# On the homology of twisted Brin-Thompson groups and Röver-Nekrashevych groups

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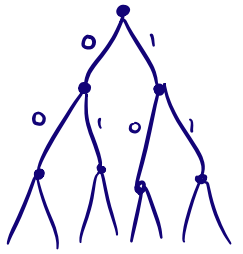
(joint with Xiaolei Wu)

## ① Thompson's group V

$X = \{0,1\}$

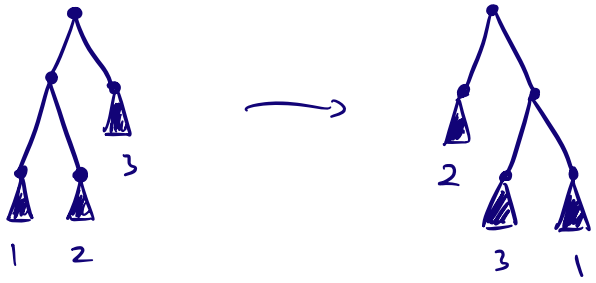
$X^* =$ words on $X$

$= T_2$




$\partial T_2 = X^{\mathbb{N}} =$ $\infty$ words on $X \cong \mathcal{C}$

**Def** $V \subset \mathrm{Homeo}(\partial T_2)$ consists of:




Thompson ('65) fin. presented, simple

Higman ('74) $V_d$ $(d \geq 2)$ $X = \{0,1,\dots,d-1\}$

Brown ('92) $\tilde{H}_*(V_2;\mathbb{Q}) = 0$

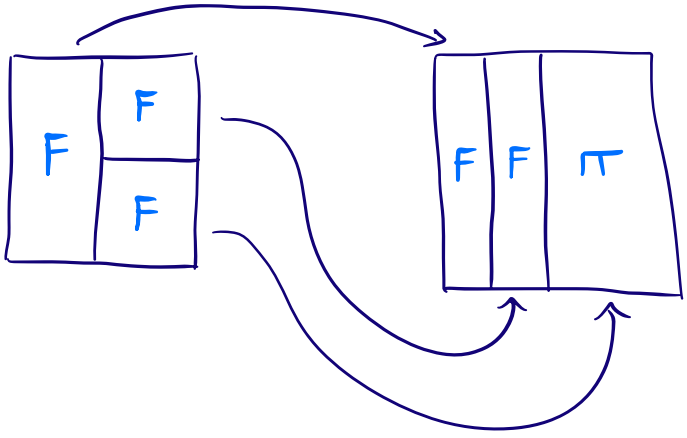
Szymik-Wahl ('19) $H_*(V_d) = H_*(\Omega^\infty_\circ M(\mathbb{Z}/(d-1)))$

## (2) Twisted Brin-Thompson groups

[Brin '04]
[Belk-Zaremsky '20]

Fix set $S$ and $G \subseteq \mathrm{Sym}(S)$

**Def** $SV_d^G \subset \mathrm{Homeo}(\mathcal{C}^S)$ consists of:




- cut into fin. many $d$-adic bricks
- rearrange & rescale (side lengths $\frac{1}{d^k}$)
- permute coordinates by $g \in G$

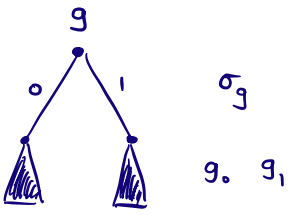
**Belk-Zaremsky ('20)**
- $SV_2^G$ simple
- examples of type $F_{n-1}$ but not type $F_n$

(cf. [Skipper-Witzel-Zaremsky '17])

