# Embedding groups into acyclic groups via étale groupoids

Joint work with
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## Plan

# I Embeddings into acyclic groups

**Def** A group $G$ is acyclic if $\tilde{H}_*(G) = 0$

**Example** (Baumslag – Gruenberg / Epstein '67/'68):

Commutator subgroup of $\langle a, b \mid a = [a, ba^{-1}b^{-1}][a, b^{-1}ab] \rangle$

[ Baumslag – Gruenberg : combinatorial construction
Epstein : geometric construction ($\pi_1$ of grope construction)
Berrick – Wong (2004): they are the same ]

**Example** (Mather '71) $\mathrm{Homeo}_c(\mathbb{R}^n)$

compact support

**Theorem** (Kan – Thurston '76)

Every countable group embeds into an acyclic countable group.

Idea: (1) $G \longrightarrow \underbrace{G^{\mathbb{Q}} \rtimes \mathrm{Aut}_c(\mathbb{Q})}_{\text{acyclic}}$

same method as [Mather]

(2) colimit argument $\longrightarrow$ countable subgroup containing $G$ and still acyclic

## Corollaries

- $\forall$ connected space $X$ [Kan-Thurston '76]
  $\exists$ group $G$ : $H_*(X) \cong H_*(G)$

  → Idea: • CW-approximation $X'$ of $X$
  • mirror the cell structure of $X'$ using groups

  (embedding into an acyclic group) ⟷ (cone of a space)

- short proof of Atiyah's $L^2$-index theorem [Chatterji-Mislin '03]
  for countable coverings of Riemannian manifolds

  → Idea: • The statement is natural w.r.t. inclusions of (countable) deck transformation groups.
  • It is easy to check if the deck transformation group is acyclic.
  • Apply [Kan-Thurston].

**Ansatz:** Every group of type $P$ embeds into an acyclic group of type $P$.

True for:

$P = \emptyset$
$P =$ countable } [Kan-Thurston '76]

$P =$ finitely generated [Baumslag-Dyer-Heller '80]
$P =$ finitely presented [Baumslag-Dyer-Miller '83]

**Thm A** (P.-Wu '25)

True for $P =$ type $F_n$, for each $n \geq 1$ and $n = \infty$

**Def** A group $G$ has <u>type $F_n$</u> if it has a classifying CW-complex (i.e. $K(G,1)$ complex) with finite $n$-skeleton.

<u>Type $F_\infty$</u> $\longleftrightarrow$ type $F_n$ for all $n \geq 1$.

**Observation**

$F_1 \longleftrightarrow$ finitely generated

$F_2 \longleftrightarrow$ finitely presented

$F_n \Rightarrow F_{n-1}$

**Thm** (Bieri '76) (Stallings '63 $n=3$)

Stallings-Bieri group

$F_n \not\Leftarrow F_{n-1}$

Specifically, $SB_n = \ker\big((F_2)^n \to \mathbb{Z}\big)$ has type $F_{n-1}$ but <u>not</u> type $F_n$.

each gen. $\mapsto 1$

**Thm** (Skipper-Witzel-Zaremsky '17)

$\exists$ <u>simple</u> group of type $F_{n-1}$ but not $F_n$

**Thm B** (P.-Wu '25)

$\exists$ <u>simple</u>, <u>acyclic</u> group of type $F_{n-1}$ but not $F_n$

# II Thompson groups & variants

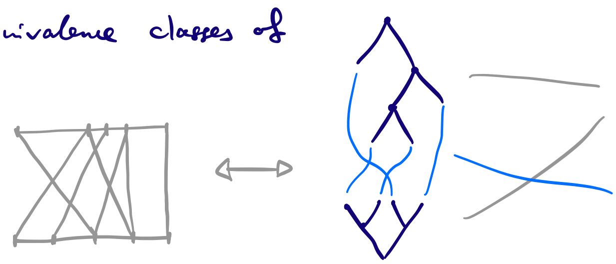
Fix $d \geq 2$

## $d^{th}$ Higman–Thompson group

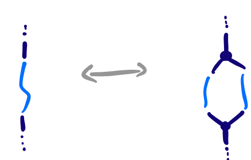
All trees planar & rooted.

