Homological stability for configuration spaces on closed manifolds VIII

GeMAT seminar
IMAR
16 July 2025

Reminder

In the last talk (i.e. yesterday) we saw:

Thm [Knudsen '17] For any d-manifold ($d \geq 2$):

$$\bigoplus_{n \geq 0} H_*(C_n(M);\mathbb{Q}) \cong H(\mathfrak{g}_M) \quad \text{as bigraded } \mathbb{Q}\text{-vspaces,}$$

where

$$\mathfrak{g}_M = H_c^{\overset{\text{degree}}{-*}}(M; \mathcal{L}(\underbrace{\mathbb{Q}^w[d-1]}_{\text{weight } 1}))$$

$\mathbb{Q}^w[d-1]$ = orientation local system on M in degree $d-1$
$\mathcal{L}(-)$ = free graded Lie algebra.

... which implies $\mathbb{Q}$-homological stability for $C_n(M)$.

Aim for today:

[Randal-Williams '23]

A more direct proof of this:
- without factorisation homology
- as a corollary of a more general result.

Setup:

Work in $\mathrm{Top}_*^{\mathbb{N}}$ = $\underbrace{\mathbb{N}\text{-indexed}}$ sequences of based spaces.

↳ weight grading from last talk

Fix M : connected d-manifold
$\cong$ interior of a compact manifold with boundary.

Definition

$C(M) \in \mathrm{Top}_*^{\mathbb{N}}$ $\qquad C(M)(n) := C_n(M)^+$ one-point compact"

This is a commutative monoid object:

$$\begin{array}{ccc} C(M)(n_1) \wedge C(M)(n_2) & \longrightarrow & C(M)(n_1+n_2) \\ \| & & \| \\ C_{n_1}(M)^+ \wedge C_{n_2}(M)^+ & \longrightarrow & C_{n_1+n_2}(M)^+ \\ \| & & \\ (C_{n_1}(M) \times C_{n_2}(M))^+ & & \end{array}$$

$$(c_1, c_2) \longmapsto \begin{cases} c_1 \cup c_2 & \text{if } \quad c_1 \cap c_2 = \emptyset \\ \infty & \text{o/w} \end{cases}$$

Definition:

For $Y \in \mathrm{Top}_*^{\mathbb{N}}$, $\quad \mathrm{Com}(Y) :=$ free commutative monoid on Y.

$$\mathrm{Com}(Y)(n) = \text{weight-}n \text{ component of } \quad SP^\infty\Big(\bigvee_{i \geq 0} Y(i)\Big)$$

$$SP^\infty(Z) = \operatorname*{colim}_{j \to \infty} \left(Z \wedge \cdots \wedge Z /_{\Sigma_j} \right)$$

$$\underset{\text{as a set}}{=} \Big\{ \underbrace{\{y_1, \dots, y_k\}}_{\text{multiset}} : \; y_i \in Y(n_i) \setminus * \,, \; \Sigma n_i = n \Big\}^+$$

operation = union of multisets

Notation:

X based space $\longrightarrow$ $X[n] \in Top_*^{\mathbb{N}}$

$$X[n](m) = \begin{cases} * & m \neq n \\ X & m = n \end{cases}$$

Eg

If $Y = M^+[k]$

Then $Com(M^+[k])(nk) = \{\{p_1, \dots, p_n\} : p_i \in M\}^+ = (M^n/\Sigma_n)^+$

$Com(M^+[k])(m) = *$ if $k \nmid m$

NB: In particular, $Com(M^+[k])(0) \cong S^0$.

Observation:

$c(M)$ can be built out of $Com(-)$:

It is the pushout of:

$$\begin{tikzcd} Com(M^+[2]) \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & S^0[0] \\ Com(M^+[1]) & \end{tikzcd}$$

double every point

(*) diagram of commutative monoids

(In every positive grading, the pushout has the effect of collapsing the "fat diagonal" of $(M^n/\Sigma_n)^+$.)

