Kernels of homological mapping class group representations

Martin Palmer-Anghel // GeMAT seminar, IMAR // May–June 2026

Abstract.

There are several different strategies for constructing interesting representations of mapping class groups of surfaces. The most topological strategy is to study their action on the homology of configuration spaces on the underlying surface. This produces a wide variety of representations, depending on the number of configuration points, whether they are ordered or unordered, the flavour of homology one considers (ordinary, Borel–Moore, etc.) and especially the choice of local system in the coefficients of homology.

This will be a short series of expository talks discussing these *homological mapping class group representations*. Motivated by the long-standing open question of whether mapping class groups are linear (admit faithful finite-dimensional representations over a field), I will focus on what is known about their kernels.

Talks.

pp. 2-16

1. Friday 22 May — Introduction/motivation and a sketch of Moriyama's 2007 proof that the kernels of the family of *Moriyama representations* coincide with the *Johnson filtration* of the mapping class group.

pp. 17-31

2. Friday 29 May — A discussion of the *Magnus representation*, which can be defined either using Fox calculus or in terms of twisted Borel–Moore homology of the surface minus a boundary point. For each characteristic subgroup N of $\pi = \pi_1(\Sigma)$, there is a Magnus representation over the ground ring $\mathbb{Z}[\pi/N]$ defined on the Johnson subgroup $J(N)$. It follow from Fox calculus that its kernel is the smaller Johnson subgroup $J([N,N])$. A survey of what is known about generating sets for different Johnson subgroups, focusing on the *Johnson filtration* of the Torelli group and the *Magnus filtration* of the Magnus kernel.

pp. 32-46

3. Friday 5 June — The construction of *Heisenberg homology representations*, which simultaneously extend the Moriyama representations and the Magnus representations, based on joint work with C. Blanchet and A. Shaukat. A priori these are *twisted* representations on the full mapping class group, but they may be untwisted in various different ways:

   (a) The subgroup on which they are untwisted (without further modification) may be identified with the *Chillingworth subgroup*, by showing that the Chillingworth homomorphism, defined in terms of winding numbers, agrees on the Torelli group with Morita's crossed homomorphism.

   (b) They may also be untwisted on the (larger) Torelli group via a certain untwisting factor. This may be constructed using the fact that the natural action of the Torelli group on the Heisenberg group is by inner automorphisms. A priori this requires us to pass to a central extension of the Torelli group, but this central extension turns out to be trivial, so we may untwist the Heisenberg homology representations on the Torelli group itself.

   (c) Replacing the Heisenberg group ring $\mathbb{Z}\mathcal{H}_g$ with the *Schrödinger representation*, which is a unitary module over $\mathbb{Z}\mathcal{H}_g$, the corresponding Heisenberg homology representations may be untwisted on (a central extension of) the *full* mapping class group via a certain untwisting factor, which is constructed using the Stone-von Neumann theorem.

   Finally, we obtain an upper bound on the kernels of the Heisenberg homology representations by comparing them to the Moriyama and Magnus representations. Explicit calculations lead us to expect that the kernels are however much smaller than this upper bound.

Kernels of homological MCG-representations I

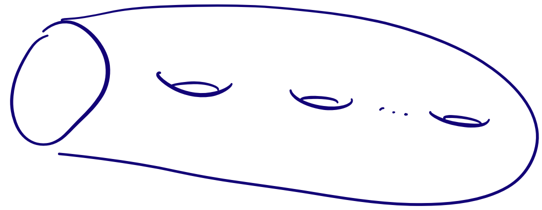
GeMAT seminar
IMAR
22 May 2026

Plan

- Intro / motivation
- Homological representations of MCGs
  - ↳ Moriyama's representation
- The Johnson filtration of MCGs
- ker (Moriyama) = Johnson
  - ↳ sketch of proof

Intro / motivation

$$\Sigma = \Sigma_{g,1} =$$



Mapping class group:

$$\mathrm{Mod}(\Sigma) = \pi_0\left(\mathrm{Homeo}_\partial(\Sigma)\right)$$

isotopy classes (π_0)

fixing a collar neighbourhood of $\partial\Sigma$ (∂)

Long-standing open question:

Is $\mathrm{Mod}(\Sigma)$ linear?

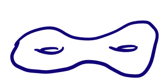
$\longleftrightarrow$ does it embed into $GL_n(K)$ for a field K?

$\longleftrightarrow$ does it admit a faithful, finite-dim representation over a field?

$g=0$: $\mathrm{Mod}(\Sigma) = \{1\}$

$g=1$: $\mathrm{Mod}(\Sigma) \cong B_3 \hookrightarrow GL_2(\mathbb{C})$

reduced Burau representation.

$g=2$: $\mathrm{Mod}\left(\Sigma_{2,0} = \right.$  $\left.\right) \hookrightarrow GL_{64}(\mathbb{C})$

[Bigelow-Budney 2001]

$g \geq 3$: ???

→ **Aim:** Construct "interesting" finite-dimensional representations of $Mod(\Sigma)$ ...

Methods:

- Quantum representations

  [Reshetikhin–Turaev '91]

  [Blanchet – Habegger – Masbaum – Vogel '95]

- Unitary representations on $L^2 X$

  for X = curve complex on Σ

  X = {measured foliations on Σ} ...

- Homological representations

  $H_*(\{\text{configurations in } \Sigma\})$

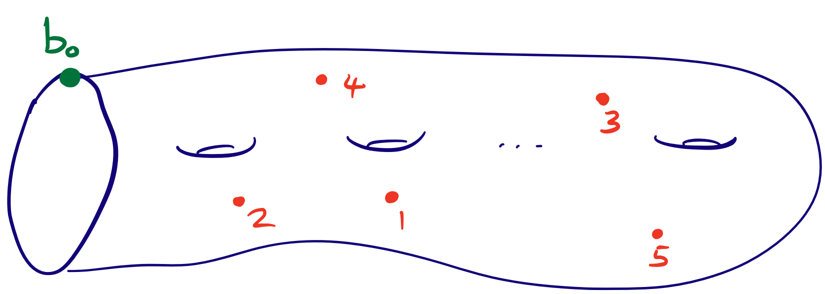
Homological representations of Mod (Σ)

Construction

Let $\Sigma' = \Sigma \setminus \{b_0\}$

boundary point

Consider $F_n(\Sigma') = \{\text{ordered } n\text{-point configurations in } \Sigma'\}$



$\in F_5(\Sigma')$

Mod(Σ) acts on $F_n(\Sigma')$ (up to isotopy of homeomorphisms)

$\Rightarrow$ on $H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$

Borel-Moore homology

This is the n^{th} Moriyama representation of Mod(Σ).

Two alternative descriptions:

- $H^n\left(F_n(\Sigma),\ \underline{F_n(\Sigma, \{b_0\})}\ ;\ \mathbb{Z}\right)$

  configurations C such that $b_0 \in C$

  (by Poincaré duality)

- $H_n\left(\Sigma^n,\ \Delta \cup B\ ;\ \mathbb{Z}\right)$

  fat diagonal

  n-tuples (multiconfigurations that contain b_0)

  (because Σ^n compactifies $F_n(\Sigma')$ with remainder $\Delta \cup B$)

Variations:

- $H_n^{BM} \rightsquigarrow H_n$
- $F_n \rightsquigarrow C_n$

  (will mention later)

  unordered configurations

- $\mathbb{Z} \rightsquigarrow$ non-trivial local systems (next talks...)

