Kernels of homological MCG-representations II

GeMAT seminar
IMAR
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Plan

Intro

Last week we studied: $\qquad \Sigma = \Sigma_{g,1}$

$$\mathrm{Mod}(\Sigma) \curvearrowright H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$$

$$\| $$

$$\pi_0 \mathrm{Homeo}_\partial(\Sigma)$$

Moriyama representation

Thm (Moriyama '07) $\quad \ker = J(n)$

Johnson filtration:

$$J(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \pi / \Gamma_n \pi\right)$$

$$\pi_1(\Sigma) = \pi = \Gamma_0 \pi \supset \Gamma_1 \pi \supset \Gamma_2 \pi \supset \cdots\cdots$$

lower central series

Cf. $\quad H_n(F_n(\Sigma))$: $\ker \supsetneq J(n)$ [Bianchi-Miller-Wilson '21]

$\quad H_*(C_n(\Sigma))$: $\ker = J(2)$ [Bianchi-Stavrou '23]

Today: • add local coefficients } $\longrightarrow$ Magnus representation
• $n=1$

(Next week: $n \geq 2$ & how to "untwist".)

The Magnus representation

[Magnus '39] Via Fox calculus

$x_1, \dots, x_{2g}$ free basis for $\pi = \pi_1(\Sigma)$

$D_j = \frac{\partial}{\partial x_j} : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ <u>Fox derivative</u>

$$\begin{aligned} \mathrm{Mod}(\Sigma) &\xrightarrow{\mu} GL_{2g}(\mathbb{Z}\pi) \\ f &\longmapsto \left(D_j(f(x_i))\right)_{i,j} \end{aligned}$$

<u>Note:</u>

This is <u>NOT</u> a homomorphism.

It is a <u>twisted</u> <u>representation</u> of $\mathrm{Mod}(\Sigma)$ over $\mathbb{Z}\pi$:

$$\mu(fg) = \mu(f).\, f_*(\mu(g))$$

[Suzuki '05] Topological interpretation

μ is given by the action of $\mathrm{Mod}(\Sigma)$ on $H_1(\Sigma, b; \mathbb{Z}\pi)$

basepoint $b \in \partial\Sigma$

local coeffs

In more detail

$$\begin{array}{ccc} \tilde b \in p^{-1}(b) \subset \widetilde{\Sigma} & & \text{universal covering} \\ \downarrow p & & \\ b \in \Sigma & & (\text{Deck transformation group} = \pi) \end{array}$$

$[f] \in \mathrm{Mod}(\Sigma) \xrightarrow{\text{lift}} f_* \curvearrowright (\widetilde{\Sigma}, p^{-1}(b))$ (commuting with Deck action)

$\longrightarrow f_* \curvearrowright H_1(\widetilde{\Sigma}, p^{-1}(b))$ ($\mathbb{Z}\pi$-linear)

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad H_1(\Sigma, b; \mathbb{Z}\pi)$

$\longrightarrow \ldots$

LES of $(\widetilde{\Sigma}, p^{-1}(b))$:

$$0 = H_1(\widetilde{\Sigma}) \to H_1(\widetilde{\Sigma}, p^{-1}(b)) \to H_0(p^{-1}(b)) \to H_0(\widetilde{\Sigma}) \to H_0(\widetilde{\Sigma}, p^{-1}(b)) = 0$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ \underbrace{\mathcal{I}_{\mathbb{Z}\pi}}_{\text{augmentation ideal}} & \longrightarrow & \mathbb{Z}\pi & \xrightarrow{\varepsilon} & \mathbb{Z} \end{array}$$

Fact: $\mathcal{I}_{\mathbb{Z}\pi} \cong (\mathbb{Z}\pi)^{\oplus 2g}$ because $\pi \cong F_{2g}$

freely gen. by $\{x_1 - 1, x_2 - 1, \ldots, x_{2g} - 1\} \subset \mathcal{I}_{\mathbb{Z}\pi}$

Topologically this corresponds to $\{\tilde x_1, \tilde x_2, \ldots, \tilde x_{2g}\} \subset H_1(\widetilde{\Sigma}, p^{-1}(b))$

where $\tilde x_i$ = unique lift of x_i starting at $\tilde b$

$\ldots \longrightarrow f_* \curvearrowright (\mathbb{Z}\pi)^{\oplus 2g}$ i.e. $f_* \in GL_{2g}(\mathbb{Z}\pi)$

Suzuki: uses Fox calculus to prove that (after conjugate-transpose)

this agrees with Magnus' definition $(D_j(f(x_i)))_{i,j}$.