Theorem (Randal-Williams)

$C(M)$ is also the homotopy pushout of $(*)$.

$$\Updownarrow$$

$$C(M) \simeq B\big(\mathrm{Com}(M^+[1]),\ \mathrm{Com}(M^+[2]),\ S^0[0]\big)$$

two-sided bar construction:

$B(X, M, Y)$ — M comm. monoid, X, Y modules over M

Notation: $H_{n,i}(Y) := \widetilde{H}_i(Y(n); \mathbb{Q})$

Fact: $H_{*,*}(B(X,M,Y)) \cong \mathrm{Tor}_*^{H_{*,*}(M)}\big(H_{*,*}(X), H_{*,*}(Y)\big)$

Mild generalisation:

$$\begin{array}{c} L \\ \pi \big\downarrow \\ M \end{array} \quad \text{real vector bundle}$$

$\rightsquigarrow$ $C(M;L) \in \mathrm{Top}_*^{\mathbb{N}}$

$$n \longmapsto C_n(M;L) = \left\{ (p_1, \ldots, p_n) \in L^n \;\middle|\; \begin{array}{c} \pi(p_i) \neq \pi(p_j) \\ \text{if } i \neq j \end{array} \right\} \Big/ \Sigma_n$$

$C(M;L)$ is again a commutative monoid, and is the pushout of the following diagram of commutative monoids:

$$\left(\begin{array}{ccc} L \oplus L & & (L\oplus L)_{\Sigma_2} = L^{(2)} \\ \downarrow & \rightsquigarrow & \downarrow \\ M & & M \end{array} \right)$$

direct sum of vector bundles

$p \in M$
unordered pair of "labels" in $L|_p$

$$\begin{array}{ccc} \mathrm{Com}\big((L^{(2)})^+[2]\big) & \xrightarrow{\ \varepsilon\ } & S^0[0] \\ \Big\downarrow \Delta & & \\ \mathrm{Com}\big(L^+[1]\big) & & \end{array} \qquad (**)$$

induced by
$$(p, \{\ell_1, \ell_2\}) \mapsto \{(p,\ell_1), (p,\ell_2)\}$$

I.e. Δ doubles each point, and the new points "share" the two labels of the original point between them.

Theorem (Randal-Williams)

$C(M;L)$ is also the homotopy pushout of $(**)$.

Plan

- Why this recovers Knudsen's formula (and hence $\mathbb{Q}$-hom. stab.).
- Proof of Theorem.

Knudsen's formula

Choose L = orientation line bundle of M
$\oplus$ trivial $(d-1)$-dim. bundle

$\rightsquigarrow$ L is orientable and $\underbrace{2d}_{\text{even}}$-dimensional

$$\begin{array}{l} C_n(M;L) \leftarrow \text{orientable } 2dn\text{-dim. manifold} \\ \quad\downarrow \quad \leftarrow \text{vector bundle} \rightsquigarrow \text{hty equivalence} \\ C_n(M) \end{array}$$

$$\rightsquigarrow \quad H^*(C_n(M);\mathbb{Q}) \cong H^*(C_n(M;L);\mathbb{Q}) \overset{\text{P.D.}}{\cong} H^{BM}_{2dn-*}(C_n(M;L);\mathbb{Q}) \cong \tilde{H}_{2dn-*}(C_n(M;L)^+;\mathbb{Q}) = H_{n,2dn-*}(C(M;L))$$

$\rightsquigarrow$ Modulo regrading, we just need to calculate
$$H_{*,*}(C(M;L))$$

By the theorem,

$$H_{*,*}(C(M;L)) \cong \mathrm{Tor}_*^{H_{**}(\mathrm{Com}((L^{(2)})^+[2]))}\left(H_{**}(\mathrm{Com}(L^+[1])),\ \mathbb{Q}[0]\right)$$

Lemma

$$C_{**}(\mathrm{Con}(L^+[1])) \overset{q.i.}{\simeq} \mathrm{Sym}^*(\tilde{C}_*(L^+;\mathbb{Q})[1])$$

$$C_{**}(\mathrm{Con}((L^{(2)})^+[2])) \simeq \mathrm{Sym}^*(\tilde{C}_*((L^{(2)})^+;\mathbb{Q})[2])$$

replace with $\tilde{H}_*$ by another quasi-isomorphism

(Every $\mathbb{Q}$-chain cx. is q.i. to its homology.)

