Kernels of homological MCG-representations III

GeMAT seminar
IMAR
5 June 2026

Plan

Recap + Intro

In the last two talks we saw:

- Moriyama representation $\mathrm{Mod}(\Sigma) \curvearrowright H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$

  [Moriyama '07]: kernel $= \mathcal{J}(n)$
  $= \mathcal{J}(\Gamma_n \pi)$

- Magnus twisted representation $\mathrm{Mod}(\Sigma) \overset{tw}{\curvearrowright} H_1^{BM}(\Sigma'; \mathbb{Z}\pi)$
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H_1(\Sigma, b; \mathbb{Z}\pi)$

  ↳ for each characteristic subgroup $N \trianglelefteq \pi$,

  $$\mu_N : \underset{\underset{\mathrm{Mod}(\Sigma)}{\cap}}{\mathcal{J}(N)} \curvearrowright H_1^{BM}(\Sigma'; \mathbb{Z}[\pi/N])$$

  untwisted representation

  [Fox '53] $\Rightarrow$ $\ker(\mu_N) = \mathcal{J}([N,N])$

Untwisting via specialising coeffs $\mathbb{Z}\pi \longrightarrow \mathbb{Z}[\pi/N]$
& restricting to a subgroup (so that the action on π/N is trivial)

Today:

- $n \geq 2$
- other methods of untwisting
- upper bounds on kernels

The Heisenberg group

For the Magnus repr. we took coeffs in $\mathbb{Z}\pi = \mathbb{Z}\pi_1(\Sigma)$

$$\begin{array}{c} \mathbb{Z}\pi \\ \downarrow\!\!\downarrow \\ \mathbb{Z}[\pi/N] \end{array} \qquad N \trianglelefteq \pi \text{ char. subgroup}$$

E.g. if $N = [\pi,\pi]$ this is $\mathbb{Z}H$ for $H = H_1(\Sigma)$.

Replace Σ with $C_n(\Sigma)$ for $n \geqslant 2$

$\rightsquigarrow$ need to choose a quotient of $\pi_1(C_n(\Sigma)) = \pi$

<u>Definition</u>

Let $\mathcal{H}_g = H \times \mathbb{Z}$ with $(x,m)(y,n) = (x+y, m+n+x\cdot y)$

intersection form on $H = H_1(\Sigma)$

(genus-g <u>Heisenberg group</u> ; central ext. of H)

<u>Proposition</u> (Blanchet - P. - Shaukat '25, Bellingeri - Gervais - Guaschi '08)

(a) $\mathcal{H}_g \cong \pi_1(C_n(\Sigma)) \big/ \langle\langle$ is central $\rangle\rangle$

(b) If $n \geqslant 3$, then $\mathcal{H}_g \cong \pi / [[\pi,\pi],\pi]$ $\quad (\pi = \pi_1(C_n(\Sigma)))$

Proof via presentation of $\pi_1 C_n(\Sigma)$ from [Bellingeri - Godelle '07].

Coro The action $\mathrm{Mod}(\Sigma) \curvearrowright \pi$ descends along the quotient $\pi \twoheadrightarrow \mathcal{H}_g$.

For $n \geq 3$, follows from (b) since $[[\pi,\pi],\pi]$ is characteristic.

For $n \geq 2$, follows from (a) since the element is fixed by the $\mathrm{Mod}(\Sigma)$-action, since it may be supported in a collar of $\partial\Sigma$.

Coro Well-defined twisted representation

$$H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$$

This is twisted via the action on the ground ring $\mathbb{Z}\mathcal{H}_g$ so it is untwisted on the subgroup:

$$\ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{H}_g\right) \subseteq \mathrm{Mod}(\Sigma)$$

What is this subgroup?

