Kernels of homological MCG-representations I

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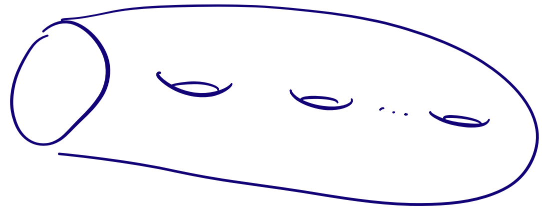
GeMAT seminar
IMAR
22 May 2026

Plan

- Intro / motivation
- Homological representations of MCGs
  ↳ Moriyama's representation
- The Johnson filtration of MCGs
- ker (Moriyama) = Johnson
  ↳ sketch of proof

Intro / motivation

$\Sigma = \Sigma_{g,1} =$ 

Mapping class group:

$$\mathrm{Mod}(\Sigma) = \pi_0\left(\mathrm{Homeo}_\partial(\Sigma)\right)$$

isotopy classes

fixing a collar neighbourhood of $\partial\Sigma$

Long-standing open question:

Is $\mathrm{Mod}(\Sigma)$ linear?

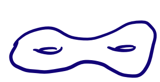
$\longleftrightarrow$ does it embed into $GL_n(K)$ for a field K?

$\longleftrightarrow$ does it admit a faithful, finite-dim representation over a field?

$g=0$: $\mathrm{Mod}(\Sigma) = \{1\}$

$g=1$: $\mathrm{Mod}(\Sigma) \cong B_3 \hookrightarrow GL_2(\mathbb{C})$

reduced Burau representation.

$g=2$: $\mathrm{Mod}\big(\Sigma_{2,0} =$  $\big) \hookrightarrow GL_{64}(\mathbb{C})$

[Bigelow-Budney 2001]

$g \geqslant 3$: ???

$\leadsto$ **Aim:** Construct "interesting" finite-dimensional representations of $\mathrm{Mod}(\Sigma)$ ...

Methods:

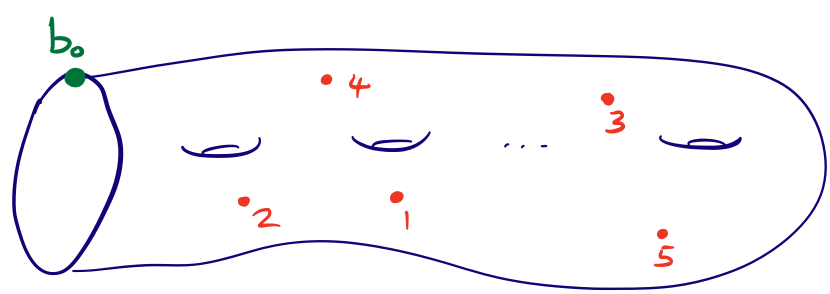
- Quantum representations
  [Reshetikhin–Turaev '91]
  [Blanchet–Habegger–Masbaum–Vogel '95]
- Unitary representations on $L^2 X$
  for X = curve complex on Σ
  $X = \{$measured foliations on $\Sigma\}$ ...
- Homological representations
  $H_*(\{\text{configurations in } \Sigma\})$

Homological representations of $\mathrm{Mod}(\Sigma)$

Construction

Let $\Sigma' = \Sigma \setminus \{b_0\}$ boundary point

Consider $F_n(\Sigma') = \{\text{ordered } n\text{-point configurations in } \Sigma'\}$



$\in F_5(\Sigma')$

$\mathrm{Mod}(\Sigma)$ acts on $F_n(\Sigma')$ (up to isotopy of homeomorphisms)

$\Rightarrow$ on $H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$ Borel-Moore homology

This is the n^{th} Moriyama representation of $\mathrm{Mod}(\Sigma)$.

Two alternative descriptions:

- $H^n\big(F_n(\Sigma),\ \underbrace{F_n(\Sigma,\{b_0\})}_{}\ ;\ \mathbb{Z}\big)$

  configurations C such that $b_0 \in C$

  (by Poincaré duality)

- $H_n\big(\Sigma^n,\ \Delta \cup B\ ;\ \mathbb{Z}\big)$

  fat diagonal

  n-tuples (multiconfigurations that contain b_0)

  (because Σ^n compactifies $F_n(\Sigma')$ with remainder $\Delta \cup B$)

Variations:

- $H_n^{BM} \rightsquigarrow H_n$
- $F_n \rightsquigarrow C_n$

  (will mention later)

  unordered configurations

- $\mathbb{Z} \rightsquigarrow$ non-trivial local systems (next talks...)