Hence

$$H_{*,*}(C(M;L)) \cong \mathrm{Tor}_*^{\mathrm{Sym}^*(\tilde{H}_*((L^{(2)})^+;\mathbb{Q})[2])}\left(\mathrm{Sym}^*(\tilde{H}_*(L^+;\mathbb{Q})[1]), \mathbb{Q}[0]\right)$$

$\Rightarrow$ Can calculate this using the "Koszul resolution" of this.

Finally, rewrite the result in terms of M instead of L and $L^{(2)}$ using the Thom isomorphism:

Thom iso for $L \downarrow M$

$$\tilde{H}_*(L^+;\mathbb{Q}) \cong \Sigma^d \tilde{H}_*(M^+;\mathbb{Q}^w)$$

Thom iso for $L\oplus L \downarrow M$

$$\Sigma_2 \circlearrowright \tilde{H}_*((L\oplus L)^+;\mathbb{Q}) \cong \Sigma^{2d}\tilde{H}_*(M^+;\mathbb{Q}) \circlearrowleft (-1)^d$$

$$\Downarrow$$

$$\tilde{H}_*((L^{(2)})^+;\mathbb{Q}) \cong \begin{cases} \Sigma^{2d}\tilde{H}_*(M^+;\mathbb{Q}) & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

Rmk This is where the dichotomy between even & odd dimensions arises.

$\rightsquigarrow$ This recovers Knudsen's formula, after re-indexing. $\square$

Proof of Theorem

Write $R := \mathrm{Com}(L^+[1])$

$S := \mathrm{Com}((L^{(2)})^+[2])$

Then R and $S^\circ[0]$ become S-modules via

$\Delta: S \to R$ (point-doubling) and $\varepsilon: S \to S^\circ[0]$ (augmentation)

Recall that we have

$$\begin{tikzcd} S \arrow[r, "\varepsilon"] \arrow[d] & S^\circ[0] \arrow[d] \\ R \arrow[r] & C(M;L) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} = R \otimes_S S^\circ[0]$$

(categorical pushout square)

There is a natural map:

$$B(R, S, S^\circ[0]) \longrightarrow C(M;L)$$

homotopy pushout / bar construction / derived tensor product

(categorical) pushout / tensor product

Aim: This is a weak equivalence.

Key lemma: R is a flat S-module,

i.e. $R \otimes_S - : S\text{-mod} \longrightarrow R\text{-mod}$ preserves weak equivalences.

[NB: sweeping under the carpet technicalities about spaces being "well-based", etc.]

Key lemma ⇒ Theorem

basic properties of the bar construction

$$B(S,S,S^\circ[0]) \xrightarrow{\ \simeq\ } S^\circ[0]$$

$$\Big\downarrow\ R \underset{S}{\otimes} -$$

$$B(R,S,S^\circ[0]) \xrightarrow{\ \simeq\ } R \underset{S}{\otimes} S^\circ[0] \;=\; C(M;L)$$

Key lemma

$\square$

Idea of proof of Key lemma

Recall that $R(n) = \left(L^n/\Sigma_n\right)^+$

Def Filtration $F_\bullet R$ of R by

$F_p R(n) :=$ subspace of $\left(L^n/\Sigma_n\right)^+$ of multisets $\{\ell_1,\dots,\ell_n\}$ where $\leq p$ of the points $\pi(\ell_1),\dots,\pi(\ell_n) \in M$ are <u>not</u> duplicated.

(together with the point at ∞)

NB: Duplicated means <u>strictly</u> duplicated, and not (e.g.) triplicated.

<u>Eg</u> For an element $\ell \in L$,

$$\{\ell,\ell,\ell\} \in F_1 R(3)$$

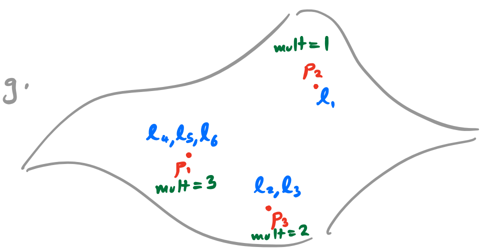
but $\{\ell,\ell,\ell\} \notin F_0 R(3)$

Rmk: This can also be described as follows:

Points of L^n/Σ_n $\longleftrightarrow$ configurations in M with multiplicities in $\mathbb{Z}_{\geq 1}$, such that the multiplicities sum to n, and where each point $p \in M$ of the configuration, with multiplicity $i \geq 1$, is equipped with i labels in $\pi^{-1}(p)$.

$$\begin{array}{c} L \\ \downarrow \pi \\ M \end{array}$$

E.g.