The Chillingworth subgroup

First, reformulate the question:

- The centre of $\mathcal{H}_g = H \times \mathbb{Z}$ is $\{0\} \times \mathbb{Z}$
- $\mathrm{Aut}^+(\mathcal{H}_g) =$ automorphisms sending $(0,1) \mapsto (0,1)$ (index-2)
- The action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{H}_g$ fixes $(0,1)$ (b/c it preserves the int. form)

$$\Rightarrow \quad \mathrm{Mod}(\Sigma) \xrightarrow{\alpha} \mathrm{Aut}^+(\mathcal{H}_g)$$

Lemma (BPS):

$$\begin{tikzcd}
 & & \mathrm{Mod}(\Sigma) \arrow[dl, dashed, "\delta", "\text{crossed hom.}"'] \arrow[d, "\alpha"] \arrow[dr, "\text{symplectic repr.}"] & & \\
1 \arrow[r] & H^1(\Sigma) \arrow[r] & \mathrm{Aut}^+(\mathcal{H}_g) \arrow[r, two heads] \arrow[l, dashed, bend left, "\text{crossed hom.}"] & Sp(H) \arrow[r] \arrow[l, bend right] & 1
\end{tikzcd}$$

Here $\delta =$ Morita's crossed homomorphism (combinatorial defn).

Crossed hom. means that: $\delta(fg) = \delta(f) + f^*(\delta(g))$

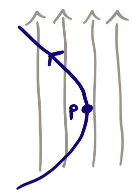
Note:
- Restricted to Torelli, δ is a homomorphism
$$\delta : \text{Torelli} \longrightarrow H^1(\Sigma)$$
- $\ker(\alpha) = \text{Torelli} \cap \ker(\delta)$ This is the subgroup we want to identify.

Definition

Fix a non-vanishing vector field X on Σ.

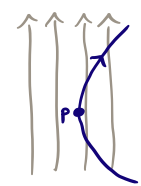
Let γ be a smooth oriented curve that is transverse to X except at a finite set $P_X(\gamma)$ of points.

Let



$\text{sign}(p) = +1$

$$\omega_X(\gamma) = \sum_{p \in P_X(\gamma)} \text{sign}(p)$$



$\text{sign}(p) = -1$

winding no. of γ w.r.t. X

($\text{sign}(p) = 0$ if $\gamma'(p)$ is opposite to X)

The Chillingworth homomorphism $e : \text{Torelli} \longrightarrow \text{Hom}(H, \mathbb{Z}) \cong H^1(\Sigma)$

is defined by $e(f)([\gamma]) = \omega_X(f \circ \gamma) - \omega_X(\gamma)$

Proposition (Chillingworth '72) This is a well-defined homomorphism.

Definition $\text{Chill}(\Sigma) = \ker(e)$ Chillingworth subgroup

Relation to Johnson filtration:

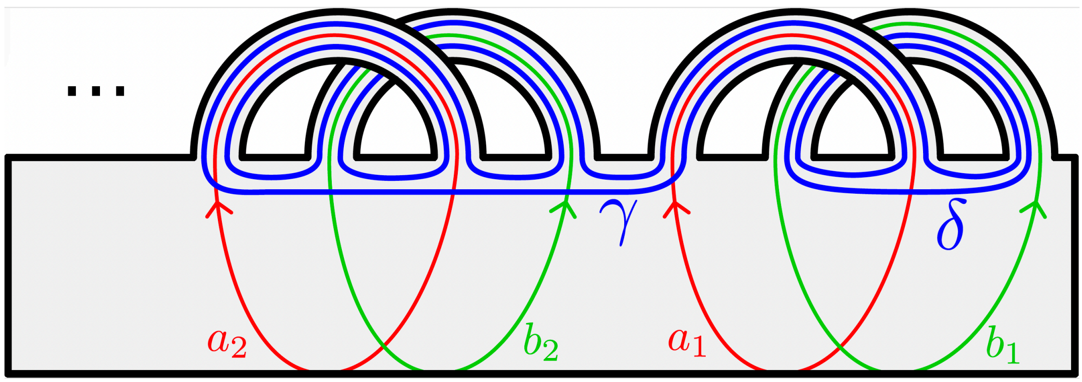
$$\underset{= J(1)}{\text{Torelli}} \supsetneq \text{Chill}(\Sigma) \supsetneq J(2)$$

