

## Description of previous work

### Summary:

The central objects of study in my previous research are *moduli spaces of submanifolds*, focusing especially on their *homology* and their *fundamental groups*, for example surface braid groups and loop (=welded) braid groups.

On the homological side, I have proven *stability* results for the (possibly twisted) homology of moduli spaces, including oriented configuration spaces, moduli spaces of manifolds with singularities and configuration-section spaces (which have connections to number theory). I have also identified the *stable homology* of oriented configuration spaces, along the way proving a twisted-homology refinement of the classical *group-completion theorem*.

On the group-theoretic side, I have studied the *lower central series* of surface, virtual and welded braid groups, and constructed new *homological representations* of motion groups and mapping class groups.

Introduction	1
Overview	2
Diagram	3
1. Configuration spaces with non-local structure	4
2. Twisted homological stability	5
3. Configurations on closed manifolds	5
4. Moduli spaces of disconnected submanifolds	6
5. Homological representations of motion groups and mapping class groups	9
6. Higher crossing diagrams in knot theory	10
7. Configuration-section spaces	11
8. Point-pushing actions	11
9. Motivic homological stability	12
10. Lawrence-Bigelow representations	13
11. Mapping class group representations via Heisenberg homology	14
12. Burau representations of loop braid groups	14
13. Lower central series of surface, virtual and welded braid groups	14
References	15

### Introduction

My research interests lie in the area of topology, including *algebraic topology* and *low-dimensional topology*, and interactions between the two. The two main themes of my previous research work are concerned with studying:

- The homology of *moduli spaces of submanifolds* of an ambient manifold (for example configuration spaces or spaces of links in  $\mathbb{R}^3$ ), as well as *diffeomorphism groups* of manifolds. This also includes *motivic cohomology* of configuration spaces on smooth algebraic varieties, and *configuration-section spaces*, where the complement of a configuration is equipped with a “field with singularities”.
- The fundamental groups of such moduli spaces (for example, *surface braid groups*, *loop braid groups* and *mapping class groups*), in particular their representation theory and their lower central series.

As well as being central objects in topology, these moduli spaces have important applications in many other domains. A key example is the *Mumford conjecture* [Mum83] – a statement from algebraic geometry about Riemann’s moduli spaces of algebraic curves – which was proven by Madsen and Weiss [MW07], using purely topological methods. More recently, homological stability for certain *configuration-mapping spaces* was proven by Ellenberg, Venkatesh and Westerland [EVW16], from which they deduced a result in analytic number theory: an asymptotic version of the *Cohen-Lenstra conjecture* for function fields. (See also §7 below.)

## Overview

Concerning the *homology of moduli spaces of submanifolds*:

- (a) I proved [Pal13] in my thesis that the *oriented configuration spaces*  $C_n^+(M)$  on open, connected manifolds  $M$  are homologically stable: a point in this space is a configuration of points in  $M$ , equipped with the *non-local* data of an ordering modulo the action of the alternating group. Together with J. Miller [MP15b], we then identified a space modelling the limiting homology of these spaces, as the number of particles goes to infinity, completing step (ii) of the above programme, and lifting the classical result of D. McDuff [McD75] and G. Segal [Seg73; Seg79] (which concerns *unordered* configuration spaces). Along the way, we also generalised their *group-completion theorem* [MS76] to the setting of twisted-homology-equivalences [MP15a]. — See §1 for more details.
- (b) The unordered configuration spaces  $C_n(M)$  on a *closed*, connected manifold  $M$  are known *not* to stabilise in general, and represent a much more difficult and complicated case. Together with F. Cantero [CP15], we proved that there are nevertheless some more subtle *stability and periodicity phenomena* in the homology of configuration spaces on closed manifolds. For example, the homology of  $C_n(M)$  with coefficients in  $\mathbb{Z}[\frac{1}{2}]$  stabilises for  $\dim(M)$  odd, whereas, for  $\dim(M)$  even, their mod- $p$  homology is unstable but *periodic* in a stable range (with an explicit period depending only on  $p$  and  $\chi(M)$ ). — See §3 for more details of these results.
- (c) Another direction in which I have taken these ideas, which opens the door to a wide variety of interesting new examples, is to generalise configuration spaces to *moduli spaces of disconnected submanifolds*, where point particles are replaced with embedded submanifolds of a specified diffeomorphism type and isotopy class (which may be parametrised, oriented, unoriented, etc.). Under a certain restriction on the codimension, I have proven [Pal21] that these moduli spaces are also *homologically stable* as the number of components of the submanifold goes to infinity. As a corollary, I also proved [Pal18a] homological stability for:
  - the diffeomorphism groups of *manifolds with conical singularities*, with respect to the number of singularities of a given type,
  - the *symmetric diffeomorphism groups* of any sequence of manifolds obtained by iterating the operation of “parametrised connected sum”, an operation which generalises both ordinary connected sum and surgery.