$\in L^6/\Sigma_6$

$$F_p R(n) = \{\leq p \text{ multiplicities are odd}\} \cup \{\infty\} \subseteq (L^n/\Sigma_n)^+ = R(n)$$

Rmk: When $L = M$, this is the same filtration used by Arnol'd ($M = \mathbb{R}^2$) and Segal to prove homological stability of $C_n(M)$.

Note that $F_0 R \cong S$, because being in filtration 0 means that all points come in pairs, which means that the configuration is in the image of $\Delta : S \to R$.

(& note that each $\Delta : S(n) \to R(n)$ is an embedding)

Claim: $\forall p \geq 0$, $F_p R \underset{S}{\otimes} -$ preserves weak equivalences.

This will complete the proof, since $F_\bullet R$ is a finite filtration in each weight-grading. ($F_n R(n) = R(n)$)

Proof of Claim by induction on p

$p=0$: ✓ since $F_0 R \cong S$

$p \geq 1$: Consider the following square:

graded wedge sum $(X \otimes Y)(n) = \bigvee_{n=i+j} X(i) \wedge Y(j)$

$$\begin{tikzcd}
F_{p-2} R(p)[p] \otimes S \arrow[r, "\text{forget}", "\text{colours}"'] \arrow[d] & F_{p-1} R \arrow[d] \\
R(p)[p] \otimes S \arrow[r, "\text{forget}", "\text{colours}"'] & F_p R
\end{tikzcd} \qquad (\star)$$

$\leq p-1$ odd multiplicities

$\leq p$ odd multiplicities

as below, but the red config must have ≥ 1 multiplicity ≥ 1

blue config of weight $n-p$ with only even multiplicities

red config of weight p

= formal definition

= interpretation in terms of config's with multiplicities

NB: In weight-grading $n < p$, the LHS is trivial (i.e. a point).

Lemma (can be checked from the description in green)

$(\star)$ is a pushout square

Hence, for any S-module V, so is:

$$\begin{tikzcd}
F_{p-2} R(p)[p] \otimes V \arrow[r] \arrow[d] & F_{p-1} R \otimes_S V \arrow[d] \\
R(p)[p] \otimes V \arrow[r] & F_p R \otimes_S V
\end{tikzcd} \qquad (\star) \otimes_S V$$

Consider

$$\begin{array}{ccc} F_{p-2}R(p) & \longrightarrow & R(p) \\ \| & & \| \\ \left\{\begin{matrix}\text{subspace where at} \\ \text{least 2 points coincide}\end{matrix}\right\}^+ & \xhookrightarrow{(\ast)} & \left(L^p/\Sigma_p\right)^+ \end{array}$$

<u>Key topological fact</u>

$\exists$ open neighborhood of in $\left(L^p/\Sigma_p\right)^+$ that deformation retracts onto it.

<u>NB</u>: Here we use the assumption on M that it is the interior of a compact manifold with boundary.

$\Rightarrow$ Hence the map $(\ast)$ is a <u>cofibration</u>.

$\Rightarrow$ The LH vertical map of $(\bigstar)\otimes_S V$ is also a cofibration.

$\Rightarrow$ $(\bigstar)\otimes_S V$ is a <u>homotopy pushout square</u>

(Lemma)

Now we have:

- The square $(\bigstar)\otimes_S V$ is <u>natural</u> w.r.t. the S-module V.
- The LH side (top and bottom) preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$.
  (by general properties of the graded wedge sum $\otimes$)
- The top-right also preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$,
  by the <u>inductive hypothesis</u>.
- Since the square $(\bigstar)\otimes_S V$ is a homotopy pushout square (by above),
  this implies that the bottom-right <u>also</u> preserves weak equivalences $V_1 \xrightarrow{\simeq} V_2$.

$\square$