[Johnson '80]

(proper inclusions for $g \geq 3$)

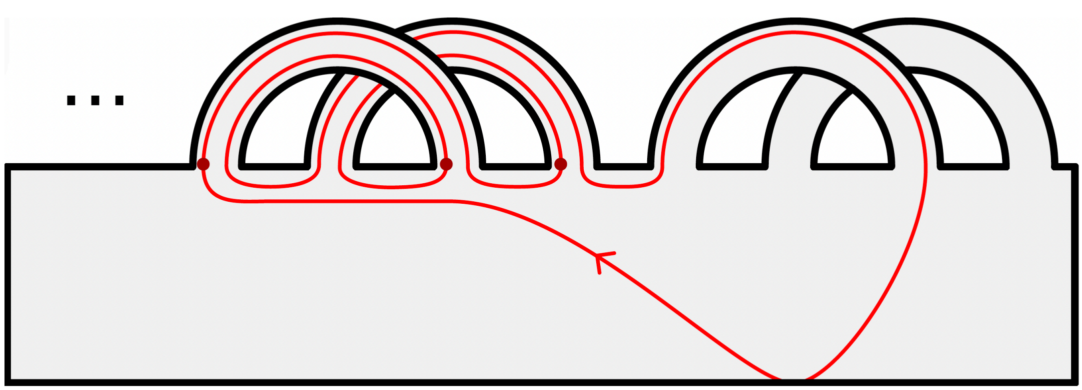
Examples:

X = constant vector field pointing upwards



$$\omega_X(a_i) = -1$$
$$\omega_X(b_i) = +1$$

$f = T_\gamma T_\delta^{-1}$ sends a_1 to $f \circ a_1$:



$$\omega_X(f \circ a_1) = 2 - 1 = +1$$

Hence $e(f): H \longrightarrow \mathbb{Z}$ $\quad (\Rightarrow e(f) \neq 0)$

$$a_1 \longmapsto 2$$

Proposition (BPS): $\quad \delta = e \; : \; \text{Torelli} \longrightarrow H^1(\Sigma)$

Proof:
- Torelli is generated by genus-1 bounding pair maps [Johnson '79].
- $\mathrm{Mod}(\Sigma)$ acts transitively on this gen. set.

$\Rightarrow$ $\{T_\gamma T_\delta^{-1}\}$ normally generates Torelli in $\mathrm{Mod}(\Sigma)$

a single, fixed bounding pair map

- δ and e both extend to crossed hom.'s

$$\mathrm{Mod}(\Sigma) \dashrightarrow H^1(\Sigma)$$

[Morita '89] for δ

[Trapp '92] for e

$\Rightarrow$ enough to check that $\delta(T_\gamma T_\delta^{-1}) = e(T_\gamma T_\delta^{-1})$

- We calculate that they are both equal to

$$\begin{aligned} H &\longrightarrow \mathbb{Z} \\ a_1 &\longmapsto 2 \\ a_i &\longmapsto 0 \quad (i \geq 2) \\ b_i &\longmapsto 0 \quad (i \geq 1) \end{aligned}$$

(See above for part of the calculation) $\square$.

Corollary 1:

$$\ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{H}_g\right) = \text{Torelli} \cap \ker(\delta)$$
$$= \text{Torelli} \cap \ker(e) = \mathrm{Chill}(\Sigma)$$

Upshot:

Restricted to $\mathrm{Chill}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$ we have an <u>untwisted</u> representation on $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$.