The Johnson filtration

Let $\pi = \pi_1(\Sigma, b_0)$ $\quad (\cong F_{2g})$

Lower central series:

$$\pi = \Gamma_0\pi \supseteq \Gamma_1\pi \supseteq \Gamma_2\pi \supseteq \cdots$$

$$\Gamma_i\pi = [\pi, \Gamma_{i-1}\pi]$$
$$= \{i\text{-fold iterated commutators}\}$$

$\mathrm{Mod}(\Sigma)$ acts on π

$\Rightarrow$ on $\pi/\Gamma_n\pi$ $\quad$ ($\Gamma_n\pi$ is a characteristic subgroup)

Def. $\mathcal{J}(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \pi/\Gamma_n\pi\right)$

This is the Johnson filtration of $\mathrm{Mod}(\Sigma)$.

$$\mathrm{Mod}(\Sigma) = \mathcal{J}(0) \supseteq \mathcal{J}(1) \supseteq \mathcal{J}(2) \supseteq \mathcal{J}(3) \supseteq \cdots$$

Torelli subgroup

Johnson kernel

Another description

$$\mathbb{Z}\pi \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \cdots\cdots$$

augmentation ideal $= \ker(\mathbb{Z}\pi \to \mathbb{Z})$

$\mathrm{Mod}(\Sigma)$ acts on $\mathbb{Z}\pi$

$\Rightarrow$ on $\mathbb{Z}\pi / I^{n+1}$

Proposition (Fox '53):

$$\ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathbb{Z}\pi / I^{n+1}\right) = J(n)$$

Key property of $J(\bullet)$:

$$\bigcap_{n=1}^{\infty} J(n) = \{1\}$$

$\varphi \in \bigcap_{n=1}^{\infty} J(n) \rightsquigarrow \varphi$ acts trivially on $\pi/\Gamma_n\pi$ $\forall n$

$\rightsquigarrow \varphi$ acts trivially on $\varprojlim \pi/\Gamma_\bullet\pi$

$$\begin{array}{c} \varprojlim \pi/\Gamma_\bullet\pi \\ \uparrow \\ \pi \end{array} \longleftarrow \text{b/c } \pi \cong F_{2g} \text{ is res. nilpotent [Magnus]}$$

$\rightsquigarrow \varphi$ acts trivially on $\pi \rightsquigarrow \varphi = 1$.

ker (Moriyama) = Johnson

Theorem (Moriyama 2007)

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright \underbrace{H_n^{BM}(F_n(\Sigma'))}_{\text{Moriyama representation}} \Big) = J(n)$$

Corollary $\mathrm{Mod}(\Sigma) \curvearrowright \bigoplus_{n=1}^{\infty} H_n^{BM}(F_n(\Sigma'))$

is faithful.

For comparison:

[Bianchi–Miller–Wilson 2021]:

ordinary homology

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright H_n(F_n(\Sigma)) \Big) \supseteq J(n)$$

but is strictly larger when $n \geq 3$, $g \geq 2$

[Bianchi–Stavrou 2023]:

ordinary homology

$$\ker\Big(\mathrm{Mod}(\Sigma) \curvearrowright H_*(C_n(\Sigma)) \Big) = J(2)$$

unordered configurations

(for $n \geq 2$)

Sketch of proof

Notation:

$$H_n = H_n^{BM}(F_n(\Sigma')) = n^{th} \text{ Moriyama representation}$$

$$K(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright H_n\right)$$

$$J(n) = n^{th} \text{ term of Johnson filtration}$$

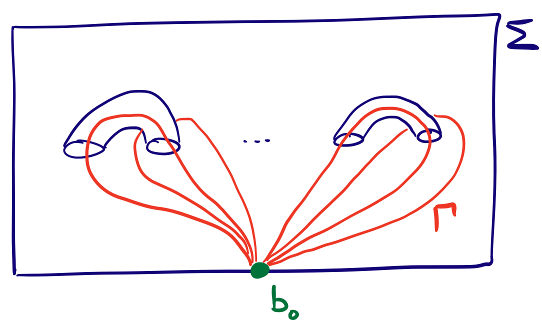
$$= \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathbb{Z}\pi / I^{n+1}\right) \qquad \text{[Fox]}$$

Aim: $J(n) = K(n)$

Steps:
(1) Module structure of H_n
(2) A filtered ring $\hat{H}$
(3) A filtered homomorphism $\mathbb{Z}\pi \longrightarrow \hat{H}$
(4) End of the proof.

(1) Module structure of H_n

$$H_n = H_n^{BM}(F_n(\Sigma \setminus \{b_0\}); \mathbb{Z})$$
$$\cong H_n^{BM}(C_n(\Sigma \setminus \{b_0\}); \mathbb{Z}[S_n])$$
$$\cong H_n^{BM}(C_n(\Gamma \setminus \{b_0\}); \mathbb{Z}[S_n]) = (*)$$



Adaptation of a lemma of Bigelow (2004)

Follows from the fact that:

- Σ def. retracts onto Γ rel. $\{b_0\}$ through a family of maps $\Sigma \xrightarrow{h_t} \Sigma$ such that:
  - h_t is 1-Lipschitz for all t
  - h_t is a topological self-embedding except for $t=1$

This general criterion is in [P.-Soulié 2025]

(Moriyama's original proof of this was different.)

Note: $C_n(\Gamma \setminus \{b_0\}) \cong C_n\left(\coprod_{2g} (0,1)\right) \cong \coprod_{\binom{2g+n-1}{n}} \mathring{\mathbb{D}}^n$

So $(*) \cong \mathbb{Z}[S_n]^{\binom{2g+n-1}{n}}$

(2) A filtered ring $\hat{H}$

$$\hat{H} = \prod_{n=1}^{\infty} H_n$$

Equipped with:

- Superposition product induced by:

$$C_m(\Sigma') \times C_n(\Sigma') \xleftarrow[\text{open}]{} C_{(m,n)}(\Sigma') \xrightarrow[\text{finite covering}]{} C_{m+n}(\Sigma')$$

$$H^{BM}_{m+n}(-) \longrightarrow H^{BM}_{m+n}(-) \longrightarrow H^{BM}_{m+n}(-)$$

$\rightsquigarrow$ ring

- Filtration $\mathcal{F}_i \hat{H} = \prod_{n=i}^{\infty} H_n$

- $Mod(\Sigma)$-action

Note: $\hat{H} / \mathcal{F}_i \hat{H} \cong H_0 \oplus H_1 \oplus \cdots \oplus H_{i-1}$ as $Mod(\Sigma)$-reps.

Proposition: There is a SES

$$0 \longrightarrow H_{n-1} \otimes H_1 \xrightarrow{\;\cdot\;} H_n \longrightarrow\!\!\!\!\rightarrow (H_{n-1})^{\oplus n-1} \longrightarrow 0$$

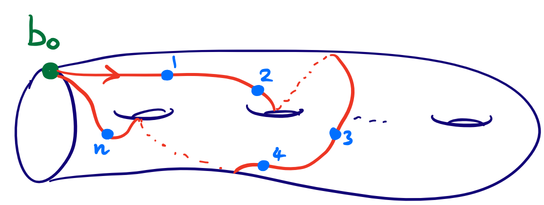
Corollary:
(1) $K(n) \subseteq K(n-1)$ (descending filtration of $Mod(\Sigma)$)

(2) $\text{mult}^n : (H_1)^{\otimes n} \longrightarrow H_n$ is injective.

(3) A filtered homomorphism $\mathbb{Z}\pi \longrightarrow \hat{H}$

$\gamma \in \pi = \pi_1(\Sigma, b_0)$

$$\longmapsto \phi_n(\gamma) \in H_n = H_n^{BM}\left(F_n(\Sigma')\right)$$



Extend linearly to group hom. $\phi_n : \mathbb{Z}\pi \longrightarrow H_n$

Define $\Phi = \prod_n \phi_n : \mathbb{Z}\pi \longrightarrow \hat{H}$

Key lemma: Φ is a ring homomorphism

Idea: $\phi_n(\gamma)$ = image of $[\Delta^n]$ under

$$(\Delta^n, \partial\Delta^n) \longrightarrow (\Sigma^n, \Delta \cup B)$$

concatenating γ and δ corresponds to breaking Δ^n into $n+1$ sub-simplices

This decomposition gives:

$$\phi_n(\gamma\delta) = \sum_{i=0}^{n} \phi_i(\gamma)\,\phi_{n-i}(\delta). \qquad \square$$

Corollary:

(*) $\Phi\left((\gamma_1 - 1) \cdots (\gamma_{n+1} - 1)\right) = \phi_1(\gamma_1) \cdots \phi_1(\gamma_{n+1})$ modulo $F_{n+2}\hat{H}$

(Clear for $n=0$, follows in general because Φ is a ring hom.)