The Johnson filtration

Let $\pi = \pi_1(\Sigma, b_0)$ $\quad (\cong F_{2g})$

Lower central series:

$$\pi = \Gamma_0\pi \supseteq \Gamma_1\pi \supseteq \Gamma_2\pi \supseteq \cdots$$

$$\Gamma_i\pi = [\pi, \Gamma_{i-1}\pi]$$
$$= \{i\text{-fold iterated commutators}\}$$

$\mathrm{Mod}(\Sigma)$ acts on π

$\Rightarrow$ on $\pi / \Gamma_n\pi$ $\quad$ ($\Gamma_n\pi$ is a characteristic subgroup)

Def. $\quad J(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \pi/\Gamma_n\pi\right)$

This is the Johnson filtration of $\mathrm{Mod}(\Sigma)$.

$$\mathrm{Mod}(\Sigma) = J(0) \supseteq J(1) \supseteq J(2) \supseteq J(3) \supseteq \cdots$$

Torelli subgroup

Johnson kernel

Another description

$$\mathbb{Z}\pi \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \cdots\cdots$$

augmentation ideal

$= \ker(\mathbb{Z}\pi \to \mathbb{Z})$

$\mathrm{Mod}(\Sigma)$ acts on $\mathbb{Z}\pi$

$\Rightarrow$ on $\mathbb{Z}\pi / I^{n+1}$

Proposition (Fox '53):

$$\ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathbb{Z}\pi / I^{n+1}\right) = \mathcal{J}(n)$$

Key property of $\mathcal{J}(\bullet)$:

$$\bigcap_{n=1}^{\infty} \mathcal{J}(n) = \{1\}$$

$\varphi \in \bigcap_{n=1}^{\infty} \mathcal{J}(n) \longrightarrow \varphi$ acts trivially on $\pi / \Gamma_n \pi$ $\forall n$

$\longrightarrow \varphi$ acts trivially on $\varprojlim \pi / \Gamma_\bullet \pi$

$\uparrow$ π $\leftarrow$ b/c $\pi \cong F_{2g}$ is res. nilpotent [Magnus]

$\longrightarrow \varphi$ acts trivially on $\pi \longrightarrow \varphi = 1$.

ker(Moriyama) = Johnson

Theorem (Moriyama 2007)

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright \underbrace{H_n^{BM}(F_n(\Sigma'))}_{\text{Moriyama representation}}\Big) = J(n)$$

Corollary $\mathrm{Mod}(\Sigma) \curvearrowright \bigoplus_{n=1}^{\infty} H_n^{BM}(F_n(\Sigma'))$

is faithful.

For comparison:

[Bianchi–Miller–Wilson 2021]:

ordinary homology

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright H_n(F_n(\Sigma))\Big) \supseteq J(n)$$

but is strictly larger when $n \geq 3$, $g \geq 2$

[Bianchi–Stavrou 2023]:

ordinary homology

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright H_*(C_n(\Sigma))\Big) = J(2)$$

unordered configurations

(for $n \geq 2$)

Sketch of proof

Notation:

$$H_n = H_n^{BM}(F_n(\Sigma')) = n^{th} \text{ Moriyama representation}$$

$$K(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright H_n\right)$$

$$J(n) = n^{th} \text{ term of Johnson filtration}$$

$$= \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathbb{Z}\pi / I^{n+1}\right) \qquad \text{[Fox]}$$

Aim: $J(n) = K(n)$

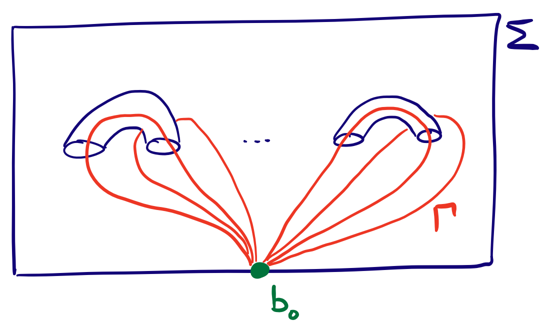
Steps:
(1) Module structure of H_n
(2) A filtered ring $\hat{H}$
(3) A filtered homomorphism $\mathbb{Z}\pi \longrightarrow \hat{H}$
(4) End of the proof.