Corollary 2:

$$\alpha : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}^{+}(\mathcal{H}_g)$$

$$\alpha^{-1}(\mathrm{Inn}(\mathcal{H}_g)) = \mathrm{Torelli}$$

Proof: ($\subseteq$) ✓ because $\mathcal{H}_g / \text{centre} = H$

($\supseteq$) $f \in \mathrm{Torelli} \Rightarrow \delta(f)$ is a multiple of 2

say $\delta(f) = 2x \cdot -$

$(x \in H)$

$$\Rightarrow \alpha(f)(y,n) = (f_*(y), n + \delta(f)(y))$$
$$= (y, n + 2x.y)$$
$$= (x,0)(y,n)(x,0)^{-1}$$

$\square$.

Corollary 2 will allow us to untwist on $\mathrm{Torelli}(\Sigma)$, not just on $\mathrm{Chill}(\Sigma)$....

Untwisting on Torelli

Twisted representation:

$$\Phi : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)\right)$$

$$\Phi(f)(x\cdot\lambda) = \Phi(f)(x)\cdot \alpha(f)(\lambda) \qquad (\lambda \in \mathcal{H}_g)$$

twisted linearity formula.

By Corollary 2:

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & & Z(\mathcal{H}_g) \cong \mathbb{Z} \arrow[d] \\ \widetilde{\mathrm{Torelli}}(\Sigma) \arrow[rr, "\tilde{\alpha}"] \arrow[d, "\pi"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{H}_g \arrow[d, "c"] \\ \mathrm{Torelli}(\Sigma) \arrow[rr, "\alpha"] & & \mathrm{Inn}(\mathcal{H}_g) \end{tikzcd}$$

$\mathcal{H}_g$ ← central extension

Untwisted representation:

$$\Phi^u : \widetilde{\mathrm{Torelli}}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}\mathcal{H}_g}\left(H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)\right)$$

$$\Phi^u(f) := \Phi(\pi(f))\cdot \tilde{\alpha}(f)$$

Check:

$$\Phi^u(f)(x\cdot\lambda) = \Phi(\pi(f))(x\cdot\lambda)\cdot\tilde{\alpha}(f) \qquad (\text{def}^n)$$

$$= \Phi(\pi(f))(x)\cdot\alpha(\pi(f))(\lambda)\cdot\tilde{\alpha}(f) \qquad (\text{twisted lin. of } \Phi)$$

$$= \Phi(\pi(f))(x)\cdot c(\tilde{\alpha}(f))(\lambda)\cdot\tilde{\alpha}(f) \qquad (\text{pullback square})$$

$$= \Phi(\pi(f))(x)\cdot\tilde{\alpha}(f)\cdot\lambda\cdot\tilde{\alpha}(f)^{-1}\cdot\tilde{\alpha}(f) \qquad (\text{write out conjugation})$$

$$= \Phi(\pi(f))(x)\cdot\tilde{\alpha}(f)\cdot\lambda \qquad (\text{cancellation})$$

$$= \Phi^u(f)(x)\cdot\lambda \qquad (\text{def}^n)$$

Prop (BPS): $\widetilde{\text{Torelli}}(\Sigma) \xrightarrow{\pi} \text{Torelli}(\Sigma)$ is a trivial extension

Idea of proof:

Assume $g \geq 3$ (can deduce for $g \leq 2$ by pullback)

Then $\text{Mod}(\Sigma) \twoheadrightarrow Sp(H)$ induces $\cong$ on $H^2(-;\mathbb{Z})$

$\Rightarrow$ $\text{Torelli}(\Sigma) \hookrightarrow \text{Mod}(\Sigma)$ induces 0 on $H^2(-;\mathbb{Z})$

Hence every $\mathbb{Z}$-central ext. of $\text{Mod}(\Sigma)$ is trivial on $\text{Torelli}(\Sigma)$.

The 2-cocycle classifying π extends to a 2-cocycle on $\text{Mod}(\Sigma)$, defined by [Morita '89].

$\Rightarrow$ π extends to a $\mathbb{Z}$-central ext. on $\text{Mod}(\Sigma)$

$\Rightarrow$ it must be trivial on $\text{Torelli}(\Sigma)$. $\square$

Upshot

We have an untwisted representation of Torelli(Σ)

on $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$.