In particular, $\circledast$ implies that:

- $\Phi(I^{n+1}) \subseteq \mathcal{F}_{n+1}\hat{H} \implies$

$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, "\Phi"] \arrow[d, two heads] & \hat{H} \arrow[d, two heads] \\ \mathbb{Z}\pi / I^{n+1} \arrow[r, "\Phi_n"] & \hat{H} / \mathcal{F}_{n+1}\hat{H} \end{tikzcd}$$

- $\phi_n(I^{n+1}) = 0 \implies$

$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, "\phi_n"] \arrow[d, two heads] & H_n \\ \mathbb{Z}\pi / I^{n+1} \arrow[ur, "\phi_n'"'] & \end{tikzcd}$$

Final ingredient: Φ_n is injective.

Idea: • By a 5-lemma argument it's enough to show that the associated graded homomorphism

$$I^n / I^{n+1} \xrightarrow{gr(\Phi_n)} \frac{\mathcal{F}_n \hat{H}}{\mathcal{F}_{n+1}\hat{H}} = H_n$$

is injective.

- $I^n / I^{n+1} \cong (H_1)^{\otimes n}$

$$[(\gamma_1 - 1) \cdots (\gamma_n - 1)] \longmapsto [\gamma_1] \otimes \cdots \otimes [\gamma_n]$$

- By $\circledast$, $gr(\Phi_n)$ is the composition

$$I^n / I^{n+1} \cong (H_1)^{\otimes n} \xrightarrow{\text{mult.}^n} H_n$$

injective (from earlier) $\square$

(4) End of the proof

Recall we want to prove that $J(n) = K(n)$.

n^{th} term of Johnson filtration $= $ kernel of action on $\frac{\mathbb{Z}\pi}{I^{n+1}}$

kernel of n^{th} Morigama representation H_n

$(\supseteq)$ $\varphi \in K(n)$

$\Rightarrow$ φ acts trivially on $H_0 \oplus H_1 \oplus \cdots \oplus H_n \cong \hat{H} / \mathcal{F}_{n+1}\hat{H}$

This contains $\frac{\mathbb{Z}\pi}{I^{n+1}}$ as a subrepresentation. (Φ_n is injective)

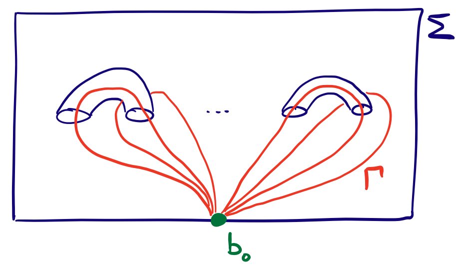
$\Rightarrow$ $\varphi \in J(n)$

$(\subseteq)$ $\varphi \in J(n)$

$\Rightarrow$ For any $1 \leq k \leq n$ we have:

$$\begin{tikzcd} \mathbb{Z}\pi \arrow[r, two heads, "\phi_k"] \arrow[d, two heads] & \mathrm{im}(\phi_k) \subseteq H_k \\ \mathbb{Z}\pi / I^{k+1} \arrow[ur, two heads, "\phi_k'"'] & \end{tikzcd}$$

$\Rightarrow$ φ acts trivially on $\mathrm{im}(\phi_1), \ldots, \mathrm{im}(\phi_n)$



Module structure of H_n $\Rightarrow$ it is generated by elements of the form $a_1 a_2 \cdots a_\ell$

with $a_j \in \operatorname{im}(\phi_{k_j})$ $\qquad \sum_j k_j = n$

$1 \leq k_j \leq n$

$\Rightarrow$ φ acts trivially on H_n

i.e. $\varphi \in K(n)$.

$\square$

Kernels of homological MCG-representations II

GeMAT seminar
IMAR
29 May 2026

Plan

Intro

Last week we studied: $\Sigma = \Sigma_{g,1}$

$$\mathrm{Mod}(\Sigma) \curvearrowright H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$$

$$\mathrm{Mod}(\Sigma) = \pi_0 \mathrm{Homeo}_\partial(\Sigma)$$

Moriyama representation

Thm (Moriyama '07) $\ker = J(n)$

Johnson filtration:

$$J(n) = \ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \pi/\Gamma_n\pi\right)$$

$$\pi_1(\Sigma) = \pi = \Gamma_0\pi \supset \Gamma_1\pi \supset \Gamma_2\pi \supset \cdots\cdots$$

lower central series

Cf. $H_n(F_n(\Sigma))$: $\ker \supsetneq J(n)$ [Bianchi-Miller-Wilson '21]

$H_*(C_n(\Sigma))$: $\ker = J(2)$ [Bianchi-Stavrou '23]

Today:
- add local coefficients $\Big\}$ $\longrightarrow$ Magnus representation
- $n=1$

(Next week: $n \geq 2$ & how to "untwist".)

The Magnus representation

[Magnus '39] Via Fox calculus

$x_1, \dots, x_{2g}$ free basis for $\pi = \pi_1(\Sigma)$

$D_j = \frac{\partial}{\partial x_j} : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ <u>Fox derivative</u>

$$\mathrm{Mod}(\Sigma) \xrightarrow{\mu} GL_{2g}(\mathbb{Z}\pi)$$

$$f \longmapsto \left(D_j(f(x_i))\right)_{i,j}$$

<u>Note:</u>

This is <u>NOT</u> a homomorphism.

It is a <u>twisted representation</u> of $\mathrm{Mod}(\Sigma)$ over $\mathbb{Z}\pi$:

$$\mu(fg) = \mu(f).\, f_*(\mu(g))$$

[Suzuki '05] Topological interpretation

μ is given by the action of $\mathrm{Mod}(\Sigma)$ on $H_1(\Sigma, b; \mathbb{Z}\pi)$

basepoint $b \in \partial\Sigma$

local coeffs

In more detail

$$\begin{array}{ccc} \tilde{b} \in p^{-1}(b) \subset \tilde{\Sigma} & & \text{universal covering} \\ \downarrow p & & \\ b \in \Sigma & & (\text{Deck transformation group} = \pi) \end{array}$$

$[f] \in \mathrm{Mod}(\Sigma) \xrightarrow{\text{lift}} f_* \curvearrowright (\tilde{\Sigma}, p^{-1}(b))$ (commuting with Deck action)

$\longrightarrow f_* \curvearrowright H_1(\tilde{\Sigma}, p^{-1}(b))$ ($\mathbb{Z}\pi$-linear)

$\qquad\qquad \| $

$\qquad H_1(\Sigma, b; \mathbb{Z}\pi)$

$\longrightarrow \ldots$

LES of $(\tilde{\Sigma}, p^{-1}(b))$:

$$0 = H_1(\tilde{\Sigma}) \longrightarrow H_1(\tilde{\Sigma}, p^{-1}(b)) \longrightarrow H_0(p^{-1}(b)) \longrightarrow H_0(\tilde{\Sigma}) \longrightarrow H_0(\tilde{\Sigma}, p^{-1}(b)) = 0$$

$$\mathcal{I}_{\mathbb{Z}\pi} \longrightarrow \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z}$$

augmentation ideal

Fact: $\mathcal{I}_{\mathbb{Z}\pi} \cong (\mathbb{Z}\pi)^{\oplus 2g}$ because $\pi \cong F_{2g}$

freely gen. by $\{x_1 - 1, x_2 - 1, \ldots, x_{2g} - 1\} \subset \mathcal{I}_{\mathbb{Z}\pi}$

Topologically this corresponds to $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2g}\} \subset H_1(\tilde{\Sigma}, p^{-1}(b))$

where $\tilde{x}_i$ = unique lift of x_i starting at $\tilde{b}$

$\ldots \longrightarrow f_* \curvearrowright (\mathbb{Z}\pi)^{\oplus 2g}$ i.e. $f_* \in GL_{2g}(\mathbb{Z}\pi)$

Suzuki: uses Fox calculus to prove that (after conjugate-transpose) this agrees with Magnus' definition $(D_j(f(x_i)))_{i,j}$.

The Johnson lattice

Definition

$N \trianglelefteq \pi$ characteristic subgroup

$$J(N) := \ker \left(\mathrm{Mod}(\Sigma) \curvearrowright \pi/N \right)$$

Note:

$J : \{\text{char. subgroups of } \pi\} \longrightarrow \{\text{subgroups of } \mathrm{Mod}(\Sigma)\}$

is a map of posets.