The Johnson lattice

Definition

$N \trianglelefteq \pi$ characteristic subgroup

$$J(N) := \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \pi/N \right)$$

Note:

$J : \{\text{char. subgroups of } \pi\} \longrightarrow \{\text{subgroups of } \mathrm{Mod}(\Sigma)\}$

is a map of posets.

(chain of char. subgroups of π) $\longmapsto$ (filtration of $\mathrm{Mod}(\Sigma)$)

E.g. LCS of π $\longmapsto$ Johnson filtration

$J(k) = J(\Gamma_k \pi)$

There is an untwisted (genuine) version of the Magnus representation defined on each term of the Johnson lattice:

$$\begin{tikzcd} \mathrm{Mod}(\Sigma) \arrow[r, "\mu"] & GL_{2g}(\mathbb{Z}\pi) \arrow[d] \\ J(N) \arrow[u, hook] \arrow[r, "\mu_N"] & GL_{2g}(\mathbb{Z}[\pi/N]) \end{tikzcd}$$

untwisted representation (i.e. group hom.)

Special cases:

$N = \pi$:

$$J(\pi) = \mathrm{Mod}(\Sigma)$$
$$\mathbb{Z}[\pi/\pi] = \mathbb{Z}$$

$$\mu_\pi : \mathrm{Mod}(\Sigma) \longrightarrow GL_{2g}(\mathbb{Z})$$

is the symplectic representation.

$N = \pi'$: (classical case, studied e.g. by [Morita '93])
$= \Gamma_1 \pi$
$= [\pi, \pi]$

$$J(\pi') = \text{Torelli}$$
$$\mathbb{Z}[\pi/\pi'] = \mathbb{Z}H \qquad H = H_1(\Sigma)$$

$$\mu_{\pi'} : \text{Torelli} \longrightarrow GL_{2g}(\mathbb{Z}H)$$

Notation: For $N = \Gamma_k \pi$ write $\mu_N = \mu_k$.

So $\mu_\pi = \mu_0$

$\mu_{\pi'} = \mu_1$.

Definition: $\mathrm{Mag}(\Sigma) =$ "Magnus kernel"

$:= \ker(\mu_1)$.

ker (Magnus)

Theorem

For any characteristic subgroup $N \trianglelefteq \pi$,

$$\begin{array}{l} \ker\left(\mu_N : J(N) \longrightarrow GL_{2g}\left(\mathbb{Z}[\pi/N]\right)\right) \\ \quad \| \\ J(N') \end{array}$$

(Note that $N \trianglelefteq \pi$ characteristic implies that $N' = [N,N] \trianglelefteq \pi$ is also characteristic, so $J(N')$ is well-defined.)

Special cases:

(a) $(N=\pi)$ $\quad \ker\left(\mu_0 : \mathrm{Mod}(\Sigma) \xrightarrow{\text{symplectic representation}} GL_{2g}(\mathbb{Z})\right) = \text{Torelli}$

(b) $(N=\pi')$ $\quad \ker\left(\mu_1 : \text{Torelli} \xrightarrow[\text{classical Magnus representation}]{} GL_{2g}(\mathbb{Z}H)\right) = J(\pi'')$

Rmk:
- (a) is tautological
- (b) is the usual statement of this theorem $\left(\mathrm{Mag}(\Sigma) = J(\pi'')\right)$ usually attributed to "by Fox calculus"
- But the proof using Fox calculus immediately generalizes to the general case.
- All necessary ingredients are in Fox's 1953 paper: "Free Differential Calculus I"
- See the Appendix for the details.

Hence understanding the kernel of the (generalised) Magnus representation $\Longleftarrow$ understanding the Johnson lattice of $\mathrm{Mod}(\Sigma)$.