Module structure:

- $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$ is a finite-rank free $\mathbb{Z}\mathcal{H}_g$-module,

  (See the first talk — the lemma works for any local coefficients.)

- But it has ∞ rank as a $\mathbb{Z}$-module

  (Because $\mathcal{H}_g$ is an infinite group.)

So before untwisting we had $\mathrm{Mod}(\Sigma) \longrightarrow GL_\infty(\mathbb{Z})$

after untwisting we have $\mathrm{Torelli}(\Sigma) \longrightarrow GL_N(\mathbb{Z}\mathcal{H}_g)$

finite!

Untwisting via Schrödinger

- For any $\mathbb{Z}\mathcal{H}_g$-module V we have a twisted representation

$$\mathrm{Mod}(\Sigma) \overset{tw}{\curvearrowright} H_n^{BM}(C_n(\Sigma'); V) \qquad (*)$$

- It can be untwisted on $\mathrm{Torelli}(\Sigma) \subset \mathrm{Mod}(\Sigma)$ for any choice of V. (By above and $- \otimes_{\mathbb{Z}\mathcal{H}_g} V$)

- It can be untwisted on all of $\mathrm{Mod}(\Sigma)$ for some choices of V.

Theorem (BPS):

$(*)$ can be untwisted on a central ext. of $\mathrm{Mod}(\Sigma)$ for $V =$ Schrödinger

∞-dim unitary

or $V =$ fin. dim. analogue of Schrödinger.

Proof idea:

Similar trick to untwisting on Torelli.

The analogue of the calculation $\alpha^{-1}(\mathrm{Inn}(\mathcal{H}_g)) = \mathrm{Torelli}$ is provided by the Stone-von Neumann theorem, to obtain:

$$\begin{tikzcd}
\widetilde{\mathrm{Mod}}(\Sigma) \arrow[rr, "\tilde{\alpha}"] \arrow[d] & & U(\mathrm{Sch.}) \arrow[d] \\
\mathrm{Mod}(\Sigma) \arrow[r, "\alpha"] & \mathrm{Aut}^+(\mathcal{H}_g) \arrow[r, "T"] & PU(\mathrm{Sch.})
\end{tikzcd}$$

Segal-Shale-Weil repr.

Upper bounds on kernels

To study kernels, pass from $C_n(\Sigma')$ to $F_n(\Sigma')$.

(Untwisting results for F_n follow from those for C_n.)

I.e. $\quad \mathrm{Mod}(\Sigma) \overset{tw}{\curvearrowright} H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) = \mathrm{Heis}_n$

Comparison to Moriyama:

$\mathbb{Z}\mathcal{H}_g \twoheadrightarrow \mathbb{Z}$ induces a surjection:

$$H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) \twoheadrightarrow H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$$

Moriyama, kernel $= \mathcal{J}(n)$

Hence $\boxed{\ker(\mathrm{Heis}_n) \subseteq \mathcal{J}(n)}$

Comparison to Magnus:

$\mathbb{Z}\mathcal{H}_g \twoheadrightarrow \mathbb{Z}H$ induces a surjection:

$$H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) \twoheadrightarrow H_n^{BM}(F_n(\Sigma'); \mathbb{Z}H) \hookleftarrow H_1^{BM}(\Sigma'; \mathbb{Z}H)^{\otimes n}$$

Lemma (P.-Soulié)

Magnus representation μ_N for $N = [\pi,\pi]$

Hence $\ker(\mathrm{Heis}_n) \subseteq \ker(\mu_{[\pi,\pi]}^{\otimes n})$

(small extra argument)

$\hookrightarrow$ $\boxed{\ker(\mathrm{Heis}_n) \subseteq \ker(\mu_{[\pi,\pi]}) = \mathrm{Mag}(\Sigma) = \mathcal{J}(\pi'')}$

Hence we have:

<u>Proposition</u> (BPS): $\ker(\mathrm{Heis}_n) \subseteq \mathrm{Mag}(\Sigma) \cap J(n)$

We expect the kernel to be much smaller....