$(\text{chain of char. subgroups of } \pi) \longmapsto (\text{filtration of } \mathrm{Mod}(\Sigma))$

E.g. LCS of $\pi \longmapsto$ Johnson filtration $J(k) = J(\Gamma_k \pi)$

There is an untwisted (genuine) version of the Magnus representation defined on each term of the Johnson lattice:

$$\begin{tikzcd}
\mathrm{Mod}(\Sigma) \arrow[r, "\mu"] & GL_{2g}(\mathbb{Z}\pi) \arrow[d] \\
J(N) \arrow[u, hook] \arrow[r, "\mu_N"] & GL_{2g}(\mathbb{Z}[\pi/N])
\end{tikzcd}$$

untwisted representation (i.e. group hom.)

Special cases:

$N = \pi$:

$$\mathcal{J}(\pi) = \mathrm{Mod}(\Sigma)$$
$$\mathbb{Z}[\pi/\pi] = \mathbb{Z}$$

$$\mu_\pi : \mathrm{Mod}(\Sigma) \longrightarrow GL_{2g}(\mathbb{Z})$$

is the symplectic representation.

$N = \pi'$: (classical case, studied e.g. by [Morita '93])
$= \Gamma_1 \pi$
$= [\pi, \pi]$

$$\mathcal{J}(\pi') = \text{Torelli}$$
$$\mathbb{Z}[\pi/\pi'] = \mathbb{Z}H \qquad H = H_1(\Sigma)$$

$$\mu_{\pi'} : \text{Torelli} \longrightarrow GL_{2g}(\mathbb{Z}H)$$

Notation: For $N = \Gamma_k \pi$ write $\mu_N = \mu_k$.

So $\mu_\pi = \mu_0$
$\mu_{\pi'} = \mu_1$.

Definition: $\mathrm{Mag}(\Sigma) =$ "Magnus kernel"
$:= \ker(\mu_1)$.

ker (Magnus)

Theorem

For any characteristic subgroup $N \trianglelefteq \pi$,

$$\ker\left(\mu_N : \mathcal{J}(N) \longrightarrow GL_{2g}(\mathbb{Z}[\pi/N])\right) = \mathcal{J}(N')$$

(Note that $N \trianglelefteq \pi$ characteristic implies that $N' = [N,N] \trianglelefteq \pi$ is also characteristic, so $\mathcal{J}(N')$ is well-defined.)

Special cases:

(a) $(N = \pi)$ $\quad \ker\left(\mu_0 : \mathrm{Mod}(\Sigma) \xrightarrow{\text{symplectic representation}} GL_{2g}(\mathbb{Z})\right) = \text{Torelli}$

(b) $(N = \pi')$ $\quad \ker\left(\mu_1 : \text{Torelli} \xrightarrow{\text{classical Magnus representation}} GL_{2g}(\mathbb{Z}H)\right) = \mathcal{J}(\pi'')$

Rmk:
- (a) is tautological
- (b) is the usual statement of this theorem $\left(\mathrm{Mag}(\Sigma) = \mathcal{J}(\pi'')\right)$ usually attributed to "by Fox calculus"
- But the proof using Fox calculus immediately generalises to the general case.
- All necessary ingredients are in Fox's 1953 paper: "Free Differential Calculus I"
- See the Appendix for the details.

Hence understanding the kernel of the (generalised) Magnus representation $\Longleftarrow$ understanding the Johnson lattice of $\mathrm{Mod}(\Sigma)$.

Picture:

$$
\begin{tikzcd}[column sep=small]
\mathrm{Mod}(\Sigma) = J(\pi) \arrow[r, phantom, "\supset"] & \overset{\text{Torelli}}{J(\Gamma_1\pi)} \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & \overset{\text{Johnson kernel}}{J(\Gamma_2\pi)} \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & J(\Gamma_3\pi) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots \quad \text{] Johnson filtration} \\
 & J(\pi') \arrow[ul, leftrightarrow] \arrow[r, phantom, "\supset"] & J(\Gamma_1\pi') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & J(\Gamma_2\pi') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots \quad \text{] Magnus filtration} \\
 & \ker(\mu_0) \arrow[u, phantom, "\parallel"] & \underset{\text{Magnus kernel}}{J(\pi'')} \arrow[ul, leftrightarrow] \arrow[r, phantom, "\supset"] & J(\Gamma_1\pi'') \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & \cdots \\
 & & \ker(\mu_1) \arrow[u, phantom, "\parallel"] & J(\pi''') \arrow[ul, leftrightarrow] \arrow[r, phantom, "\supset"] & \cdots \\
 & & & \ker(\mu_2) \arrow[u, phantom, "\parallel"] & \cdots \quad \text{] derived filtration}
\end{tikzcd}
$$

(*Note*: This is only part of the Johnson lattice; there are characteristic $N \trianglelefteq \pi$ that are not of the form $\Gamma_k \pi^{(\ell)}$.)

Notation for Johnson & Magnus filtrations:

$$J(k) = J(\Gamma_k \pi)$$

$$M(k) = J(\Gamma_k \pi')$$

$$
\begin{tikzcd}[column sep=small]
\text{Torelli} & \overset{\text{Johnson kernel}}{\mathcal{K}(\Sigma)} & & \\
J(1) \arrow[u, phantom, "\parallel"] \arrow[r, phantom, "\supset"] & J(2) \arrow[u, phantom, "\parallel"] \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & J(3) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cup"] & \cdots \\
 & M(1) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\parallel"] & M(2) \arrow[r, phantom, "\supset"] & \cdots \\
 & \mathrm{Mag}(\Sigma) & &
\end{tikzcd}
$$

Plan for rest of the talk:

Partial survey of what is known about $J(*)$ and $M(*)$, mainly about how to generate them.

Interlude: higher Johnson homomorphisms

Classical Johnson homomorphisms [Johnson]

$$\begin{aligned} J(k) &\xrightarrow{\tau_k} \mathrm{Hom}\left(H, \Gamma_k\pi / \Gamma_{k+1}\pi\right) \\ f &\longmapsto \left([x] \longmapsto [f(x)x^{-1}] \right) \\ & \qquad\qquad x \in \pi \end{aligned}$$

Lemma: τ_k is well-defined and $\ker(\tau_k) = J(k+1)$

Picture:

$$\begin{tikzcd}[column sep=small]
& \ker(\tau_1) \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "="] & & \ker(\tau_2) \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "="] & & \ker(\tau_3) \arrow[dl, phantom, sloped, "\subset"] & \\
J(1) \arrow[d, "\tau_1"'] & \supset & J(2) \arrow[d, "\tau_2"'] & \supset & J(3) \arrow[d, "\tau_3"'] & \supset & \cdots \\
{} & & {} & & {} & &
\end{tikzcd}$$

Higher versions [McNeill 2013]

Fix $N \trianglelefteq \pi$ characteristic

extending [Church-Farb 2009] in the case $N = \pi'$, $k=1$

$$J(\Gamma_k N) \xrightarrow{\tau_k^N} \mathrm{Hom}_{\mathbb{Z}[\pi/N]}\left(N/N', \Gamma_k N / \Gamma_{k+1} N \right)$$

same formula as above

Lemma: τ_k^N is well-defined and $\ker(\tau_k^N) \supseteq J(\Gamma_{k+1}N)$

Picture:

$$\begin{tikzcd}[column sep=small]
& \ker(\tau_1^N) \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\cup"] & & \ker(\tau_2^N) \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\cup"] & & \ker(\tau_3^N) \arrow[dl, phantom, sloped, "\subset"] & \\
J(\Gamma_1 N) \arrow[d, "\tau_1^N"'] & \supset & J(\Gamma_2 N) \arrow[d, "\tau_2^N"'] & \supset & J(\Gamma_3 N) \arrow[d, "\tau_3^N"'] & \supset & \cdots \\
{} & & {} & & {} & &
\end{tikzcd}$$