Picture:

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathrm{Mod}(\Sigma) = J(\pi) \arrow[r, phantom, "\supset"] \arrow[dr, leftrightarrow] & \overset{\text{Torelli}}{J(\Gamma_1\pi)} \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & \overset{\substack{\text{Johnson}\\ \text{kernel}}}{J(\Gamma_2\pi)} \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & J(\Gamma_3\pi) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots & \text{] Johnson filtration} \\
 & J(\pi') = \ker(\mu_0) \arrow[r, phantom, "\supset"] \arrow[dr, leftrightarrow] & J(\Gamma_1\pi') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & J(\Gamma_2\pi') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots & \text{] Magnus filtration} \\
 & & \underset{\substack{\text{Magnus}\\ \text{kernel}}}{J(\pi'') = \ker(\mu_1)} \arrow[r, phantom, "\supset"] \arrow[dr, leftrightarrow] & J(\Gamma_1\pi'') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & \cdots & \\
 & & & J(\pi''') = \ker(\mu_2) \arrow[r, phantom, "\supset"] \arrow[dr, phantom, "\ddots"] & \cdots & \\
 & & & & & \text{] derived filtration}
\end{tikzcd}$$

(Note: This is only part of the Johnson lattice; there are characteristic $N \trianglelefteq \pi$ that are not of the form $\Gamma_k \pi^{(\ell)}$.)

Notation for Johnson & Magnus filtrations:

$$J(k) = J(\Gamma_k \pi)$$

$$M(k) = J(\Gamma_k \pi')$$

$$\begin{tikzcd}[column sep=small, row sep=small]
\text{Torelli} \arrow[d, phantom, "\parallel"] & \overset{\text{Johnson kernel}}{\mathcal{K}(\Sigma)} \arrow[d, phantom, "\parallel"] & & \\
J(1) \arrow[r, phantom, "\supset"] & J(2) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & J(3) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots \\
 & M(1) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & M(2) \arrow[r, phantom, "\supset"] & \cdots \\
 & \mathrm{Mag}(\Sigma) & &
\end{tikzcd}$$

Plan for rest of the talk:

Partial survey of what is known about $J(*)$ and $M(*)$, mainly about how to generate them.

Interlude: higher Johnson homomorphisms

Classical Johnson homomorphisms [Johnson]

$$\begin{array}{ccc} J(k) & \xrightarrow{\tau_k} & \mathrm{Hom}\left(H, \Gamma_k\pi/\Gamma_{k+1}\pi\right) \\ f & \longmapsto & \left([x] \longmapsto [f(x)x^{-1}] \right) \\ & & x \in \pi \end{array}$$

Lemma: τ_k is well-defined and $\ker(\tau_k) = J(k+1)$

Picture:

$$\begin{tikzcd}[column sep=small]
 & \ker(\tau_1) \arrow[dr, phantom, sloped, "="] & & \ker(\tau_2) \arrow[dr, phantom, sloped, "="] & & \ker(\tau_3) & \\
J(1) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_1"'] & \supset & J(2) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_2"'] & \supset & J(3) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_3"'] & \supset & \cdots \\
 & & & & & &
\end{tikzcd}$$

Higher versions [McNeill 2013] Fix $N \trianglelefteq \pi$ characteristic

extending [Church-Farb 2009] in the case $N = \pi'$, $k=1$

$$J(\Gamma_k N) \xrightarrow{\tau_k^N} \mathrm{Hom}_{\mathbb{Z}[\pi/N]}\left(N/N', \Gamma_k N/\Gamma_{k+1}N \right)$$

same formula as above

Lemma: τ_k^N is well-defined and $\ker(\tau_k^N) \supseteq J(\Gamma_{k+1}N)$

Picture:

$$\begin{tikzcd}[column sep=small]
 & \ker(\tau_1^N) \arrow[dr, phantom, sloped, "\supset"] & & \ker(\tau_2^N) \arrow[dr, phantom, sloped, "\supset"] & & \ker(\tau_3^N) & \\
J(\Gamma_1 N) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_1^N"'] & \supset & J(\Gamma_2 N) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_2^N"'] & \supset & J(\Gamma_3 N) \arrow[ur, phantom, sloped, "\supset"] \arrow[d, "\tau_3^N"'] & \supset & \cdots \\
 & & & & & &
\end{tikzcd}$$

Generating the Johnson filtration

$$\begin{array}{ccccccccc} J(0) & \supset & J(1) & \supset & J(2) & \supset & J(3) & \supset & \cdots \\ \| & & \| & & \| & & & & \\ \mathrm{Mod}(\Sigma) & & \text{Torelli} & & \text{Johnson kernel} & & & & \end{array}$$