**Thm A** (P.-Wu '25) Every $SV_2^G$ is acyclic

## ③ Röver - Nekrashevych groups

$X = \{0, 1, \dots, d-1\}$




$$Aut(T_d) \xrightarrow[\cong]{\phi} Sym(X) \wr Aut(T_d)$$

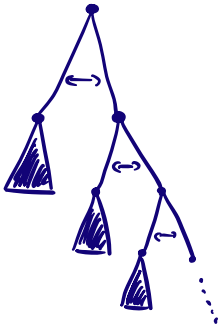
**Def** $\Gamma \subseteq Aut(T_d)$ **self-similar** if

$$\phi(\Gamma) \subseteq Sym(X) \wr \Gamma^X$$

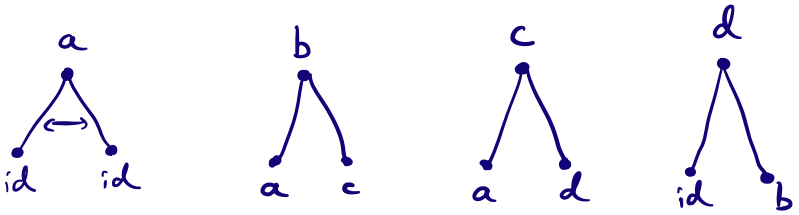
**Ex** $(d=2)$

- $\langle \text{reflection} \rangle \cong \mathbb{Z}/2$ $\qquad \phi(r) = \big((01); (r, r)\big)$
- $\langle \text{odometer} \rangle \cong \mathbb{Z}$ $\qquad \phi(a) = \big((01); (id, a)\big)$

  $\langle \text{odometer} \rangle = \langle a \rangle$

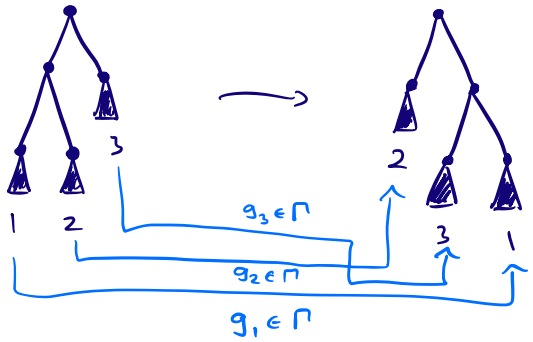


- Grigorchuk $= \langle a, b, c, d \rangle$ ('80)



**Def** [Nekrashevych '05]

$V_d(\Gamma) \subset \mathrm{Homeo}(\partial T_d)$ consists of:




**Def** An abstract recursion is

$$G \xrightarrow{\phi} \mathrm{Sym}(X) \ltimes G^X$$

**Rmk**

$$G \xrightarrow{\alpha} \mathrm{Aut}(T_d)$$

$\Gamma = \alpha(G)$ is self-similar

**Ex**

$$G = \mathbb{Z}/2 * (\mathbb{Z}/2)^2 \xrightarrow{\alpha} \Gamma \subset \mathrm{Aut}(T_d)$$

$a$ $\quad$ $b, c, d$

$\Gamma$ = Grig.

↑ not finitely presentable

Any abstract recursion defines

$$V_d(G, \phi) \xrightarrow{\pi} V_d(\Gamma) \subset \mathrm{Homeo}(\partial T_d)$$

[Miller - Steinberg '24]

$\phi$ "loosely faithful" $\Rightarrow$ $\pi$ isomorphism

↑ true for $\langle a \rangle * \langle b,c,d \rangle$

**Thm B** (P. - Wu '26)

$$H_*(V_d(G,\phi)) \cong H_*(\Omega^\infty_\circ M_\phi)$$

↑ compatible spectrum

**Ex**

| $V_d(G,\phi)$ | $M_\phi$ |
|---|---|
| $V_2$ (reflection) | $*$ |
| $V_2$ (odometer) | $\Sigma^\infty(S^1 \vee S^2)$ |
| Röver $=$ $V_2$ (Grigorchuk) | $(\Sigma^\infty BA_4)_{(2)} \wedge (S^0 \vee S^1)$ |

$V_2$ (reflection) $\cong V_2$ [Bleak, Donoven, Jonušas '17]

Röver rationally acyclic [Miller - Steinberg '24]

## ④ Étale groupoids

**Def** ↓ = topological groupoid $\mathcal{G}$

$s, t : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}$ local homeo.