Generating the Johnson filtration

$$\begin{array}{ccccccccc} J(0) & \supset & J(1) & \supset & J(2) & \supset & J(3) & \supset & \cdots \\ \| & & \| & & \| & & & & \\ \mathrm{Mod}(\Sigma) & & \text{Torelli} & & \text{Johnson kernel} & & & & \end{array}$$

Quantitative

- $\mathrm{Mod}(\Sigma) = J(0)$ is fin. generated [Dehn 1938]
- Torelli $= J(1)$ is fin. gen. for $g \geq 3$ [Johnson 1983]

  (NOT fin. gen. for $g=2$) [McCullogh-Miller 1986]
- Johnson kernel $= J(2)$:
  - Originally conjectured to be ∞ generated.
  - [Dimca-Papadima 2013]: $J(2)^{ab} \otimes \mathbb{Q}$ fin. dim. $(g \geq 4)$
  - [Ershov-He 2018]: $J(2)$ is fin. gen. $(g \geq 12)$

    improved to $g \geq 4$ by [Church-Ershov-Putman 2021]

- For $k \geq 3$, $J(k)$ is fin. gen. for $g \geq 2k+1$

  [Church-Ershov-Putman 2021]

Upshot: $\forall k \geq 0$, $J(k)$ is fin. generated for $g \gg k$.

hence so is $J(k)/J(k+1)$

Qualitative

- $\mathrm{Mod}(\Sigma) = \mathcal{J}(0)$

$$\{T_\gamma \;:\; \gamma \text{ non-separating scc}\}$$

simple closed curve

Dehn twist

[Likorish 1964]
[Mumford 1967]

- Torelli $= \mathcal{J}(1)$

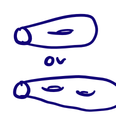
$$\left\{ T_\gamma T_\delta^{-1} \;:\; \begin{array}{l} \gamma, \delta \text{ non-separating scc} \\ \gamma \sqcup \delta \text{ cuts off} \end{array} \right\}$$

$(g \geq 3)$

[Johnson 1979]

- Johnson kernel $= \mathcal{J}(2)$

$$\{T_\gamma \;:\; \gamma \text{ scc cutting off}\}$$



[Johnson 1985]

- $k \geq 3$: No "nice" generating set of $\mathcal{J}(k)$ is known.

Thm [Church-Putman 2015]

$\exists$ function $G : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$\forall g \geq 1$
$\forall k \geq 1$ $\quad \mathcal{J}(k)$ is generated by elements supported on subsurfaces of genus $\leq G(k)$ and exactly 1 ∂-component.

independent of g!

Rmk • By above, one can take $G(1) = 2$
$G(2) = 2$

• (Thm of [Church-Putman]) $G(k)$ must be $\geq \frac{k}{2}$

Generating the Magnus filtration

$$\begin{array}{c} \mathcal{M}(1) \supset \mathcal{M}(2) \supset \mathcal{M}(3) \supset \cdots \\ \| \\ \text{Magnus} \\ \text{kernel} \end{array}$$

Open question for a while (Problem 6.23 in Morita's 1999 survey on MCGs)

Is $\mathcal{M}(1)$ trivial?

$\longrightarrow$ I.e. is $\mu_1 : \text{Torelli} \to GL_{2g}(\mathbb{Z}H)$ faithful?

Since $\mathbb{Z}H \hookrightarrow \mathbb{C}$ this would have implied that the Torelli group is linear.

Thm [Suzuki 2001] $\mathcal{M}(1)$ is non-trivial $(g \geq 2)$

Thm [Church-Farb 2009] $\operatorname{rank}\left(\mathcal{M}(1)^{ab}\right) = \infty$ $(g \geq 2)$

$\hookrightarrow$ $\mathcal{M}(1)$ not fin. generated

Proof uses the higher Johnson homomorphism

$$\begin{array}{ccc} J(\Gamma, \pi') & \xrightarrow{\tau_1^{\pi'}} & \operatorname{Hom}_{\mathbb{Z}H}\left(\pi'/\pi'', \pi''/\Gamma_2\pi'\right) \\ \| & & \\ \mathcal{M}(1) & & \end{array}$$

They construct an ∞ family of elements of $\mathcal{M}(1)$ whose images in $\operatorname{Hom}_{\mathbb{Z}H}\left(\pi'/\pi'', \pi''/\Gamma_2\pi'\right)$ are linearly independent.

ab. group

Thm [McNeill 2013]

$$\forall k \geq 1, \qquad \mathrm{rank}\left(\left(M(k)/M(k+1)\right)^{ab}\right) = \infty \qquad (g \geq 3)$$

$\hookrightarrow$ $M(k)/M(k+1)$ is not fin. generated

Proof also uses the higher Johnson homomorphisms, generalising [Church – Farb 2009].

Contrast:

k	$g \geq$
0	0
1	3
2	4
≥ 3	$2k+1$

$\forall k \geq 1$:

$J(k)$ is fin. generated for $g \gg k$ $\quad$ ($\Rightarrow$ so is $J(k)/J(k+1)$)

$M(k)/M(k+1)$ is <u>not</u> fin. generated for $g \geq 3$ $\quad$ ($\Rightarrow$ neither is $M(k)$)

Fundamental difference:

(why this is not unexpected)

$J(k)$ is defined from the LCS of $\pi \cong F_{2g}$

$M(k)$ is defined from the LCS of $\pi' \cong F_\infty$

Appendix — Proof of $\ker(\mu_N) = \mathcal{J}(N')$

(Application of [Fox, 1953, "Free Differential Calculus, I"])

Notation:

$\pi = \pi_1(\Sigma, b)$

$x_1, x_2, \ldots, x_{2g}$ free basis of π

Two operations:

$$\{\text{normal subgroups of } \pi\} \underset{s}{\overset{i}{\rightleftarrows}} \{\text{2-sided ideals of } \mathbb{Z}\pi\}$$

$$i(N) = \ker\left(\mathbb{Z}\pi \longrightarrow \mathbb{Z}[\pi/N]\right)$$

$$s(I) = \{x \in \pi : x - 1 \in I\}$$

Directly from the definitions one can check that:

Lemma: $s(i(N)) = N$

Two key results from Fox's paper:

Thm 4.5: For any ideal I of $\mathbb{Z}\pi$ contained in $\mathcal{I}_{\mathbb{Z}\pi}$, augmentation ideal

$$I\mathcal{I}_{\mathbb{Z}\pi} = \{a \in \mathcal{I}_{\mathbb{Z}\pi} : D_j a \in I \text{ for all } 1 \leq j \leq 2g\}$$

Thm 4.9: For any normal subgroup $N \trianglelefteq \pi$,

$$s(i(N)\mathcal{I}_{\mathbb{Z}\pi}) = [N, N]$$

Corollary: For any characteristic subgroup $N \trianglelefteq \pi$,

$$\ker(\mu_N) = \mathcal{J}([N,N])$$

<u>Proof</u>: Let $f \in \mathcal{J}(N)$ ← domain of μ_N

Then $f \in \ker(\mu_N)$

$\Longleftrightarrow$ $D_j f(x_i) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ for all $1 \leq i,j \leq 2g$ modulo $i(N)$ — by the Fox calculus def^n of μ_N

$\Longleftrightarrow$ $D_j\big(f(x_i) - x_i\big) = 0$ for all $1 \leq i,j \leq 2g$ modulo $i(N)$ — because $D_j x_i = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ & D_j is $\mathbb{Z}$-linear

$\Longleftrightarrow$ $D_j\big(f(x_i) - x_i\big) \in i(N)$ for all $1 \leq i,j \leq 2g$

$\Longleftrightarrow$ $f(x_i) - x_i \in i(N)\mathcal{I}_{\mathbb{Z}\pi}$ for all $1 \leq i \leq 2g$ — by [Fox, Thm 4.5]

$\Longleftrightarrow$ $f(x_i)x_i^{-1} - 1 \in i(N)\mathcal{I}_{\mathbb{Z}\pi}$ for all $1 \leq i \leq 2g$ — since x_i is a unit in $\mathbb{Z}\pi$