Quantitative

- $\mathrm{Mod}(\Sigma) = J(0)$ is fin. generated [Dehn 1938]
- Torelli $= J(1)$ is fin. gen. for $g \geq 3$ [Johnson 1983]

  (NOT fin. gen. for $g=2$) [McCullogh-Miller 1986]
- Johnson kernel $= J(2)$:
  - Originally conjectured to be ∞ generated.
  - [Dimca-Papadima 2013]: $J(2)^{ab} \otimes \mathbb{Q}$ fin. dim. $(g \geq 4)$
  - [Ershov-He 2018]: $J(2)$ is fin. gen. $(g \geq 12)$

    improved to $g \geq 4$ by [Church-Ershov-Putman 2021]
- For $k \geq 3$, $J(k)$ is fin. gen. for $g \geq 2k+1$

  [Church-Ershov-Putman 2021]

Upshot: $\forall k \geq 0$, $J(k)$ is fin. generated for $g \gg k$.

hence so is $J(k)/J(k+1)$

Qualitative

- $\mathrm{Mod}(\Sigma) = \mathcal{J}(0)$

  simple closed curve

  $\{T_\gamma \;:\; \gamma \text{ non-separating scc}\}$ [Likorish 1964] [Mumford 1967]

  Dehn twist

- Torelli $= \mathcal{J}(1)$

  $\left\{ T_\gamma T_\delta^{-1} \;:\; \begin{array}{l} \gamma, \delta \text{ non-separating scc} \\ \gamma \sqcup \delta \text{ cuts off } \end{array} \right\}$  $(g \geq 3)$ [Johnson 1979]

- Johnson kernel $= \mathcal{J}(2)$

  $\{ T_\gamma \;:\; \gamma \text{ scc cutting off }$  or  $\}$ [Johnson 1985]

- $k \geq 3$: No "nice" generating set of $\mathcal{J}(k)$ is known.

Thm [Church-Putman 2015]

$\exists$ function $G : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$\forall g \geq 1$, $\forall k \geq 1$ $\quad \mathcal{J}(k)$ is generated by elements supported on subsurfaces of genus $\leq G(k)$ and exactly 1 ∂-component.

independent of g!

Rmk
- By above, one can take $G(1) = 2$, $G(2) = 2$
- (Thm of [Church-Putman]) $G(k)$ must be $\geq \frac{k}{2}$

Generating the Magnus filtration

$$\begin{array}{l} \mathcal{M}(1) \supset \mathcal{M}(2) \supset \mathcal{M}(3) \supset \cdots \\ \parallel \\ \text{Magnus} \\ \text{kernel} \end{array}$$

Open question for a while (Problem 6.23 in Morita's 1999 survey on MCGs)

Is $\mathcal{M}(1)$ trivial?

$\rightarrow$ I.e. is $\mu_1 : \text{Torelli} \to GL_{2g}(\mathbb{Z}H)$ faithful?

Since $\mathbb{Z}H \hookrightarrow \mathbb{C}$ this would have implied that the Torelli group is linear.

Thm [Suzuki 2001] $\mathcal{M}(1)$ is non-trivial $(g \geq 2)$

Thm [Church-Farb 2009] $\operatorname{rank}(\mathcal{M}(1)^{ab}) = \infty$ $(g \geq 2)$

$\hookrightarrow$ $\mathcal{M}(1)$ not fin. generated

Proof uses the higher Johnson homomorphism

$$\begin{array}{l} J(\Gamma, \pi') \xrightarrow{\tau_1^{\pi'}} \operatorname{Hom}_{\mathbb{Z}H}\left(\pi'/\pi'', \pi''/\Gamma_2\pi'\right) \\ \parallel \\ \mathcal{M}(1) \end{array}$$

They construct an ∞ family of elements of $\mathcal{M}(1)$ whose images in $\operatorname{Hom}_{\mathbb{Z}H}\left(\pi'/\pi'', \pi''/\Gamma_2\pi'\right)$ are linearly independent.

ab. group

Thm [McNeill 2013]

$$\forall k \geq 1, \qquad \mathrm{rank}\left(\left(M(k)/M(k+1)\right)^{ab}\right) = \infty \qquad (g \geq 3)$$

$\hookrightarrow$ $M(k)/M(k+1)$ is not fin. generated

Proof also uses the higher Johnson homomorphisms, generalising [Church – Farb 2009].