See §4 for more details of these results.

- (d) *Configuration-section spaces* model point-particles moving in a background field that is undefined (and may be singular) at the particles themselves. Particular examples of these include the classical *Hurwitz spaces*, which classify branched coverings of the 2-disc. In joint work with U. Tillmann [PT21], we proved homological stability for configuration-section spaces, subject to a condition on the permitted “charges” of the particles. This has some intersection with, but is mostly complementary to, a homological stability result of Ellenberg, Venkatesh and Westerland [EVW16], which they used to deduce an asymptotic version of the *Cohen-Lenstra conjecture* for function fields. See §7 for more details.
- (e) Related to the previous point, in another joint work with U. Tillmann [PT20], we studied in detail the *point-pushing action* for manifolds with boundary  $M$ , namely the monodromy of the universal bundle  $U_k(M) \rightarrow C_k(\mathring{M})$  whose fibre over a configuration  $z$  in the interior of  $M$  is the complement  $M \setminus z$ , obtaining explicit formulas when the dimension of  $M$  is at least 3, and identifying precisely the kernel of the action. As a corollary, we obtained explicit formulas for the natural monodromy action of  $\pi_1(C_k(M))$  on the fibres of configuration-mapping spaces on  $M$ . See §8 for more details.
- (f) Together with G. Horel [HP20], we have proven an analogue of classical homological stability for the *motivic* and *étale motivic* cohomology of configuration schemes on a smooth scheme  $X$ . See §9.

Concerning *motion groups* (the fundamental groups of moduli spaces of submanifolds) and *mapping class groups*:

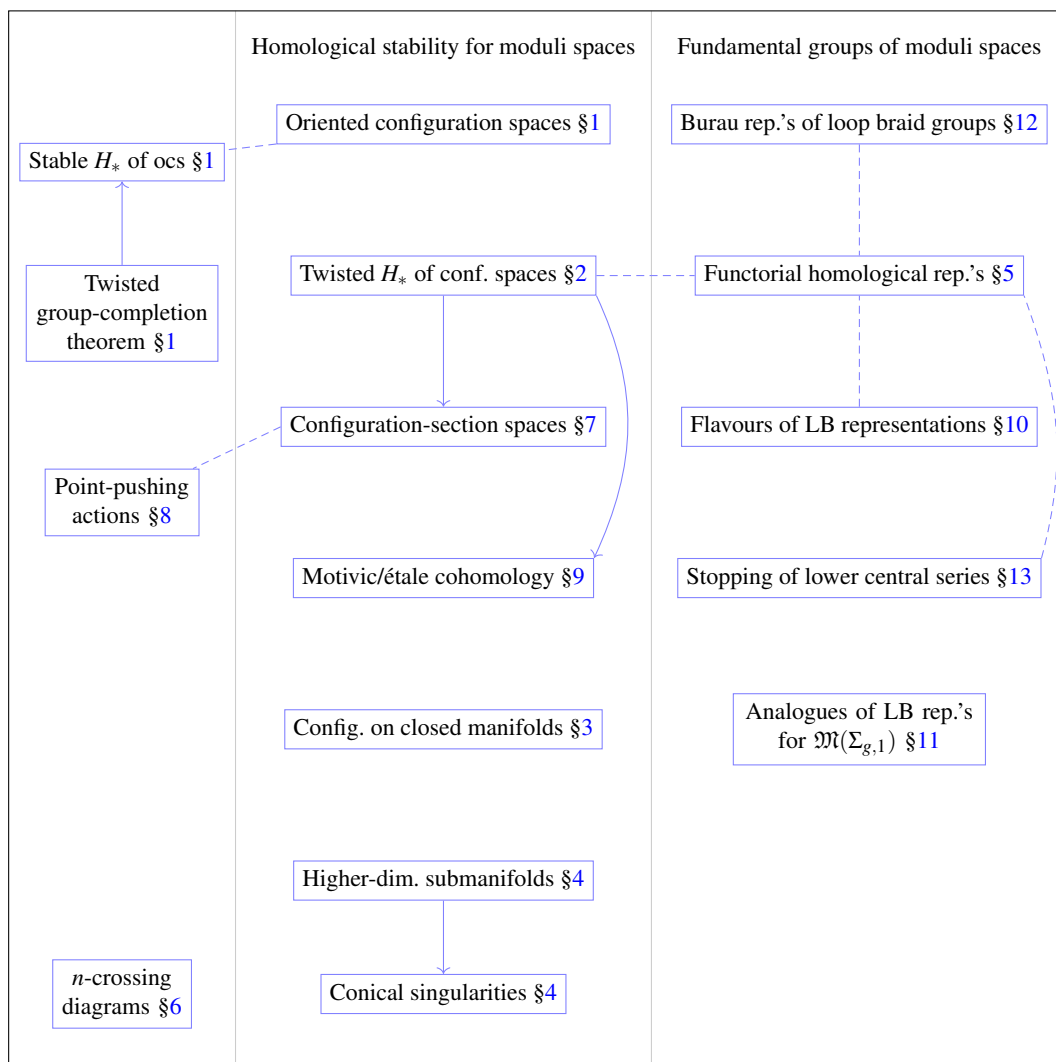
- (g) I have proven [Pal18b] that the unordered configuration spaces  $C_n(M)$  are homologically stable with coefficients in any *polynomial twisted coefficient system* (this in particular includes a choice of representation of  $\pi_1(C_n(M))$  for each  $n$ ). — See §2 for more details.
- (h) In joint work with A. Soulié [PS19], we have set up a general topological construction of representations of motion groups (fundamental groups of moduli spaces of submanifolds) and mapping class groups. This generalises and unifies several known constructions, including the *Lawrence-Bigelow representations* and the *Long-Moody construction*. It also produces many new families of representations,