$V_d := \{ \varphi \in \mathrm{Homeo}(\mathfrak{C}) \mid \varphi$ given by
- cut $\mathfrak{C}$ into fin. many $d$-adic pieces
- rearrange + rescale by factors of $d$ $\}$

= equivalence classes of

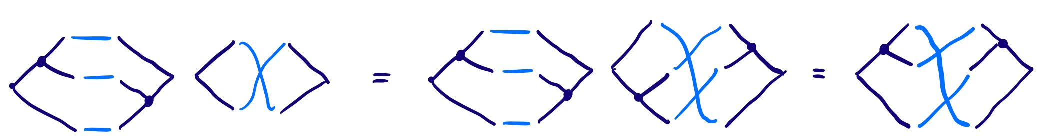



eg. relation generated by



Composition in $\mathrm{Homeo}(\mathfrak{C})$

$\longleftrightarrow$
- expand until middle trees are the same
- compose bijections



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**Timeline**

$\langle a,b,c,d \mid b^a=b^2, c^b=c^2, \underbrace{d^c=d^2, a^d=a^2}_{\text{max. normal}} \rangle$

1951 (Higman) $\exists$ infinite, fin. pres., simple group ("Higman group")

1965 (Thompson) $V = V_2$ ———— " ————
type $F_\infty$

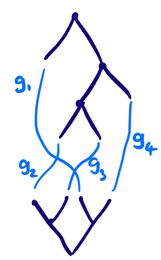
1974 (Higman) $V_d$ type $F_\infty$
$d$ even : simple
$d$ odd : $V_d \underset{(2)}{\trianglerighteq} [V_d, V_d]$ simple

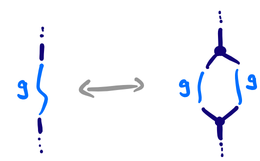
$\}$ [Brown '87]

# Variants: Thompson-like groups

## (1) Labelled Higman-Thompson groups

Replace $\Sigma_n$ with $G \wr \Sigma_n = G^n \rtimes \Sigma_n$ $\rightsquigarrow$ $V_d(G)$




**Note** $G$ embeds into $V_d(G)$ via $g \longmapsto$ 


**Theorem** [Wu-Wu-Zhao-Zhou, 2024]

$G$ is of type $F_n$

$\Updownarrow$

$V_2(G)$ is of type $F_n$

## (2) Brin-Thompson groups [Brin, 2004]

Replace $\mathcal{C}$ with $\mathcal{C}^k$, $k \geq 1$

$\varphi \in kV_d$ acts on $\mathcal{C}^k$ by

- cutting it into finitely many bricks
- rearrange + rescale by factors of $d$

cuboids of side-lengths $1/d^n$ & vertices = $d$-adic rationals

**Eg**  $\in 2V = 2V_2$

- $kV$ is fin. presentable & simple [Brin, Hennig-Matucci]
- $kV \not\cong \ell V$ for $k \neq \ell$ [Brin, Bleak-Lanoue]

## ③ Twisted Brin–Thompson groups [Belk–Zaremsky, 2020]

Replace $\mathfrak{C}$ with $\mathfrak{C}^S$ ($S$ any set)

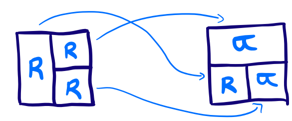
Choose $G \subseteq \mathrm{Sym}(S)$

- Bricks have length 1 in all but fin. many dimensions
- Rearrange, rescale and rotate/reflect each brick.

  permute its coordinates by any $g \in G$.

$\rightsquigarrow$ $SV_d^G$

Eg




**Theorem** [Belk-Zaremsky, 2020]:

- $SV_2^G$ is simple
- For certain choices of $G \subseteq \mathrm{Sym}(S)$, $SV_2^G$ is of type $F_{n-1}$ but not of type $F_n$.

**Thm C** (P. – Wu '25)

(i) For any group $G$, $V_2(G)$ is acyclic.

(ii) For any subgroup $G \subseteq \mathrm{Sym}(S)$, $SV_2^G$ is acyclic.