$\Longleftrightarrow$ $f(x_i)x_i^{-1} \in s\big(i(N)\mathcal{I}_{\mathbb{Z}\pi}\big)$ for all $1 \leq i \leq 2g$ — by def^n of the operation $s(-)$

$$s\big(i(N)\mathcal{I}_{\mathbb{Z}\pi}\big) = [N,N] \quad \text{[Fox, Thm 4.9]}$$

$\Longleftrightarrow$ $f(x_i) = x_i$ modulo $[N,N]$ for all $1 \leq i \leq 2g$

$\Longleftrightarrow$ f acts trivially on $\pi/[N,N]$ — since $x_1, \ldots, x_{2g}$ generate π

$\Longleftrightarrow$ $f \in \mathcal{J}([N,N])$ — by definition of the Johnson lattice $\mathcal{J}(-)$

$\blacksquare$

Kernels of homological MCG-representations III

GeMAT seminar
IMAR
5 June 2026

Plan

Recap + Intro

In the last two talks we saw:

- Moriyama representation $\mathrm{Mod}(\Sigma) \curvearrowright H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$

  [Moriyama '07]: $\text{kernel} = \mathcal{J}(n)$
  $= \mathcal{J}(\Gamma_n\pi)$

- Magnus twisted representation $\mathrm{Mod}(\Sigma) \overset{\text{tw}}{\curvearrowright} H_1^{BM}(\Sigma'; \mathbb{Z}\pi)$
  $$H_1^{BM}(\Sigma'; \mathbb{Z}\pi) = H_1(\Sigma, b; \mathbb{Z}\pi)$$

  $\rightarrow$ for each characteristic subgroup $N \trianglelefteq \pi$,

  $$\mu_N : \mathcal{J}(N) \curvearrowright H_1^{BM}(\Sigma'; \mathbb{Z}[\pi/N]) \qquad \text{untwisted representation}$$
  $$\mathcal{J}(N) \subset \mathrm{Mod}(\Sigma)$$

  [Fox '53] $\Rightarrow$ $\ker(\mu_N) = \mathcal{J}([N,N])$

Untwisting via specialising coeffs $\mathbb{Z}\pi \longrightarrow \mathbb{Z}[\pi/N]$
& restricting to a subgroup (so that the action on π/N is trivial)

Today:

- $n \geq 2$
- other methods of untwisting
- upper bounds on kernels

The Heisenberg group

For the Magnus repr. we took coeffs in $\mathbb{Z}\pi = \mathbb{Z}\pi_1(\Sigma)$
$$\mathbb{Z}\pi \twoheadrightarrow \mathbb{Z}[\pi/N]$$
$N \trianglelefteq \pi$ char. subgroup

E.g. if $N = [\pi,\pi]$ this is $\mathbb{Z}H$ for $H = H_1(\Sigma)$.

Replace Σ with $C_n(\Sigma)$ for $n \geq 2$

$\rightsquigarrow$ need to choose a quotient of $\pi_1(C_n(\Sigma)) = \pi$

Definition

Let $\mathcal{H}_g = H \times \mathbb{Z}$ with $(x,m)(y,n) = (x+y, m+n+x\cdot y)$

intersection form on $H = H_1(\Sigma)$

(genus-g Heisenberg group ; central ext. of H)

Proposition (Blanchet-P.-Shaukat '25, Bellingeri-Gervais-Guaschi '08)

(a) $\mathcal{H}_g \cong \pi_1(C_n(\Sigma)) / \langle\langle \sigma \text{ is central} \rangle\rangle$

(b) If $n \geq 3$, then $\mathcal{H}_g \cong \pi / [[\pi,\pi],\pi]$ $\quad (\pi = \pi_1(C_n(\Sigma)))$

Proof via presentation of $\pi_1 C_n(\Sigma)$ from [Bellingeri-Godelle '07].

Coro The action $\mathrm{Mod}(\Sigma) \curvearrowright \pi$ descends along the quotient $\pi \twoheadrightarrow \mathcal{H}_g$.

For $n \geq 3$, follows from (b) since $[[\pi,\pi],\pi]$ is characteristic.

For $n \geq 2$, follows from (a) since the element is fixed by the $\mathrm{Mod}(\Sigma)$-action, since it may be supported in a collar of $\partial\Sigma$.

Coro Well-defined twisted representation

$$H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$$

This is twisted via the action on the ground ring $\mathbb{Z}\mathcal{H}_g$ so it is untwisted on the subgroup:

$$\ker(\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{H}_g) \subseteq \mathrm{Mod}(\Sigma)$$

What is this subgroup?

The Chillingworth subgroup

First, reformulate the question:

- The centre of $\mathcal{H}_g = H \times \mathbb{Z}$ is $\{0\} \times \mathbb{Z}$
- $\mathrm{Aut}^+(\mathcal{H}_g)$ = automorphisms sending $(0,1) \mapsto (0,1)$ (index-2)
- The action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{H}_g$ fixes $(0,1)$ (b/c it preserves the int. form)

$$\Rightarrow \quad \mathrm{Mod}(\Sigma) \xrightarrow{\alpha} \mathrm{Aut}^+(\mathcal{H}_g)$$

<u>Lemma</u> (BPS):

$$\begin{tikzcd}
 & & \mathrm{Mod}(\Sigma) \arrow[dl, dashed, "\delta" near end, "\text{crossed hom.}"'] \arrow[d, "\alpha"] \arrow[dr, "\text{symplectic repr.}"] & & \\
1 \arrow[r] & H^1(\Sigma) \arrow[r] & \mathrm{Aut}^+(\mathcal{H}_g) \arrow[r, two heads] \arrow[l, dashed, bend left, "\text{crossed hom.}"] & Sp(H) \arrow[r] \arrow[l, bend right] & 1
\end{tikzcd}$$

Here δ = Morita's crossed homomorphism (combinatorial defn).

Crossed hom. means that: $\delta(fg) = \delta(f) + f^*(\delta(g))$

<u>Note</u>:
- Restricted to Torelli, δ is a homomorphism

$$\delta : \text{Torelli} \longrightarrow H^1(\Sigma)$$

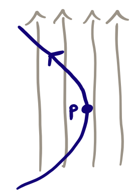
- $\ker(\alpha) = \text{Torelli} \cap \ker(\delta)$ ← This is the subgroup we want to identify.

Definition

Fix a non-vanishing vector field X on Σ.

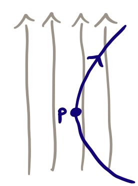
Let γ be a smooth oriented curve that is transverse to X except at a finite set $P_X(\gamma)$ of points.

Let



$\mathrm{sign}(p) = +1$

$$\omega_X(\gamma) = \sum_{p \in P_X(\gamma)} \mathrm{sign}(p)$$



$\mathrm{sign}(p) = -1$

winding no. of γ w.r.t. X

($\mathrm{sign}(p) = 0$ if $\gamma'(p)$ is opposite to X)

The Chillingworth homomorphism $e : \mathrm{Torelli} \longrightarrow \mathrm{Hom}(H, \mathbb{Z}) \cong H^1(\Sigma)$

is defined by $e(f)([\gamma]) = \omega_X(f \circ \gamma) - \omega_X(\gamma)$

Proposition (Chillingworth '72) This is a well-defined homomorphism.