in particular for the loop-braid groups, which are studied in more detail in [PS21]. See §5 and §12 for more details.

- (i) The *Lawrence-Bigelow representations*, and other representations of mapping class groups of surfaces, come in many different “flavours”, depending on which part of the boundary of the surface is removed, and which type of homology (such as locally finite homology, ordinary homology relative to the boundary, etc.) is considered. In joint work with C. Anghel [AP20], we have investigated the fundamental relationships between these different flavours, establishing various non-degenerate pairings and embeddings between them. See §10 for details.
- (j) In joint work with C. Blanchet and A. Shaukat [BPS21], we have constructed *analogues of the Lawrence-Bigelow representations for the mapping class groups of the surfaces  $\Sigma_{g,1}$* , as well as for their Torelli, Chillingworth and Morita subgroups. See §11 for details.
- (k) In joint work with J. Darné and A. Soulié [DPS22], we study the *lower central series of surface braid groups, virtual braid groups and welded (=loop) braid groups*, as well as their finite-index subgroups preserving a given partition of the strands. A fundamental question for any group is whether and when its lower central series *stops*, to which we give a complete answer for these families of groups. See §13 for details.

In addition, I have also studied *multi-crossing diagrams* for links, in joint work with C. Adams and J. Hoste: our main result is the construction of a complete set of Reidemeister moves for *triple-crossing diagrams*. — See §6 for more details.

### Diagram



## 1. Configuration spaces with non-local structure

For a space  $M$ , the  $n$ th ordered configuration space  $\tilde{C}_n(M)$  is defined to be the subspace of  $M^n$  consisting of all  $n$ -tuples of pairwise distinct points in  $M$ . The symmetric group  $\mathfrak{S}_n$  acts on this space, and we define

$$C_n(M) = \tilde{C}_n(M)/\mathfrak{S}_n \quad \text{and} \quad C_n^+(M) = \tilde{C}_n(M)/A_n,$$

where  $A_n < \mathfrak{S}_n$  is the alternating group. These are called, respectively, the *unordered* configuration space on  $M$  and the *oriented* configuration space on  $M$ . It is a classical result, going back to McDuff [McD75] and Segal [Seg73; Seg79], that the sequence  $C_n(M)$ , when  $M$  is a connected, open manifold, is *homologically stable*. This means that, for each degree  $i$ , there are isomorphisms  $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$  once  $n$  is sufficiently large (depending on  $i$ ).

**Aside.** The condition that  $M$  is connected is clearly necessary, as one can see by considering  $H_0$ . On the other hand, the condition that  $M$  is open is not obviously necessary, and the situation in this case is much more subtle. This is the subject of another part of my previous work, see §3.