ample $\longleftrightarrow$ étale

$\mathcal{G}^{(0)}$ compact, Hausdorff, totally disconnected

$\mathcal{G}$ ample gpd $\rightsquigarrow$ $F(\mathcal{G}) \subset \mathrm{Homeo}(\mathcal{G}^{(0)})$

$$= \left\{ \text{cpt open } U \subseteq \mathcal{G} \;\middle|\; \mathcal{G}^{(0)} \xleftarrow[\cong]{t|_U} U \xrightarrow{s|_U} \mathcal{G}^{(0)} \right\}$$

topological full group

**Ex** • $X = \{0, 1, \dots, d-1\}$

$\mathcal{V}_d \subset X^{\mathbb{N}} \times \mathbb{Z} \times X^{\mathbb{N}}$ consists of $(w, n, w')$ :

$\exists\, \ell, m \geq 0 : \begin{cases} w_{\geq \ell} = w'_{\geq m} \\ n = \ell - m \end{cases}$

$F(\mathcal{V}_d) = V_d$

• [Nekrashevych] $F(\mathcal{G}_\phi) = V_d(G, \phi)$

**Thm** [Li '22] (mild cond's on $\mathcal{G}$) $\mathcal{G} \rightsquigarrow \mathbb{K}(\mathcal{G})$ connective spectrum

ⓐ $H_*(\mathbb{K}(\mathcal{G})) \cong H_*(\mathcal{G})$ [Crainic - Moerdijk '99]

ⓑ $H_*(\Omega^\infty_0 \mathbb{K}(\mathcal{G})) \cong H_*(F(\mathcal{G}))$ also [KLMMS '25]

**Prop** [Matsui '12] $\quad H_*(\mathcal{V}_d) \cong \begin{cases} \mathbb{Z}/(d-1) & *=0 \\ 0 & *>0 \end{cases}$

**Corollary** $\quad H_*(V_d) \cong H_*(\Omega^\infty_0 M(\mathbb{Z}/(d-1)))$

(recovering [Szymik-Wahl])

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## (5) Proof of Thm A (optional) ($SV_2^G$ acyclic)

Build $S\mathcal{V}_2$ = restricted product of $S$ copies of $\mathcal{V}_2$ relative to $\mathcal{V}_2^{(0)}$

$\rightsquigarrow S\mathcal{V}_2 \rtimes G$

**Prop** $\quad F(S\mathcal{V}_2 \rtimes G) \cong SV_2^G$

**Swindle**: $\quad S\mathcal{V}_2 \cong \mathcal{V}_2 \times (S \setminus \{s\})\mathcal{V}_2$

Künneth $\Rightarrow$ $H_*(S\mathcal{V}_2) = 0$

spec. seq. [Matsui] $\Rightarrow$ $H_*(S\mathcal{V}_2 \rtimes G) = 0$

[Li] $\Rightarrow$ $SV_2^G$ acyclic. $\square$

# (6) Proof of Thm B

## Key idea

For fixed $G$,

$$\left\{ G \xrightarrow[\phi]{\text{abstract recursions}} \mathrm{Sym}(X) \ltimes G^X \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{sym. mon. functors} \\ \mathrm{Fin}_G \xrightarrow[\Phi]{} \mathrm{Fin}_G \end{array} \right\}$$

finite sets
$G$-labelled bijections

$$\Rightarrow \quad \begin{array}{ccc} K(\mathrm{Fin}_G) & \xrightarrow{K(\Phi)} & K(\mathrm{Fin}_G) \\ \text{IS} & & \text{IS} \\ \Sigma^\infty_+ BG & \longrightarrow & \Sigma^\infty_+ BG \end{array}$$

Barratt - Priddy - Quillen

explicit formula (later)

**Thm B'**: $K(\mathcal{G}_\phi) \simeq \mathrm{hocofib}\left(K(\Phi) - \mathrm{id}\right) =: M_\phi$

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**Rmks** (1) + [Li] (b) $\Rightarrow$ Thm B $\quad \left( H_*(V_d(G,\phi)) \cong H_*(\Omega^\infty_0 M_\phi) \right)$