Definition $\mathrm{Chill}(\Sigma) = \ker(e)$ Chillingworth subgroup

Relation to Johnson filtration:

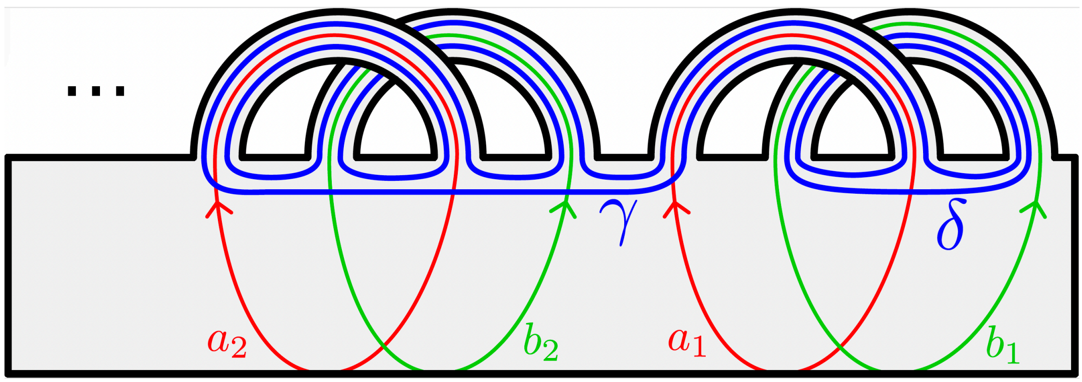
$$\underset{\substack{\parallel \\ J(1)}}{\mathrm{Torelli}} \supsetneq \mathrm{Chill}(\Sigma) \supsetneq J(2)$$

[Johnson '80]

(proper inclusions for $g \geq 3$)

Examples:

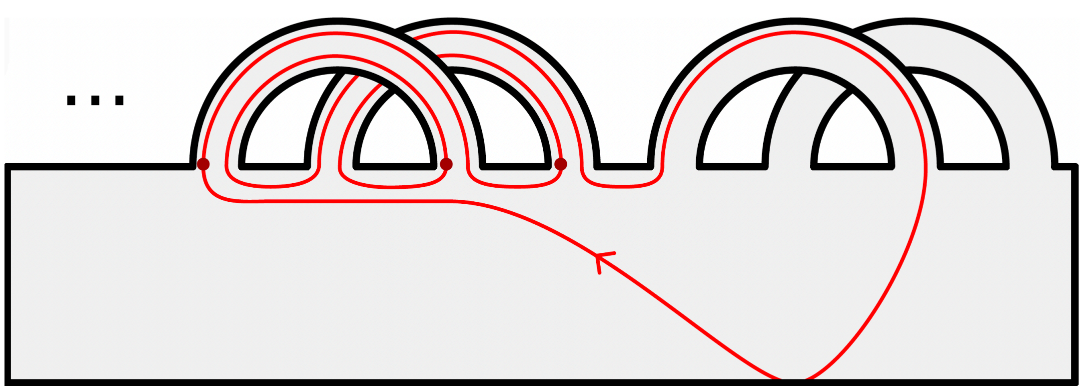
X = constant vector field pointing upwards



$\omega_X(a_i) = -1$

$\omega_X(b_i) = +1$

$f = T_\gamma T_\delta^{-1}$ sends a_1 to $f \circ a_1$:



$\omega_X(f \circ a_1) = 2 - 1 = +1$

Hence $e(f) : H \longrightarrow \mathbb{Z}$
$a_1 \longmapsto 2$

$(\Rightarrow e(f) \neq 0)$

<u>Proposition</u> (BPS): $\delta = e \;:\; \text{Torelli} \longrightarrow H^1(\Sigma)$

<u>Proof</u> :
- Torelli is generated by genus-1 bounding pair maps [Johnson '79].
- $\mathrm{Mod}(\Sigma)$ acts transitively on this gen. set.

$\Rightarrow$ $\{T_\gamma T_\delta^{-1}\}$ normally generates Torelli in $\mathrm{Mod}(\Sigma)$

a single, fixed bounding pair map

- δ and e both extend to crossed hom.'s

$\mathrm{Mod}(\Sigma) \dashrightarrow H^1(\Sigma)$ [Morita '89] for δ [Trapp '92] for e

$\Rightarrow$ enough to check that $\delta(T_\gamma T_\delta^{-1}) = e(T_\gamma T_\delta^{-1})$

- We calculate that they are both equal to

$$\begin{aligned} H &\longrightarrow \mathbb{Z} \\ a_1 &\longmapsto 2 \\ a_i &\longmapsto 0 \quad (i \geq 2) \\ b_i &\longmapsto 0 \quad (i \geq 1) \end{aligned}$$

(See above for part of the calculation) $\square$.

<u>Corollary 1</u> :

$$\ker\left(\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{H}_g\right) = \text{Torelli} \cap \ker(\delta)$$
$$= \text{Torelli} \cap \ker(e) = \mathrm{Chill}(\Sigma)$$

<u>Upshot</u>:

Restricted to $\mathrm{Chill}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$ we have an <u>untwisted</u> representation on $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$.

<u>Corollary 2</u>:

$$\alpha : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}^{f}(\mathcal{H}_g)$$

$$\alpha^{-1}(\mathrm{Inn}(\mathcal{H}_g)) = \mathrm{Torelli}$$

<u>Proof</u>: $(\subseteq)$ ✓ because $\mathcal{H}_g/\text{centre} = H$

$(\supseteq)$ $f \in \mathrm{Torelli} \Rightarrow \delta(f)$ is a multiple of 2

say $\delta(f) = 2x \cdot -$

$(x \in H)$

$$\Rightarrow \alpha(f)(y,n) = (f_*(y), n + \delta(f)(y))$$
$$= (y, n + 2x.y)$$
$$= (x,0)(y,n)(x,0)^{-1}$$

$\square$.

Corollary 2 will allow us to untwist on $\mathrm{Torelli}(\Sigma)$, not just on $\mathrm{Chill}(\Sigma)$....

Untwisting on Torelli

Twisted representation:

$$\Phi : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)\right)$$

$$\Phi(f)(x \cdot \lambda) = \Phi(f)(x) \cdot \alpha(f)(\lambda) \qquad (\lambda \in \mathcal{H}_g)$$

twisted linearity formula.

By Corollary 2:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] & & Z(\mathcal{H}_g) \cong \mathbb{Z} \arrow[d] \\
\widetilde{\mathrm{Torelli}}(\Sigma) \arrow[rr, "\tilde{\alpha}"] \arrow[d, "\pi"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{H}_g \arrow[d, "c"] \\
\mathrm{Torelli}(\Sigma) \arrow[rr, "\alpha"] & & \mathrm{Inn}(\mathcal{H}_g)
\end{tikzcd}$$

central extension

Untwisted representation:

$$\Phi^u : \widetilde{\mathrm{Torelli}}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}\mathcal{H}_g}\left(H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)\right)$$

$$\Phi^u(f) := \Phi(\pi(f)) \cdot \tilde{\alpha}(f)$$

Check:

$$
\begin{aligned}
\Phi^u(f)(x\cdot\lambda) &= \Phi(\pi(f))(x\cdot\lambda)\cdot\tilde{\alpha}(f) && (\text{def}^n) \\
&= \Phi(\pi(f))(x)\cdot\alpha(\pi(f))(\lambda)\cdot\tilde{\alpha}(f) && (\text{twisted lin. of } \Phi) \\
&= \Phi(\pi(f))(x)\cdot c(\tilde{\alpha}(f))(\lambda)\cdot\tilde{\alpha}(f) && (\text{pullback square}) \\
&= \Phi(\pi(f))(x)\cdot\tilde{\alpha}(f)\cdot\lambda\cdot\tilde{\alpha}(f)^{-1}\cdot\tilde{\alpha}(f) && (\text{write out conjugation}) \\
&= \Phi(\pi(f))(x)\cdot\tilde{\alpha}(f)\cdot\lambda && (\text{cancellation}) \\
&= \Phi^u(f)(x)\cdot\lambda && (\text{def}^n)
\end{aligned}
$$

Prop (BPS): $\widetilde{\text{Torelli}}(\Sigma) \xrightarrow{\pi} \text{Torelli}(\Sigma)$ is a trivial extension

Idea of proof:

Assume $g \geq 3$ (can deduce for $g \leq 2$ by pullback)

Then $\text{Mod}(\Sigma) \twoheadrightarrow Sp(H)$ induces $\cong$ on $H^2(-;\mathbb{Z})$

$\Rightarrow$ $\text{Torelli}(\Sigma) \hookrightarrow \text{Mod}(\Sigma)$ induces 0 on $H^2(-;\mathbb{Z})$

Hence every $\mathbb{Z}$-central ext. of $\text{Mod}(\Sigma)$ is trivial on $\text{Torelli}(\Sigma)$.

The 2-cocycle classifying π extends to a 2-cocycle on $\text{Mod}(\Sigma)$, defined by [Morita '89].