The ordered configuration spaces  $\tilde{C}_n(M)$  are not homologically stable: for example, the first homology of  $\tilde{C}_n(\mathbb{R}^2)$  is the abelianisation of the pure braid group, which is  $\mathbb{Z}^{\binom{n}{2}}$ . This raises the question of whether there is an intermediate covering space between  $\tilde{C}_n(M)$  and  $C_n(M)$  for which homological stability still holds. I proved in [Pal13] that the answer is positive for the oriented configuration spaces  $C_n^+(M)$ , which doubly cover the unordered configuration spaces  $C_n(M)$ :

**Theorem ([Pal13])** *The natural stabilisation map  $C_n^+(M) \rightarrow C_{n+1}^+(M)$  induces isomorphisms on homology in degrees  $* \leq \frac{n-5}{3}$  and surjections in degrees  $* \leq \frac{n-2}{3}$ .*

The theorem also holds more generally for *labelled configuration spaces*, where each point is equipped with a “label” in some fixed, path-connected parameter space  $X$ . The so-called *slope* of the stability range here is  $\frac{1}{3}$  (and one may calculate explicitly for certain  $M$  to see that this is sharp), in contrast to the slope of  $\frac{1}{2}$  that holds for the unordered configuration spaces.

**Note.** The integral homology of  $C_n^+(M)$  may be interpreted as a certain twisted homology group  $H_*(C_n(M); \mathbb{Z}[\mathbb{Z}/2])$ , where  $\pi_1(C_n(M))$  acts on the group ring of  $\mathbb{Z}/2$  via the natural projection  $\pi_1(C_n(M)) \rightarrow \mathfrak{S}_n$  followed by the sign homomorphism. This is an example of an *abelian twisted coefficient system* on  $C_n(M)$ , and is a precursor (in a special case) of the notion of *abelian homological stability*, which has been developed more recently by Randal-Williams and Wahl [RW17] and Krannich [Kra19]. See §2 for more twisted homological stability results.

This result leads to the question of whether one can identify the *stable homology* of the sequence  $\{C_n^+(M) \mid n \in \mathbb{N}\}$ , in other words the colimit  $\lim_{n \rightarrow \infty} H_*(C_n^+(M))$ , in terms of other well-understood spaces. In joint work with **Jeremy Miller**, we answered this question positively by lifting the classical *scanning map* [Seg73; McD75] to a homology equivalence between appropriate covering spaces.

- First, in our paper [MP15a], we generalise the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff’s *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients (more precisely to a setting where we consider homology with respect to all twisted coefficient systems in a fixed class  $\mathcal{C}$  that is closed under pullbacks).
- Using these tools, we proved in [MP15b] a new kind of scanning result, lifting the classical scanning map to covering spaces and showing that it remains a homology equivalence after doing so. This identifies the stable homology of oriented configuration spaces on  $M$  with the homology of an explicit double cover of the section space of a certain bundle over  $M$ :

**Theorem ([MP15b])** *Writing  $\dot{T}M \rightarrow M$  for the fibrewise one-point compactified tangent bundle of  $M$  and  $\Gamma_c(\dot{T}M \rightarrow M)_\circ$  for its space of compactly-supported sections of degree zero, we have:*

$$\lim_{n \rightarrow \infty} H_*(C_n^+(M)) \cong H_*(\tilde{\Gamma}_c(\dot{T}M \rightarrow M)_\circ), \quad (1)$$

for a certain explicit double cover  $\tilde{\Gamma}_c(\dot{T}M \rightarrow M)_\circ \rightarrow \Gamma_c(\dot{T}M \rightarrow M)_\circ$ .

This double cover may be defined as the connected covering space of  $\Gamma_c = \Gamma_c(\dot{T}M \rightarrow M)_\circ$  corresponding to the projection

$$\pi_1(\Gamma_c) \rightarrow H_1(\Gamma_c) \cong H_1(C_2(M)) \rightarrow H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/(2).$$

Here, the isomorphism  $H_1(\Gamma_c) \cong H_1(C_2(M))$  arises from homological stability and the identification of the stable homology for *unordered* configuration spaces, and the map  $H_1(C_2(M)) \rightarrow H_1(C_2(\mathbb{R}^\infty))$  is induced by any embedding  $M \hookrightarrow \mathbb{R}^\infty$ . For example, when  $M = \mathbb{R}^\infty$  the right-hand side of (1) is the universal cover of (one component of) the infinite loop space  $\Omega^\infty S^\infty = QS^0$ . When  $M = \mathbb{R}^2$  it is the unique connected double cover of  $\Omega^2 S^3$ .

## 2. Twisted homological stability

As well as the notion of abelian homological stability mentioned in the previous section, another sense in which a sequence of spaces (or groups) can satisfy *twisted homological stability* is with respect to a so-called *polynomial twisted coefficient system*. This consists of a choice of local coefficient system on each space  $X_n$ , together with additional morphisms of local coefficient systems between them, organised into a functor  $\mathcal{C} \rightarrow \text{Ab}$ , where the automorphism groups of  $\mathcal{C}$  are the fundamental groups  $\pi_1(X_n)$ . In addition, the *degree* of this functor (defined using extra structure on  $\mathcal{C}$ ) is required to be finite.