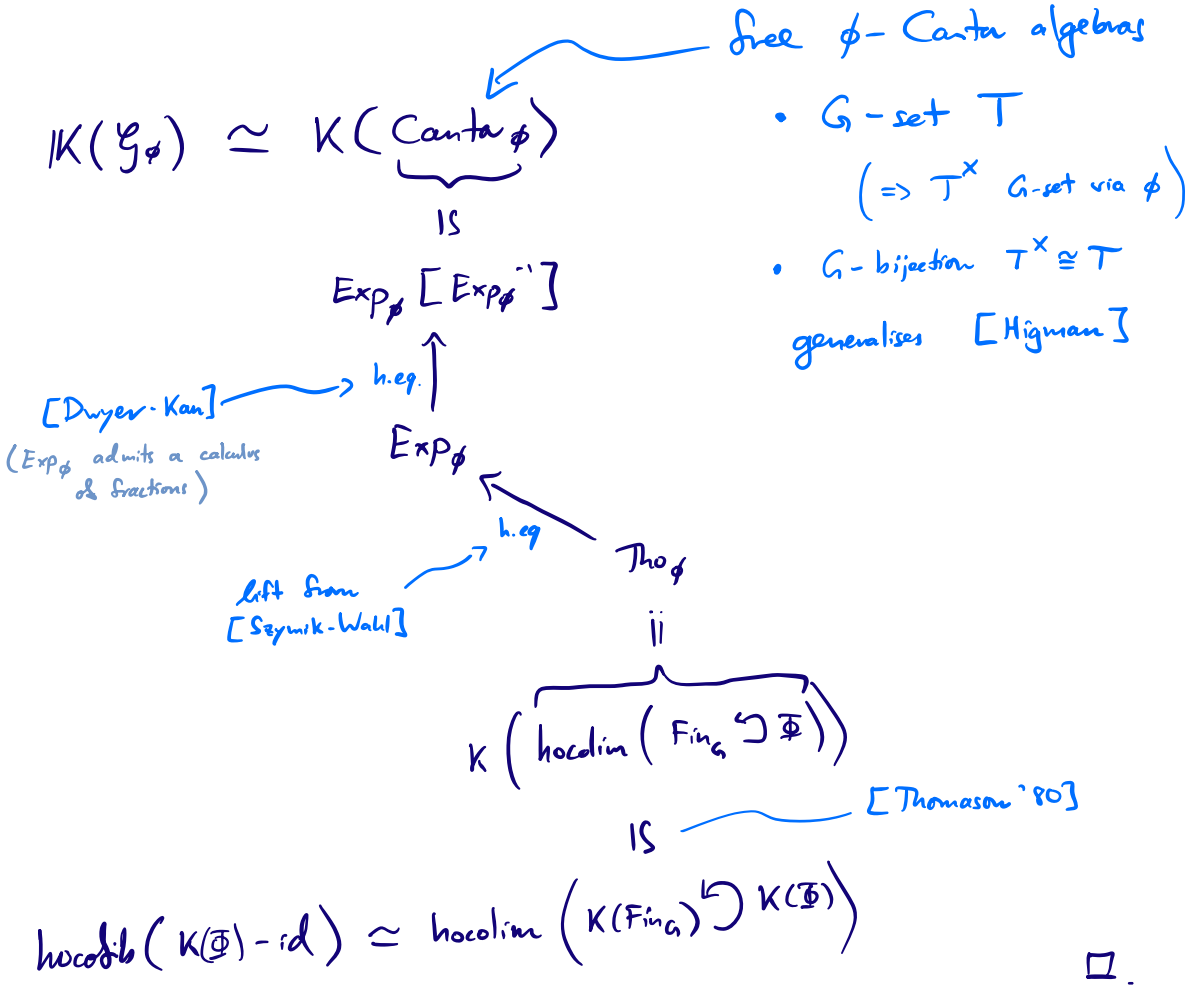
(2) + [Li] (a) $\Rightarrow$ LES:

$$\cdots \longrightarrow H_i(G) \xrightarrow{\Phi_i - \mathrm{id}} H_i(G) \longrightarrow H_i(\mathcal{G}_\phi) \longrightarrow \cdots$$