**Corollaries**

(i) + [Wu-Wu-Zhao-Zhou] $\Rightarrow$ Thm A

and $V_2(SB_n)$ is acyclic & of type $F_{n-1}$ but not type $F_n$

(ii) + [Belk-Zaremsky] $\Rightarrow$ Thm B

# III Homology of Thompson-like groups

**Thm** (Brown '92) $\tilde{H}_*(V_2;\mathbb{Q}) = 0$

**Thm** (Szymik–Wahl '19) $H_*(V_d;\mathbb{Z}) \cong H_*(\Omega^\infty_\circ \underbrace{M(\mathbb{Z}/_{d-1})}_{\text{Moore spectrum}};\mathbb{Z})$

↓

**Cor** $\tilde{H}_*(V_2;\mathbb{Z}) = 0$

$\tilde{H}_*(V_d;\mathbb{Q}) = 0 \quad \forall d \geq 2$

**Thm** (Li '22) $\forall k \geq 1$, $\left.\begin{array}{l}\tilde{H}_*(kV_2;\mathbb{Z}) = 0 \\ \tilde{H}_*(kV_d;\mathbb{Q}) = 0\end{array}\right\}$ Re-proven by [KLMMS'24]

Kupers
Lemann
Malkiewich
Miller
Sroka

**Methods:**

algebraic K-theory & homological stability

↓

[SW] of a category of "Cantor algebras"
[Li] of étale groupoids → We use this approach for Thm C.
[KLMMS] of a category of "scissors congruences"

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Work in progress (P.–Wu '26+)

$H_*(V_d(G);\mathbb{Z}) \cong H_*(\Omega^\infty_\circ \operatorname{hocofib}(\Sigma^\infty BG_+ \xrightarrow{d-1} \Sigma^\infty BG_+);\mathbb{Z})$

↳ acyclicity of $V_2(G)$

$\mathbb{Q}$-acyclicity of $V_d(G)$

$H_*(V_d(G);\mathbb{Z})$ depends only on $H_*(G;\mathbb{Z})$ as a coalgebra

$H_1(V_d(G);\mathbb{Z}) \cong \left\{\begin{array}{ll} G^{ab}/(d-1) & d \text{ even} \\ G^{ab}/(d-1) \oplus \mathbb{Z}/2 & d \text{ odd}\end{array}\right\}$ Can also be computed "by hand".

**Def**

Étale groupoid :

$$\begin{tikzcd} G^{(0)} \arrow[r, "id"] & G \arrow[loop above, "inv."] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G^{(0)} \end{tikzcd} \qquad \begin{tikzcd} G \times_{G^{(0)}} G \arrow[d] \\ G \end{tikzcd}$$

continuous

$s, t$ are local homeomorphisms

**Examples**

- discrete groups
- $\mathcal{V}_2$ "one-sided full shift"

$$\mathcal{V}_2^{(0)} = \{0,1\}^{\mathbb{N}} = X \qquad X \xrightarrow{T} X$$
$$(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$$

$$\mathcal{V}_2 \subseteq X \times \mathbb{Z} \times X$$

$$\{(x, n, y) : \exists\, \ell, m \geq 0 : \begin{array}{l} n = \ell - m \\ T^{\ell}(x) = T^{m}(y) \end{array}\}$$

**Topological full group**

$F(G)$ = group of homeo of of $G^{(0)}$ that locally look like elements of $G$

**Ex** $F(\mathcal{V}_2) \cong V_2$

**Groupoid homology** [Crainic – Moerdijk '99]

**Ex** $H_*(\mathcal{V}_2) = 0$ — exercise using excision

More generally

$(d-1)$-torsion

$$H_*(\mathcal{V}_d; A) \cong \begin{cases} A/(d-1) & * = 0 \\ A_{(d-1)} & * = 1 \\ 0 & * \geq 2 \end{cases}$$

[Matui '12]

**Construction** (Li '22)

$G$ étale groupoid

$G^{(0)}$ locally compact Hausdorff
totally disconnected

$\longrightarrow$ symmetric monoidal category $\mathcal{B}_G$

$\longrightarrow$ spectrum $K\mathcal{B}_G$

**Theorem** (Li '22)

if $G$ is purely infinite minimal — $\forall$ cpt open $U, V \subseteq G^{(0)}$, $V \neq \emptyset$

$G^{(0)}$ has no isolated points — $\exists$ cpt open bisection $\sigma \subseteq G$ with $s(\sigma) = U$, $r(\sigma) \subseteq V$.

then $H_*(G) \cong \widetilde{H}_*(K\mathcal{B}_G)$

[Crainic-Moerdijk]

$$H_*(F(G)) \cong H_*(\Omega_0^\infty K\mathcal{B}_G)$$

group homology

hom. stab^y hidden in proof

**Coro** (via Hurewicz Thm)

$H_*(G) = 0 \implies F(G)$ is acyclic.