Many families of groups  $G = \{G_n\}$  are known to be homologically stable in this sense (where we set  $X_n = BG_n$  in the above paragraph), for appropriately-defined categories  $\mathcal{C}_G$ , for example the symmetric groups  $\mathfrak{S}_n$ , braid groups  $\beta_n$ , general linear groups, automorphism groups of free groups  $\text{Aut}(F_n)$  and mapping class groups of surfaces and of 3-manifolds.<sup>1</sup> In [Pal18b], I proved the first such result for a sequence of *spaces*, namely the unordered configuration spaces  $C_n(M)$  on any connected, open manifold  $M$ . Note that, when  $\dim(M) \geq 3$ , these spaces are not aspherical (in contrast to the case of surfaces), so this does not reduce to a statement about the homology of their fundamental groups. When  $\dim(M) = 2$ , these configuration spaces are aspherical, so the result may be thought of as twisted homological stability for their fundamental groups, the surface braid groups  $B_n(M) = \pi_1(C_n(M))$ .

**Theorem ([Pal18b])** *Let  $M$  be an open, connected manifold and let  $T$  be a twisted coefficient system for  $\{C_n(M)\}$ . This includes in particular the data of a local coefficient system  $T_n$  for each  $C_n(M)$ , as well as a homomorphism*

$$H_*(C_n(M); T_n) \longrightarrow H_*(C_{n+1}(M); T_{n+1}).$$

*This homomorphism is split-injective in all degrees, and, if  $T$  has finite degree  $d$ , it is an isomorphism in the range  $* \leq \frac{n-d}{2}$ .*

This theorem also generalises to configuration spaces with labels in any path-connected space  $X$ , as in the previous section.

**Twisted coefficient systems.** In connection with these results, I have also explored in more depth [Pal17] the notion of *twisted coefficient system* (a.k.a. finite-degree or polynomial functor), and in particular the *degree* of a twisted coefficient system. The main results of [Pal17] are:

- A comparison and unification of various different notions of “finite-degree” functor  $\mathcal{C} \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is an abelian category and  $\mathcal{C}$  is a category with various kinds of additional structure.
- The development of a functorial construction of (*injective* or *partial*) *braid categories*, which were used in [Pal18b] as the domain of definition of twisted coefficient systems, and of finite-degree functors on such braid categories.

## 3. Configurations on closed manifolds

When  $M$  is closed, homological stability for the unordered configuration spaces  $C_n(M)$  is not true in general, for example one may calculate that  $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2n-2)$ , which does not stabilise as  $n \rightarrow \infty$ . Moreover, the stabilisation maps mentioned in the theorem in §1 do not exist, since these depend on adding a new configuration point in  $M$  “near infinity”. In joint work with **Federico Cantero** [CP15], we prove three main results which show that the homology of configuration spaces on closed manifolds exhibits a large amount of stability despite these issues.

<sup>1</sup> Symmetric groups: [Bet02]; braid groups: [CF13; RW17; Pal18b]; general linear groups: [Dwy80; Kal80]; automorphism groups of free groups: [RW17]; mapping class groups of surfaces: [Iva93; CM09; Bol12; RW17]; mapping class groups of 3-manifolds: [RW17]. Note that these are references for the proofs of *twisted* homological stability; in each case, homological stability with untwisted coefficients was known earlier.

(1) When the Euler characteristic of  $M$  is zero, we construct *replication maps*  $C_n(M) \rightarrow C_{\lambda n}(M)$  for any integer  $\lambda \geq 2$ , and prove that they induce homological stability after inverting  $\lambda$ :

**Theorem** ([CP15]) *These maps induce isomorphisms on  $H_i(-, \mathbb{Z}[\frac{1}{\lambda}])$  in the range  $2i \leq \lambda$ .*

**Note.** A construction related to our replication maps has also been used in the paper [Ber<sup>+</sup>06], in which they use something similar to a replication map in §3 to build a crossed simplicial group out of the configuration spaces on any given manifold  $M$  that admits a non-vanishing vector field.

(2) When the manifold  $M$  is odd-dimensional, the configuration spaces  $C_n(M)$  do in fact satisfy homological stability after inverting 2.

