

$$H_*(\mathfrak{g}) \cong H_*(\Lambda^* \mathfrak{g})$$

differential induced by $[-,-] : \Lambda^2 \mathfrak{g} \longrightarrow \mathfrak{g}$

Can use this to generalise to graded Lie algebras:

Def $\mathfrak{g}$ graded Lie algebra

$$H_*(\mathfrak{g}) := H_*(\mathrm{Sym}^*(\mathfrak{g}[1]))$$

Chevalley–Eilenberg complex $C_*^{CE}(\mathfrak{g})$

differential induced by $[-,-]$

Note: $\mathfrak{g}$ ungraded $\longleftrightarrow$ concentrated in degree zero

$\Rightarrow$ $\mathfrak{g}[1]$ is concentrated in degree 1

$\Rightarrow$ $\mathrm{Sym}^*(\mathfrak{g}[1]) = \Lambda^* \mathfrak{g}$

Plan

- Some ideas of the proof
- How to deduce stability
- Example calculations

Idea of proof

Uses <u>factorisation homology</u>

Def

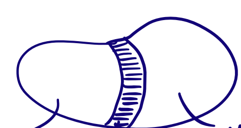
- $\mathrm{Disk}_d = \begin{cases} \text{manifolds diffeo to } \coprod_n \mathbb{R}^d \\ \text{smooth embeddings} \end{cases}$

- n-disk algebra A = symmetric monoidal functor $(\mathrm{Disk}_d, \amalg) \xrightarrow{A} (\mathrm{Ch}_{\mathbb{Q}}, \otimes)$

- $\underline{\int} A : (\mathrm{Mfld}_d, \amalg) \longrightarrow (\mathrm{Ch}_{\mathbb{Q}}, \otimes)$

  is the unique symmetric monoidal functor s.t.

  - $\int_{\mathbb{R}^d} A \simeq A$

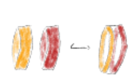
  - if  $= M$

    M_1 M_2 $M_0 \times \mathbb{R}$

    then $\int_M A \simeq \int_{M_1} A \underset{\int_{M_0 \times \mathbb{R}} A}{\otimes} \int_{M_2} A$

$\exists$ and uniqueness
<u>Thm</u> (Ayala-Francis '15)

$\int_{M_0 \times \mathbb{R}} A$ has a ring structure and $\int_{M_i} A$ is a module over it via applying $\underline{\int} A$ to the inclusions

 $(M_0 \times \mathbb{R}) \amalg (M_0 \times \mathbb{R}) \hookrightarrow M_0 \times \mathbb{R}$

and $(M_0 \times \mathbb{R}) \amalg M_i \hookrightarrow M_i$ 

- $\int_M A$ =: factorisation homology of M with coefficients in A.

From d-disk algebras to Lie algebras

Part of the structure of a d-disk algebra is

$$C_*\left(\mathrm{Emb}\left(\coprod_2 \mathbb{R}^d, \mathbb{R}^d\right)\right) \otimes A \otimes A \longrightarrow A$$

by abuse of notation, A denotes $A(\mathbb{R}^d)$

Note that $H_*\left(\mathrm{Emb}\left(\coprod_2 \mathbb{R}^d, \mathbb{R}^d\right)\right)$

$$\begin{aligned} &\cong \\ &H_*(F_2(\mathbb{R}^d);\mathbb{Q}) \\ &\cong \\ &H_*(S^{d-1};\mathbb{Q}) \\ &\cong \\ &\mathbb{Q}[0] \oplus \mathbb{Q}[d-1] \end{aligned}$$

Restricting to (a cycle representing a generator of) $\mathbb{Q}[d-1]$, we get:

$$A[d-1] \otimes A \longrightarrow A$$

$$\downarrow$$

$$m_{d-1} : A[d-1] \otimes A[d-1] \longrightarrow A[d-1]$$

This is a Lie bracket on $A[d-1]$.