[Miller - Steinberg '24]

This is what made us conjecture Thm B'

## Idea of proof (adapting [Szymik - Wahl])



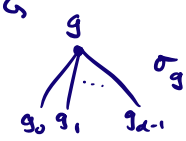

□.

# (7) Examples

**Prop** Suppose $G$ acts transitively on $X = \{0, 1, \dots, d-1\}$

$$G_0 = \mathrm{stab}_G(0) \xrightarrow{s_0} G$$

$$g \longmapsto \sigma_g \longmapsto g_0$$




$$\Sigma^\infty_+ BG \xrightarrow{tr} \Sigma^\infty_+ BG_0 \xrightarrow{(s_0)_*} \Sigma^\infty_+ BG$$

$K(\Phi)$

## (1) $V_2$ (reflection)

$G = \mathbb{Z}/2$

$G_0 = *$

Kahn-Priddy map $\lambda$

$$\mathbb{S} \vee \Sigma^\infty BG \xrightarrow{\begin{pmatrix} 2 & \cdot \\ 0 & 0 \end{pmatrix}} \mathbb{S} \vee \Sigma^\infty BG$$

$$M_\phi = \mathrm{hocofib}\begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix} \simeq *$$

## (2) $V_2$ (odometer)

$G = \mathbb{Z} = \langle a \rangle$

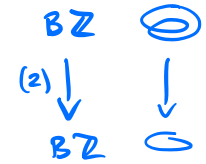
$G_0 = \langle a^2 \rangle \xrightarrow[\cong]{s_0} \langle a \rangle = G$

$a \longrightarrow ((01); (1, a))$

$a^2 \longrightarrow (id; (a, a))$

$$\mathbb{S} \vee \Sigma^\infty B\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 & \cdot \\ 0 & 1 \end{pmatrix}} \mathbb{S} \vee \Sigma^\infty B\mathbb{Z}$$

Hopf map $\eta$




$$M_\phi = \mathrm{hocofib}\begin{pmatrix} 1 & \eta \\ 0 & 0 \end{pmatrix} = \mathrm{hocofib}\left(\Sigma\mathbb{S} \xrightarrow{0} \Sigma\mathbb{S}\right)$$

$$= \Sigma\mathbb{S} \vee \Sigma^2\mathbb{S} = \Sigma^\infty(S^1 \vee S^2).$$

$$H_*(V_2(\text{odometer})) \cong H_*(\Omega^\infty_0 \Sigma^\infty(S^1 \vee S^2))$$
$$= H_*(\Omega S^1 \times \Omega S^2)$$

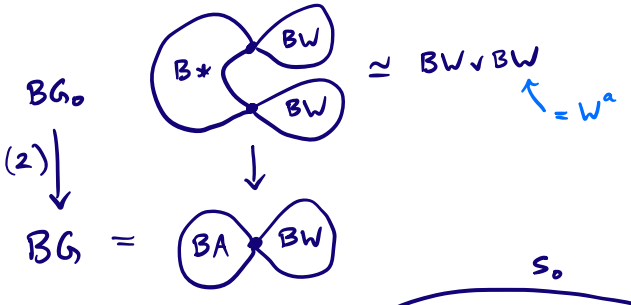
| $*$ | $H_*(\Omega S^1)$ | $H_*(\Omega S^2)$ | $H_*(V_2(\text{od}))$ |
|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ |
| 3 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}/2$ |
| 4 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ |

(3) $V_2(G_{\text{rig}}) \cong V_2(G_1, \phi)$ (optional)