<u>Sample calculation</u>:

$n=2$ $\qquad \mathrm{Heis}_2 = H_2^{BM}(C_2(\Sigma'); \mathbb{Z}\mathcal{H}_1)$

$g=1$ $\qquad$ has rank 3 over $\mathbb{Z}\mathcal{H}_1 = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle / ab = u^2 ba$

$T_{\partial\Sigma}$ acts by the following element of $GL_3(\mathbb{Z}\mathcal{H}_1)$:

$$\begin{pmatrix}
\begin{matrix} u^{-8}b^{-2}+u^{-4}a^2-ua^2b^{-2}+ \\ (u^{-1}-u^{-2})a^2b^{-1}+(u^{-3}-u^{-4})ab^{-2}+ \\ (u^{-4}-u^{-5})ab^{-1} \end{matrix} &
\begin{matrix} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^2b^{-2}-ua^2b^{-4}+ \\ (-u^2+u+u^{-1}-u^{-2})a^2b^{-3}-u^{-3}a^2+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^2b^{-1} \end{matrix} &
\begin{matrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^2b^{-1}+ \\ (u-1)a^2b^{-3}+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^2b^{-2}+ \\ (-u^{-3}+u^{-4})ab^{-1}+(u^{-4}-u^{-5})ab^{-3}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})ab^{-2}+ \\ (-u^{-3}+u^{-4})a^2 \end{matrix} \\
\begin{matrix} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^{-2}+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-1}+u^{-6}a^2+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a \end{matrix} &
\begin{matrix} (1+u^{-2}-u^{-3}+u^{-6})+u^{-6}a^2b^{-2}-u^{-1}b^{-2}+ \\ (u^{-3}-u^{-4})ab^{-2}+(-1+u^{-1}+u^{-3}-u^{-4})b^{-1}+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})ab^{-1}-u^{-5}a^2+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a+(u^{-5}-u^{-6})a^2b^{-1} \end{matrix} &
\begin{matrix} (-u^{-6}+u^{-7})a^2b^{-1}+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b^{-1}+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})ab^{-1}+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a+ \\ (u^{-2}-u^{-3})a^{-1}b^{-1}+(-1+u^{-1}+u^{-3}-u^{-4})a^{-1}+ \\ (-u^{-5}+u^{-6})a^2 \end{matrix} \\
\begin{matrix} -u^{-6}a^{-1}b^{-1}+(-u^{-3}+u^{-4}-u^{-7})b^{-1}-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})ab^{-1}+u^{-2}a^2b^{-1}+ \\ (-u^{-3}+u^{-6})a+u^{-5}a^2 \end{matrix} &
\begin{matrix} (-1-u^{-2}+2u^{-3}-u^{-6})ab^{-1}+u^{-1}ab^{-3}+ \\ u^{-2}a^2b^{-3}+(1-u^{-1}-u^{-3}+u^{-4})ab^{-2}+ \\ (u^{-1}-u^{-2}+u^{-5})a^2b^{-2}+ \\ (-u^{-1}+u^{-4}-u^{-5})a^2b^{-1}+(u^{-2}-u^{-5})a-u^{-4}a^2 \end{matrix} &
\begin{matrix} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})ab^{-2}+(-u^{-2}+u^{-3})a^2b^{-2}+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})ab^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^2b^{-1}+(-u^{-4}+u^{-5})b^{-2}+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b^{-1}+(-u^{-4}+u^{-5})a^2 \end{matrix}
\end{pmatrix}$$

Setting $a=b=u^2=1$ we get $H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathbb{Z}/2])$

$= H_2^{BM}(F_2(\Sigma'); \mathbb{Z}) = \mathrm{Moriyama}_2$

Exercise: the matrix above reduces to the identity in this specialisation.

(it has to since $T_{\partial\Sigma} \in J(2) = \ker(\mathrm{Moriyama}_2)$)