(Jacobi identity $\longleftrightarrow$ Yang-Baxter relation in $H_*(F_3(\mathbb{R}^d))$.)

$$F : \{d\text{-disk algebras}\} \longrightarrow \{\text{graded Lie algebras}\}$$

U_d

left adjoint

"d-enveloping algebra"

Main steps of the proof

$$(*) = \int_M \underbrace{U_d}_{d\text{-enveloping algebra}} \Big(\underbrace{\mathcal{L}}_{\text{free Lie algebra — bigraded}} \Big(\underbrace{\widetilde{C}_*((\mathbb{R}^d)^+)[-1]}_{\substack{\text{graded vector space} \\ \text{(weight 1)}}} \Big)\Big)$$ bigraded chain cx.

Calculate this in two different ways:

$$(*) \simeq \bigoplus_{n \geq 0} C_*(C_n(M); \mathbb{Q})$$

$$(*) \simeq C_*^{CE}\left(H_c^{-*}(M; \mathcal{L}(\mathbb{Q}^w[d-1]))\right)$$

Then take homology. $\square$

Deducing stability (M connected)

By the main theorem,

$$(*) \qquad \bigoplus_n H_*(C_n(M);\mathbb{Q}) \cong H_*(C^{CE}_*(\mathfrak{g}_M))$$

$$C^{CE}_*(\mathfrak{g}_M) = \mathrm{Sym}^*(\mathfrak{g}_M[1])$$

$$= \mathrm{Sym}^*\Big(\underbrace{H_c^{-*}(M;\ \underbrace{\mathbb{Q}^{\omega}[d-1]}_{\text{weight } 1} \oplus \underbrace{\mathbb{Q}[2d-2]}_{\text{weight } 2})}_{H_*(M;\mathbb{Q})}[1]\Big)$$

The $\begin{pmatrix}\text{degree-}0\\ \text{weight-}1\end{pmatrix}$ summand is one-dimensional.

Choose a generator p and extend it to a bihomogeneous basis B of $\mathfrak{g}_M[1]$.

Key proposition:

For $x \in C^{CE}_*(\mathfrak{g}_M)$, if $\mathrm{weight}(x) \geq |x| + 1$ (homological degree)
then x is divisible by p.

(Algebraic incarnation of hom. stab.^{y}).
(With a bit of work, implies hom. stab^y on LHS of $(*)$.)

Proof Assume $d \geq 3$. ($d=2$ is slightly more complicated)

Write $x = x_1 \cdots x_r \qquad x_i \in B$

$\Rightarrow \quad \text{weight}(x_j) \geq |x_j| + 1 \quad$ for some j

Since $x_j \in \mathfrak{g}_M[1]$, $\text{weight}(x_j) = 1$ or 2

Suppose $\text{weight}(x_j) = 2$.

$\longrightarrow \quad d-1 \leq |x_j| \leq 2d-1$

$\longrightarrow$ contradiction to $2 \geq |x_j| + 1$ and $d \geq 3$.

Hence $\text{weight}(x_j) = 1$.

$\longrightarrow \quad 0 \leq |x_j| \leq d$

$\longrightarrow$ hence $|x_j| = 0$

$\longrightarrow \quad x_j = \lambda p. \qquad \square$

Rmk Proof just uses the formal structure of the bigrading of $\mathfrak{g}_M[1]$.

Reminder:

$$\bigoplus_n H_*(C_n(M);\mathbb{Q}) \cong H_*\Big(\overbrace{\mathrm{Sym}^*(\mathfrak{g}_M[1])}^{C^{CE}_*(\mathfrak{g}_M)}\Big)$$

$$\mathfrak{g}_M[1] \cong \underbrace{H_*(M;\mathbb{Q})}_{wt=1} \Big(\oplus \underbrace{H_*(M;\mathbb{Q}^w)[d-1]}_{wt=2}\Big)$$

if d is even

Examples

$M = \mathbb{R}^d$

d odd

no differentials b/c odd dim.