**Theorem** ([CP15]) *When  $\dim(M)$  is odd, there are isomorphisms*

$$H_i(C_n(M); \mathbb{Z}[\frac{1}{2}]) \cong H_i(C_{n+1}(M); \mathbb{Z}[\frac{1}{2}]) \quad \text{and} \quad H_i(C_n(M); \mathbb{Z}) \cong H_i(C_{n+2}(M); \mathbb{Z})$$

*in the range  $2i \leq n$ , induced by a zigzag of maps.*

This strengthens a result of [BM14].

(3) When the manifold  $M$  is even-dimensional, and  $\mathbb{F}$  is a field of characteristic 0 or 2, it is known by the work of many people [BCT89; ML88; Chu12; Ran13; BM14; Knu17] that homological stability holds for  $C_n(M)$  with coefficients in  $\mathbb{F}$ , even when  $M$  is closed. When  $\mathbb{F}$  has odd characteristic  $p$ , however, this is false, as one can see from the example of  $M = S^2$  mentioned above. In fact:

$$H_1(C_n(S^2); \mathbb{F}) \cong \begin{cases} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{cases} \quad \text{for } n \geq 2.$$

From this example we see that the first homology of  $C_n(S^2)$  is not stable, but it is  $p$ -periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic  $\chi$  of  $M$  is non-zero. Write  $a = v_p(\chi)$  for the  $p$ -adic valuation of  $\chi$ , in other words  $\chi = p^a b$  with  $b$  coprime to  $p$ .

**Theorem** ([CP15]) *Suppose that  $\dim(M)$  is even. For each fixed  $i$ , the sequence*

$$H_i(C_n(M); \mathbb{F}) \quad \text{for } n \geq 2i \tag{2}$$

*is  $p^{a+1}$ -periodic and takes on at most  $a+2$  values. Moreover, if  $\chi \equiv 1 \pmod p$  then the above sequence is 1-periodic, i.e. homological stability holds with coefficients in  $\mathbb{F}$ .*

The  $p^{a+1}$ -periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of  $p$  depending on  $i$  rather than on  $\chi$ . This result was later improved by [KM16] to  $p$ -periodicity, independent of  $i$  or  $\chi$ . Combining this with (a slightly more precise statement of) our result, a corollary is that in fact the sequence (2) above takes on only two different values.

## 4. Moduli spaces of disconnected submanifolds

Instead of configurations of points in  $M$  (closed 0-dimensional submanifolds), one may consider configurations of closed submanifolds of  $M$  of higher dimension, which are diffeomorphic to the disjoint union of finitely many copies of a fixed (“model”) manifold  $L$ . In this setting, I proved [Pal21] that *moduli spaces of disconnected submanifolds* of this kind are also homologically stable as the number of components goes to infinity, just as in the classical setting of points [Seg73; McD75; Seg79] – under a certain hypothesis on the relative dimensions of the manifolds involved.

Let  $\bar{M}$  be a connected manifold with non-empty boundary and of dimension at least 2, and denote its interior by  $M$ . Also fix a closed manifold  $L$  and an embedding  $t_0: L \hookrightarrow \partial\bar{M}$ . Choose a self-embedding  $e: \bar{M} \hookrightarrow \bar{M}$  which is isotopic to the identity and such that  $e(t_0(L))$  is contained in the interior  $M \subset \bar{M}$ . We then obtain a sequence of pairwise-disjoint embeddings of  $L$  into  $M$  by defining  $t_n := e^n \circ t_0$  for  $n \geq 0$ .

Let  $nL = \{1, \dots, n\} \times L$  and write  $t_{1, \dots, n}: nL \hookrightarrow M$  for the embedding  $(i, x) \mapsto t_i(x)$ .

**Definition** Define  $C_{nL}(M)$  to be the path-component of

$$\text{Emb}(nL, M)/\text{Diff}(nL)$$

containing  $[t_{1, \dots, n}]$ . Here, the embedding space is given the Whitney topology and  $C_{nL}(M)$  the quotient topology. There is a natural *stabilisation map*

$$C_{nL}(M) \longrightarrow C_{(n+1)L}(M) \quad (3)$$

defined by adjoining the embedding  $t_0$  to a given embedding  $nL \hookrightarrow M$ , to obtain a new embedding  $(n+1)L \hookrightarrow \bar{M}$ , and then composing with the self-embedding  $e$ . In symbols, this may be written as  $[\phi] \mapsto [\phi^+]$ , where  $\phi^+(i, x) = e \circ \phi(i, x)$  for  $1 \leq i \leq n$  and  $\phi^+(n+1, x) = t_1(x)$ .