**Coro** (via Milnor-Moore Thm)

$H_*(G;\mathbb{Q})$ determines $H_*(F(G);\mathbb{Q})$

**Example**

$F(\mathcal{V}_2) \cong V_2$ is acyclic.

## Thm C(ii)

For any subgroup $G \subseteq \mathrm{Sym}(S)$, $SV_2^G$ is acyclic.

## Sketch of proof

Let $$S\mathcal{V}_2 = \begin{cases} \text{obj} = \prod_S \mathcal{V}_2^{(0)} \\ \text{mor} = \left\{ (x_s) \in \prod_S \mathcal{V}_2 \;\middle|\; \begin{array}{l} x_s \in \mathcal{V}_2^{(0)} \text{ for all but} \\ \text{fin. many } s \in S \end{array} \right\} \end{cases}$$

⚠ Do not give this the subspace topology induced from $\prod_S \mathcal{V}_2$. Instead: "<u>restricted product topology</u>".

Sim. to topology on the ring of adèles in algebraic number theory.

strictly finer whenever $S$ is infinite

Basis given by $\prod_{s \in S} U_s$ for $U_s \subseteq \mathcal{V}_2$ open with $U_s = \mathcal{V}_2^{(0)}$ for all but fin. many $s$

The action $G \curvearrowright S$ induces $G \curvearrowright S\mathcal{V}_2$

$$\longrightarrow S\mathcal{V}_2 \rtimes G$$

**Thm** (P.-Wu '25) $F(S\mathcal{V}_2 \rtimes G) \cong SV_2^G$

This would be false if we used the subspace topology!

**Thm** [Matui '12] $\exists$ spectral sequence

$$H_*(G; H_*(S\mathcal{V}_2)) \Rightarrow H_*(S\mathcal{V}_2 \rtimes G)$$

**Lemma** Let $T = S \setminus \{s_0\}$. Then $S\mathcal{V}_2 \cong \mathcal{V}_2 \times T\mathcal{V}_2$.

- $H_*(\mathcal{V}_2) = 0$
- Künneth $\longrightarrow H_*(S\mathcal{V}_2) = 0$
- Matui $\longrightarrow H_*(S\mathcal{V}_2 \rtimes G) = 0$
- Li $\longrightarrow F(S\mathcal{V}_2 \rtimes G)$ is acyclic.

$$\begin{array}{c} F(S\mathcal{V}_2 \rtimes G) \\ \cong \\ SV_2^G \end{array}$$

$\square$

# Extras

**Proof of Thm C(i):**

- $F(\mathcal{V}_2 \times G) \cong V_2(G)$
- Künneth $\rightsquigarrow$ $H_*(\mathcal{V}_2 \times G) = 0$
- Li $\rightsquigarrow$ $F(\mathcal{V}_2 \times G)$ acyclic.

## Speculation about $\mathrm{br}V_d$

Replace $\Sigma_n$ with $B_n$ in the definition of $V_d$

**Q:** What is $H_*(\mathrm{br}V_d)$?

maybe define braided topological full groups ??

**Note:**

- $\mathrm{br}V_d$ is not a topological full group
- methods of Szymik–Wahl also do not apply
  — they assume a symmetric monoidal structure

**Guess:** $H_*(\Omega_0 \,\mathrm{hofib}(S^2 \xrightarrow{d-1} S^2))$

$\Downarrow$

$$H_1(\mathrm{br}V_d) \cong \pi_2 \,\mathrm{hofib}(S^2 \xrightarrow{d-1} S^2) \cong \mathbb{Z}/(d-1)$$

Note:

- $\mathrm{br}V_d \twoheadrightarrow \mathbb{Z}/(d-1)$ — writhe mod $d-1$
- $H_1(V_d) \cong \mathbb{Z}/(d-1) \otimes \mathbb{Z}/2$