$$C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x] \qquad w(x) = 1, \quad |x| = 0$$

$$\| $$

$$H_*(C_*^{CE}(\mathfrak{g}_M))$$

$$H_*(C_n(\mathbb{R}^d); \mathbb{Q}) \cong \text{weight-}n \text{ part of } \mathbb{Q}[x]$$

$$= \mathbb{Q}\{x^n\} \qquad |x^n| = 0$$

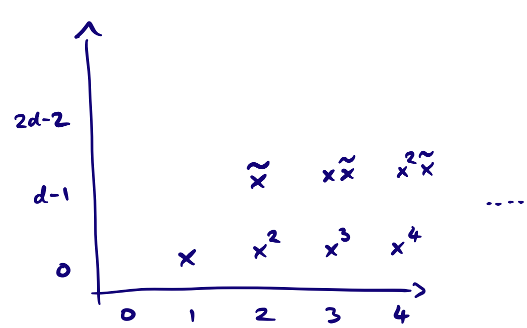
d even

$$H_*(C_*^{CE}(\mathfrak{g}_M)) = C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x] \otimes \Lambda[\tilde{x}]$$

no differentials b/c no non-0 cup products

$w(x) = 1 \qquad |x| = 0$

$w(\tilde{x}) = 2 \qquad |\tilde{x}| = d-1$

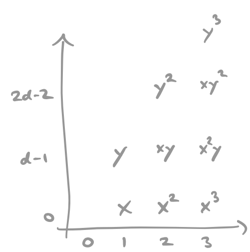


$M=\mathbb{R}^n\setminus \text{point}$

d odd

$$H_*(C_*^{CE}(\mathfrak{g}_M)) \cong \mathbb{Q}[x,y] \qquad \begin{aligned} w(x)=1&=w(y)\\ |x|&=0\\ |y|&=d-1\end{aligned}$$

$$H_*(C_n(\mathbb{R}^d\setminus\text{point});\mathbb{Q}) \cong \text{weight-}n \text{ piece of this}$$



$$= \mathbb{Q}\{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$$

degrees $\quad 0,\ d-1,\ 2d-2, \dots, nd-n$

$$\cong \begin{cases} \mathbb{Q} & * = i(d-1) \quad 0\le i\le n \\ 0 & \text{o/w.}\end{cases}$$

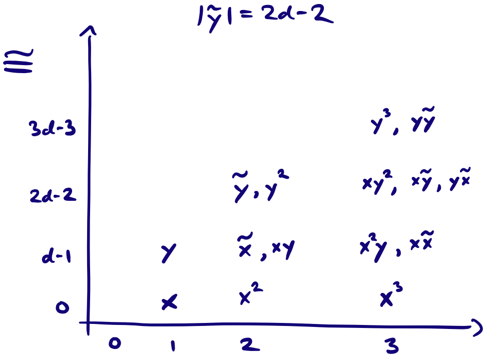
d even

$$H_*(C_*^{CE}(\mathfrak{g}_M)) = C_*^{CE}(\mathfrak{g}_M) = \mathbb{Q}[x,\tilde{y}]\otimes\Lambda[\tilde{x},y]$$

no differentials b/c no non-0 cup products

$$\begin{aligned} w(x)=1&=w(y)\\ w(\tilde{x})=2&=w(\tilde{y})\\ |x|&=0\\ |y|&=d-1\\ |\tilde{x}|&=d-1\\ |\tilde{y}|&=2d-2\end{aligned}$$

$$\bigoplus_n H_*(C_n(\mathbb{R}^d\setminus\text{point});\mathbb{Q}) \cong$$



[Drummond-Cole–Knudsen] use this Lie algebra model to calculate explicitly $\dim_{\mathbb{Q}} H_i(C_n(\Sigma);\mathbb{Q})$ for $\underline{\text{all}}$ i, n, Σ.

finite-type surface.