**Theorem ([Pal21])** Assume that the dimensions  $m = \dim(M)$  and  $\ell = \dim(L)$  satisfy

$$2\ell \leq m - 3. \quad (4)$$

Then (3) induces isomorphisms on homology in degrees  $*$   $\leq \frac{n-2}{2}$  and surjections in degrees  $*$   $\leq \frac{n}{2}$ .

This theorem may be extended further:

- There is a more general version of this setting, in which the submanifolds  $L \subset M$  are parametrised modulo a subgroup of  $\text{Diff}(L)$  and come equipped with labels in some bundle over  $\text{Emb}(L, M)$ . The theorem proved in [Pal21] includes this more general setting.
- This setting is compatible with the techniques of §1 above, so we also have homological stability for “oriented” (in the sense of §1) versions of these moduli spaces, in which the submanifolds  $L \subset M$  are ordered modulo even permutations.

**Applications to diffeomorphism groups.** In the sequel [Pal18a], I used homological stability for the moduli spaces  $C_{nL}(M)$  (and their more refined versions mentioned in the first point above) to prove homological stability for:

- Symmetric diffeomorphism groups, with respect to parametrised connected sum.
- Diffeomorphism groups of manifolds with conical singularities, with respect to the number of singularities.

**Definition** Given two embeddings  $L \hookrightarrow M$  and  $L \hookrightarrow Q$  with isomorphic normal bundles, one may cut out a tubular neighbourhood of each embedding and glue the resulting boundaries to obtain the *parametrised connected sum*  $M \#_L Q$ . If  $L$  is a point this corresponds to the ordinary connected sum of  $M$  and  $Q$ . Other examples include the following.

- If  $L \hookrightarrow Q$  is the canonical embedding  $S^k \hookrightarrow S^m$ , where  $k \leq m = \dim(M)$ , then  $M \#_L Q$  is the result of a *k-surgery* on  $M$ .
- If  $L \hookrightarrow Q$  is the embedding  $S^1 \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{T} \cup_{p/q} \mathbf{T} = L(p, q)$ , where  $\mathbf{T} = D^2 \times S^1$  denotes the solid torus, then  $M \#_L Q$  is the result of a *Dehn surgery of slope  $p/q$*  on the 3-manifold  $M$ .

If we now iterate the operation  $-\#_L Q$  many times, using a different copy of  $Q$  and a disjoint embedding  $L \hookrightarrow M$  each time (but always using the same copy of  $M$ ), we obtain a sequence

$$M \quad M \#_L Q \quad M \#_L Q \#_L Q \quad M \#_L Q \#_L Q \#_L Q \quad \dots \quad (5)$$

of manifolds, which we abbreviate to  $M \#_L^n Q$ , the  *$n$ th iterated parametrised connected sum*.

A diffeomorphism of  $M \#_L^n Q$  is called *symmetric* if it fixes the boundary of  $M$  and preserves the decomposition of  $M \#_L^n Q$  into pieces of the form  $M \setminus n\mathcal{T}(L)$  and  $Q \setminus \mathcal{T}(L)$ , where  $\mathcal{T}(L)$  is a tubular neighbourhood of  $L$  in  $M$  or  $Q$ . The corresponding subgroup

$$\Sigma\text{Diff}(M \#_L^n Q) \leq \text{Diff}(M \#_L^n Q)$$

is called the *symmetric diffeomorphism group* of  $M \#_L^n Q$ .<sup>2</sup>

<sup>2</sup> A mild technical condition has been elided from the definition of symmetric diffeomorphism group and in the statement of the theorem below, in order to simplify the discussion.

**Theorem ([Pal18a])** *If  $M$  is connected and has non-empty boundary and  $\dim(L) \leq \frac{1}{2}(\dim(M) - 3)$ , the sequence*

$$\dots \longrightarrow B\Sigma\text{Diff}(M_{\#L}^n Q) \longrightarrow B\Sigma\text{Diff}(M_{\#L}^{\#n+1} Q) \longrightarrow \dots$$

*of (classifying spaces of) symmetric diffeomorphism groups is homologically stable.*

This generalises a result of Tillmann [Til16], which corresponds to the case  $L = \text{point}$  (i.e. the usual connected sum operation).