$\Rightarrow$ π extends to a $\mathbb{Z}$-central ext. on $\text{Mod}(\Sigma)$

$\Rightarrow$ it must be trivial on $\text{Torelli}(\Sigma)$. $\square$

Upshot

We have an untwisted representation of $\mathrm{Torelli}(\Sigma)$ on $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$.

Module structure :

- $H_n^{BM}(C_n(\Sigma'); \mathbb{Z}\mathcal{H}_g)$ is a finite-rank free $\mathbb{Z}\mathcal{H}_g$-module,

  (See the first talk — the lemma works for any local coefficients.)

- But it has ∞ rank as a $\mathbb{Z}$-module

  (Because $\mathcal{H}_g$ is an infinite group.)

So before untwisting we had $\mathrm{Mod}(\Sigma) \longrightarrow GL_\infty(\mathbb{Z})$

after untwisting we have $\mathrm{Torelli}(\Sigma) \longrightarrow GL_N(\mathbb{Z}\mathcal{H}_g)$

finite!

Untwisting via Schrödinger

- For any $\mathbb{Z}\mathcal{H}_g$-module V we have a twisted representation

$$\mathrm{Mod}(\Sigma) \overset{tw}{\curvearrowright} H_n^{BM}(C_n(\Sigma'); V) \qquad (*)$$

- It can be untwisted on $\mathrm{Torelli}(\Sigma) \subset \mathrm{Mod}(\Sigma)$ for any choice of V. (By above and $- \otimes_{\mathbb{Z}\mathcal{H}_g} V$)

- It can be untwisted on all of $\mathrm{Mod}(\Sigma)$ for some choices of V.

Theorem (BPS):

$(*)$ can be untwisted on a central ext. of $\mathrm{Mod}(\Sigma)$ for $V =$ Schrödinger

or $V =$ fin. dim. analogue of Schrödinger.

∞-dim unitary

Proof idea:

Similar trick to untwisting on Torelli.

The analogue of the calculation $\alpha^{-1}(\mathrm{Inn}(\mathcal{H}_g)) = \mathrm{Torelli}$

is provided by the Stone-von Neumann theorem,

to obtain:

$$\begin{tikzcd}
\widetilde{\mathrm{Mod}}(\Sigma) \arrow[rr, "\tilde{\alpha}"] \arrow[d] & & U(\mathrm{Sch.}) \arrow[d] \\
\mathrm{Mod}(\Sigma) \arrow[r, "\alpha"] & \mathrm{Aut}^+(\mathcal{H}_g) \arrow[r, "T"] & PU(\mathrm{Sch.})
\end{tikzcd}$$

Segal-Shale-Weil repr.

Upper bounds on kernels

To study kernels, pass from $C_n(\Sigma')$ to $F_n(\Sigma')$.

(Untwisting results for F_n follow from those for C_n.)

I.e. $\quad Mod(\Sigma) \overset{tw}{\curvearrowright} H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) = \mathrm{Heis}_n$

Comparison to Moriyama:

$\mathbb{Z}\mathcal{H}_g \twoheadrightarrow \mathbb{Z}$ induces a surjection:

$$H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) \twoheadrightarrow H_n^{BM}(F_n(\Sigma'); \mathbb{Z})$$

Moriyama, kernel $= \mathcal{J}(n)$

Hence $\boxed{\ker(\mathrm{Heis}_n) \subseteq \mathcal{J}(n)}$

Comparison to Magnus:

$\mathbb{Z}\mathcal{H}_g \twoheadrightarrow \mathbb{Z}H$ induces a surjection:

$$H_n^{BM}(F_n(\Sigma'); \mathbb{Z}\mathcal{H}_g) \twoheadrightarrow H_n^{BM}(F_n(\Sigma'); \mathbb{Z}H)$$

$$\hat{\cup} \quad \text{Lemma (P.-Soulié)}$$

$$H_1^{BM}(\Sigma'; \mathbb{Z}H)^{\otimes n}$$

Magnus representation μ_N for $N = [\pi,\pi]$

Hence $\ker(\mathrm{Heis}_n) \subseteq \ker(\mu_{[\pi,\pi]}^{\otimes n})$

(small extra argument)

$\hookrightarrow$ $\boxed{\ker(\mathrm{Heis}_n) \subseteq \ker(\mu_{[\pi,\pi]}) = Mag(\Sigma) = \mathcal{J}(\pi'')}$

Hence we have:

Proposition (BPS): $\ker(\mathrm{Heis}_n) \subseteq \mathrm{Mag}(\Sigma) \cap J(n)$

We expect the kernel to be much smaller....

Sample calculation:

$n = 2$ $\qquad \mathrm{Heis}_2 = H_2^{BM}(C_2(\Sigma'); \mathbb{Z}\mathcal{H}_1)$

$g = 1$

has rank 3 over $\mathbb{Z}\mathcal{H}_1 = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle / ab = u^2 ba$

$T_{\partial\Sigma}$ acts by the following element of $GL_3(\mathbb{Z}\mathcal{H}_1)$:

$$\begin{pmatrix}
\begin{array}{c} u^{-8}b^{-2}+u^{-4}a^2-ua^2b^{-2}+ \\ (u^{-1}-u^{-2})a^2b^{-1}+(u^{-3}-u^{-4})ab^{-2}+ \\ (u^{-4}-u^{-5})ab^{-1} \end{array} &
\begin{array}{c} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^2b^{-2}-ua^2b^{-4}+ \\ (-u^2+u+u^{-1}-u^{-2})a^2b^{-3}-u^{-3}a^2+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^2b^{-1} \end{array} &
\begin{array}{c} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^2b^{-1}+ \\ (u-1)a^2b^{-3}+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^2b^{-2}+ \\ (-u^{-3}+u^{-4})ab^{-1}+(u^{-4}-u^{-5})ab^{-3}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})ab^{-2}+ \\ (-u^{-3}+u^{-4})a^2 \end{array} \\
\begin{array}{c} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^{-2}+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-1}+u^{-6}a^2+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a \end{array} &
\begin{array}{c} (1+u^{-2}-u^{-3}+u^{-6})+u^{-6}a^2b^{-2}-u^{-1}b^{-2}+ \\ (u^{-3}-u^{-4})ab^{-2}+(-1+u^{-1}+u^{-3}-u^{-4})b^{-1}+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})ab^{-1}-u^{-5}a^2+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a+(u^{-5}-u^{-6})a^2b^{-1} \end{array} &
\begin{array}{c} (-u^{-6}+u^{-7})a^2b^{-1}+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b^{-1}+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})ab^{-1}+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a+ \\ (u^{-2}-u^{-3})a^{-1}b^{-1}+(-1+u^{-1}+u^{-3}-u^{-4})a^{-1}+ \\ (-u^{-5}+u^{-6})a^2 \end{array} \\
\begin{array}{c} -u^{-6}a^{-1}b^{-1}+(-u^{-3}+u^{-4}-u^{-7})b^{-1}-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})ab^{-1}+u^{-2}a^2b^{-1}+ \\ (-u^{-3}+u^{-6})a+u^{-5}a^2 \end{array} &
\begin{array}{c} (-1-u^{-2}+2u^{-3}-u^{-6})ab^{-1}+u^{-1}ab^{-3}+ \\ u^{-2}a^2b^{-3}+(1-u^{-1}-u^{-3}+u^{-4})ab^{-2}+ \\ (u^{-1}-u^{-2}+u^{-5})a^2b^{-2}+ \\ (-u^{-1}+u^{-4}-u^{-5})a^2b^{-1}+(u^{-2}-u^{-5})a-u^{-4}a^2 \end{array} &
\begin{array}{c} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})ab^{-2}+(-u^{-2}+u^{-3})a^2b^{-2}+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})ab^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^2b^{-1}+(-u^{-4}+u^{-5})b^{-2}+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b^{-1}+(-u^{-4}+u^{-5})a^2 \end{array}
\end{pmatrix}$$

Setting $a = b = u^2 = 1$ we get $H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2])$

$= H_2^{BM}(F_2(\Sigma'); \mathbb{Z}) = \mathrm{Moriyama}_2$

Exercise: the matrix above reduces to the identity in this specialisation.

(it has to since $T_{\partial\Sigma} \in J(2) = \ker(\mathrm{Moriyama}_2)$)