(1) Module structure of H_n

$$H_n = H_n^{BM}(F_n(\Sigma \setminus \{b_0\}); \mathbb{Z})$$

$$\cong H_n^{BM}(C_n(\Sigma \setminus \{b_0\}); \mathbb{Z}[S_n])$$

$$\cong H_n^{BM}(C_n(\Gamma \setminus \{b_0\}); \mathbb{Z}[S_n]) = (*)$$



Adaptation of a lemma of Bigelow (2004)

Follows from the fact that:

- Σ def. retracts onto Γ rel. $\{b_0\}$ through a family of maps $\Sigma \xrightarrow{h_t} \Sigma$ such that:
  - h_t is 1-Lipschitz for all t
  - h_t is a topological self-embedding except for $t=1$

this general criterion is in [P.-Soulié 2025]

(Moriyama's original proof of this was different.)

Note: $C_n(\Gamma \setminus \{b_0\}) \cong C_n\left(\coprod_{2g} (0,1)\right) \cong \coprod_{\binom{2g+n-1}{n}} \mathring{\mathbb{D}}^n$

So $(*) \cong \mathbb{Z}[S_n]^{\binom{2g+n-1}{n}}$

(2) A filtered ring $\hat{H}$

$$\hat{H} = \prod_{n=1}^{\infty} H_n$$

Equipped with:

- Superposition product induced by:

$$C_m(\Sigma') \times C_n(\Sigma') \xleftarrow[\text{open}]{} C_{(m,n)}(\Sigma') \xrightarrow[\substack{\text{finite}\\ \text{covering}}]{} C_{m+n}(\Sigma')$$

$$H^{BM}_{m+n}(-) \longrightarrow H^{BM}_{m+n}(-) \longrightarrow H^{BM}_{m+n}(-)$$

$\rightsquigarrow$ ring

- Filtration $\mathcal{F}_i \hat{H} = \prod_{n=i}^{\infty} H_n$

- $Mod(\Sigma)$-action

Note: $\hat{H} / \mathcal{F}_i \hat{H} \cong H_0 \oplus H_1 \oplus \cdots \oplus H_{i-1}$ as $Mod(\Sigma)$-reps.

Proposition: There is a SES

$$0 \longrightarrow H_{n-1} \otimes H_1 \xrightarrow{\ \cdot\ } H_n \longrightarrow (H_{n-1})^{\oplus n-1} \longrightarrow 0$$

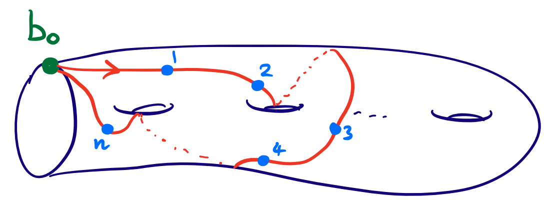
Corollary: (1) $K(n) \subseteq K(n-1)$ (descending filtration of $Mod(\Sigma)$)

(2) mult.^{n}: $(H_1)^{\otimes n} \longrightarrow H_n$ is injective.

(3) A filtered homomorphism $\mathbb{Z}\pi \longrightarrow \hat{H}$

$\gamma \in \pi = \pi_1(\Sigma, b_0)$

$\longrightarrow \phi_n(\gamma) \in H_n = H_n^{BM}(F_n(\Sigma'))$



Extend linearly to group hom. $\phi_n : \mathbb{Z}\pi \longrightarrow H_n$

Define $\Phi = \prod_n \phi_n : \mathbb{Z}\pi \longrightarrow \hat{H}$

Key lemma: Φ is a ring homomorphism

Idea: $\phi_n(\gamma)$ = image of $[\Delta^n]$ under

$$(\Delta^n, \partial\Delta^n) \longrightarrow (\Sigma^n, \Delta \cup B)$$

concatenating γ and δ corresponds to breaking Δ^n into $n+1$ sub-simplices

This decomposition gives:

$$\phi_n(\gamma\delta) = \sum_{i=0}^{n} \phi_i(\gamma)\,\phi_{n-i}(\delta). \qquad \square$$