Contrast:

k	$g \geq$
0	0
1	3
2	4
≥ 3	$2k+1$

$\forall k \geq 1$:

$J(k)$ is fin. generated for $g \gg k$ $\quad$ ($\Rightarrow$ so is $J(k)/J(k+1)$)

$M(k)/M(k+1)$ is not fin. generated for $g \geq 3$ $\quad$ ($\Rightarrow$ neither is $M(k)$)

Fundamental difference:

(why this is not unexpected)

$J(k)$ is defined from the LCS of $\pi \cong F_{2g}$

$M(k)$ is defined from the LCS of $\pi' \cong F_\infty$

Appendix — Proof of $\ker(\mu_N) = J(N')$

(Application of [Fox, 1953, "Free Differential Calculus, I"])

Notation:

$\pi = \pi_1(\Sigma, b)$

$x_1, x_2, \ldots, x_{2g}$ free basis of π

Two operations:

$$\{\text{normal subgroups of } \pi\} \underset{s}{\overset{i}{\rightleftarrows}} \{\text{2-sided ideals of } \mathbb{Z}\pi\}$$

$$i(N) = \ker\left(\mathbb{Z}\pi \longrightarrow \mathbb{Z}[\pi/N]\right)$$

$$s(I) = \{x \in \pi : x - 1 \in I\}$$

Directly from the definitions one can check that:

Lemma: $s(i(N)) = N$

Two key results from Fox's paper:

augmentation ideal

Thm 4.5: For any ideal I of $\mathbb{Z}\pi$ contained in $\mathcal{I}_{\mathbb{Z}\pi}$,

$$I\,\mathcal{I}_{\mathbb{Z}\pi} = \{a \in \mathcal{I}_{\mathbb{Z}\pi} : D_j a \in I \text{ for all } 1 \leq j \leq 2g\}$$

Thm 4.9: For any normal subgroup $N \trianglelefteq \pi$,

$$s(i(N)\,\mathcal{I}_{\mathbb{Z}\pi}) = [N, N]$$

Corollary: For any characteristic subgroup $N \trianglelefteq \pi$,

$$\ker(\mu_N) = J([N,N])$$

<u>Proof</u>: Let $f \in \mathcal{J}(N)$ ← domain of μ_N

Then $f \in \ker(\mu_N)$

$\iff D_j f(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ for all $1 \leq i,j \leq 2g$ modulo $i(N)$ — by the Fox calculus defⁿ of μ_N

$\iff D_j\big(f(x_i) - x_i\big) = 0$ for all $1 \leq i,j \leq 2g$ modulo $i(N)$ — because $D_j x_i = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ & D_j is $\mathbb{Z}$-linear

$\iff D_j\big(f(x_i) - x_i\big) \in i(N)$ for all $1 \leq i,j \leq 2g$

$\iff f(x_i) - x_i \in i(N)\,\mathcal{I}_{\mathbb{Z}\pi}$ for all $1 \leq i \leq 2g$ — by [Fox, Thm 4.5]

$\iff f(x_i)\,x_i^{-1} - 1 \in i(N)\,\mathcal{I}_{\mathbb{Z}\pi}$ for all $1 \leq i \leq 2g$ — since x_i is a unit in $\mathbb{Z}\pi$

$\iff f(x_i)\,x_i^{-1} \in s\big(i(N)\,\mathcal{I}_{\mathbb{Z}\pi}\big)$ for all $1 \leq i \leq 2g$ — by defⁿ of the operation $s(-)$

$$s\big(i(N)\,\mathcal{I}_{\mathbb{Z}\pi}\big) = [N,N] \quad \text{[Fox, Thm 4.9]}$$

$\iff f(x_i) = x_i$ modulo $[N,N]$ for all $1 \leq i \leq 2g$

$\iff f$ acts trivially on $\pi/[N,N]$ — since $x_1, \ldots, x_{2g}$ generate π

$\iff f \in \mathcal{J}([N,N])$ — by definition of the Johnson lattice $\mathcal{J}(-)$

$\square$