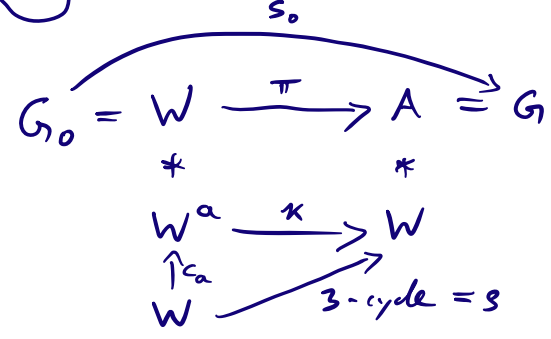
$$G_1 = \overbrace{\mathbb{Z}/2}^{A} * \overbrace{(\mathbb{Z}/2)^2}^{W}$$
$$a \qquad b, c, d$$

$G_0 = \ker(G_1 \twoheadrightarrow A)$

$a \longrightarrow ((01); id, id)$
$b \longrightarrow (id; a, c)$
$c \longrightarrow (id; a, d)$
$d \longrightarrow (id; id, b)$




$aba \longrightarrow (id; c, a)$
$aca \longrightarrow (id; d, a)$
$ada \longrightarrow (id; b, id)$



$$S \vee \Sigma^\infty BA \vee \Sigma^\infty BW$$

transfer $\quad \begin{pmatrix} 2 & \lambda & 0 \\ 0 & 0 & id \\ 0 & 0 & c_a \end{pmatrix}$

$$S \vee \Sigma^\infty BW \vee \Sigma^\infty BW^a$$

$(\sigma_\circ)_* \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & x \end{pmatrix}$

$$S \vee \Sigma^\infty BA \vee \Sigma^\infty BW$$

$$K(\Phi) = \begin{pmatrix} 2 & \lambda & 0 \\ 0 & 0 & \tau \\ 0 & 0 & s \end{pmatrix}$$

invertible

$$K(\Phi) - id = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & -1 & \tau \\ 0 & 0 & s - id \end{pmatrix}$$

$$M_\phi = \text{hocofib}\left(K(\Phi) - id\right)$$

$$= \text{hocofib}\left(s - id : \Sigma^\infty BW \to \Sigma^\infty BW\right)$$

Split $\Sigma^\infty BW$ further:

$$e = \tfrac{1}{3}\left(id + s + s^2\right) : \Sigma^\infty BW \to \Sigma^\infty BW$$

$$e^2 = e$$

$$\Sigma^\infty BW = \left(\Sigma^\infty BW\right)_e \vee \left(\Sigma^\infty BW\right)_{1-e}$$

$g - id$ is $0$ on $(\Sigma^\infty BW)_e$

$\cong$ on $(\Sigma^\infty BW)_{1-e}$

$$\Rightarrow \quad M_\phi \simeq (S^0 \vee S^1) \wedge (\Sigma^\infty BW)_e$$

**Fact** [Mitchell-Priddy '84]

$$(\Sigma^\infty BW)_e \simeq \left(\Sigma^\infty B\left(\underbrace{W \times \langle s \rangle}_{\cong S_4}\right)\right)_{(2)}$$

$$\Rightarrow \quad M_\phi \simeq (S^0 \vee S^1) \wedge (\Sigma^\infty BA_4)_{(2)}$$

$$\Omega^\infty_0 M_\phi \simeq (QBA_4)_{(2)} \times (Q\Sigma BA_4)_{(2)}$$

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**Rmk** $N_d = V_d(\mathrm{Aut}(T_d))$ [Nevetin]

$$H_*(N_d) \cong H_*\left(\Omega^\infty_0 \mathrm{hocofib}\left(\underbrace{\Sigma^\infty_+ B\mathrm{Aut}(T_d)}_{\text{v. complicated!}} \circlearrowright K(\mathbb{D}) - id\right)\right)$$

**known:** $H_1(N_d) = 0$ [Kapoudjian '99]

$H_2(N_2) \twoheadrightarrow \mathbb{Z}/2$ [Kapoudjian-Sergiescu '04]