Corollary:

$$\circledast \quad \Phi\big((\gamma_1 - 1)\cdots\cdots(\gamma_{n+1}-1)\big) = \phi_1(\gamma_1)\cdots\cdots\phi_1(\gamma_{n+1}) \quad \text{modulo } F_{n+2}\hat{H}$$

(Clear for $n=0$, follows in general because Φ is a ring hom.)

In particular, $\circledast$ implies that:

- $\Phi(I^{n+1}) \subseteq \mathcal{F}_{n+1}\hat{H} \implies$
$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, "\Phi"] \arrow[d, two heads] & \hat{H} \arrow[d, two heads] \\ \mathbb{Z}\pi / I^{n+1} \arrow[r, "\Phi_n"] & \hat{H} / \mathcal{F}_{n+1}\hat{H} \end{tikzcd}$$

- $\phi_n(I^{n+1}) = 0 \implies$
$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, "\phi_n"] \arrow[d, two heads] & H_n \\ \mathbb{Z}\pi / I^{n+1} \arrow[ur, "\phi_n'"'] & \end{tikzcd}$$

Final ingredient: Φ_n is injective.

Idea: • By a 5-lemma argument it's enough to show that the associated graded homomorphism
$$I^n / I^{n+1} \xrightarrow{gr(\Phi_n)} \frac{\mathcal{F}_n\hat{H}}{\mathcal{F}_{n+1}\hat{H}} = H_n$$
is injective.

- $I^n / I^{n+1} \cong (H_1)^{\otimes n}$
$$[(\gamma_1 - 1)\cdots(\gamma_n - 1)] \longmapsto [\gamma_1] \otimes \cdots \otimes [\gamma_n]$$

- By $\circledast$, $gr(\Phi_n)$ is the composition
$$I^n / I^{n+1} \cong (H_1)^{\otimes n} \xrightarrow{\text{mult.}^n} H_n$$
injective (from earlier) $\square$

(4) End of the proof

Recall we want to prove that $J(n) = K(n)$.

n^{th} term of Johnson filtration $= $ kernel of action on $\mathbb{Z}\pi / I^{n+1}$

kernel of n^{th} Morigama representation H_n

$(\supseteq)$ $\varphi \in K(n)$

$\Rightarrow$ φ acts trivially on $H_0 \oplus H_1 \oplus \cdots \oplus H_n \cong \hat{H} / \mathcal{F}_{n+1}\hat{H}$

This contains $\mathbb{Z}\pi / I^{n+1}$ as a subrepresentation. (Φ_n is injective)

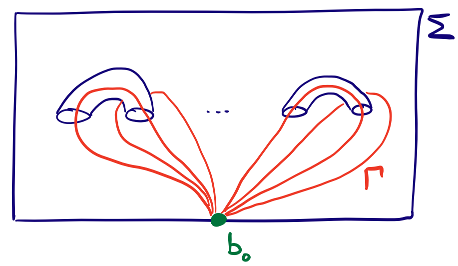
$\Rightarrow$ $\varphi \in J(n)$

$(\subseteq)$ $\varphi \in J(n)$

$\Rightarrow$ For any $1 \leq k \leq n$ we have:

$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, two heads, "\phi_k"] \arrow[d, two heads] & \mathrm{im}(\phi_k) \subseteq H_k \\ \mathbb{Z}\pi / I^{k+1} \arrow[ur, two heads, "\phi_k'"'] & \end{tikzcd}$$

$\Rightarrow$ φ acts trivially on $\mathrm{im}(\phi_1), \ldots, \mathrm{im}(\phi_n)$



Module structure of H_n $\Rightarrow$ it is generated by elements of the form $a_1 a_2 \cdots a_\ell$

with $a_j \in \operatorname{im}(\phi_{k_j})$ $\quad \sum_j k_j = n$

$1 \leq k_j \leq n$

$\Rightarrow$ φ acts trivially on H_n

i.e. $\varphi \in K(n)$.

$\square$