**Informal definition** Fix an  $(m - 1)$ -dimensional manifold  $T$ . Let  $\text{cone}(T) = (T \times [0, \infty)) / (T \times \{0\})$  be the open cone on  $T$ . An  $m$ -dimensional *manifold with conical  $T$ -singularities* is a space  $M$  that is locally homeomorphic to  $\text{cone}(T)$ , together with a smooth atlas on the subset  $M_{\text{mfd}} \subseteq M$  of locally Euclidean points of  $M$ . A *diffeomorphism* of  $M$  is a homeomorphism  $M \rightarrow M$  that restricts to a diffeomorphism  $M_{\text{mfd}} \rightarrow M_{\text{mfd}}$  and is of the form  $\text{cone}(\varphi)$  for some diffeomorphism  $\varphi: T \rightarrow T$  near each point of the discrete subset  $M \setminus M_{\text{mfd}} \subseteq M$ . These form a subgroup

$$\text{Diff}^T(M) \leq \text{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood  $\mathcal{T}(L)$  of any submanifold  $L \subset M$ . The quotient  $M_L = M / \mathcal{T}(L)$  is then a manifold with a single conical  $\partial\mathcal{T}(L)$ -singularity. In particular, using the setting at the beginning of this subsection, we may collapse a tubular neighbourhood of each submanifold  $\iota_i(L) \subset M$  for  $1 \leq i \leq n$ , to obtain a manifold  $M_{n,L}$  with (precisely  $n$ ) conical  $\partial\mathcal{T}(L)$ -singularities.

**Theorem ([Pal18a])** *If  $M$  is connected and has non-empty boundary and  $\dim(L) \leq \frac{1}{2}(\dim(M) - 3)$ , the sequence of classifying spaces  $B\text{Diff}^{\partial\mathcal{T}(L)}(M_{n,L})$  is homologically stable.*

**Special values of  $\ell$  and  $m$ .** The condition (4) on the relative dimensions of  $L$  and  $M$  excludes some interesting special cases of the moduli space  $C_{nL}(M)$ . One such special case is  $\ell = 1, m = 3$ , in other words, moduli spaces of links in a 3-manifold. If one considers moduli spaces of *unlinks* in a 3-manifold  $M$ , then this is known to be homologically stable as the number of components of the unlink goes to infinity, by a result of Kupers [Kup20]. However, the techniques of Kupers do not generalise to moduli spaces of non-trivial links (even if the components are pairwise unlinked), so this question is open for general links.

Another special case is  $\ell = 0, m = 2$ . If  $L$  is a point, this corresponds to configuration spaces of points on a surface  $M$ , for which homological stability is known classically. However, the only assumption that we have made about  $L$  is that it is closed, not necessarily connected, so we could also take  $L = \{1, \dots, \xi\}$  for any positive integer  $\xi$ . The moduli space  $C_{nL}(M)$  is then the covering space of  $C_{n\xi}(M)$  with

$$p_\xi(n) = \frac{(n\xi)!}{n!(\xi!)^n}$$

sheets, whose fibres correspond to all ways of partitioning the  $n\xi$  points of a configuration into  $n$  subsets of size  $\xi$ . Since configuration spaces on  $M$  are aspherical ( $M$  is a connected surface with non-empty boundary), this is equivalent to studying the corresponding index- $p_\xi(n)$  subgroup of the surface braid group  $B_{n\xi}(M)$ , called the *partitioned surface braid group*  $B_{\xi|n}(M)$ , consisting of all braids that preserve a given partition  $n\xi = \xi + \xi + \dots + \xi$  of their endpoints. In joint work with **TriThang Tran**, we have shown that homological stability holds also in this special case, corresponding to  $(\ell, m) = (0, 2)$ .

**Theorem ([PT14])** *Let  $M$  be a connected surface with non-empty boundary. Then the sequence of partitioned surface braid groups  $B_{\xi|n}(M)$  is homologically stable as  $n \rightarrow \infty$ , for any fixed  $\xi \geq 1$ .*

This recovers the classical case of homological stability for configuration spaces of points when  $\xi = 1$ , and is new for  $\xi \geq 2$ .



## 5. Homological representations of motion groups and mapping class groups

Mapping class groups, braid groups and more generally *motion groups* (fundamental groups of moduli spaces of submanifolds) typically have “wild” representation theory. It is therefore very useful to be able to construct representations of these groups by topological or geometric means, in order to be able to understand them with topological or geometric tools. As one very important example, Lawrence [Law90] and Bigelow [Big04] constructed families of linear representations of the classical braid groups starting from actions on the twisted homology of configuration spaces — these were used by Bigelow [Big01] and Krammer [Kra02] to prove the linearity of the braid groups.

In joint work [PS19] with Arthur Soulié, we give a unified construction of such topological representations, which:

- applies simultaneously to all mapping class groups and motion groups in a given dimension  $d$ ,
- produces a much wider family of representations,
- produces interesting representations also over non-commutative rings.

In more detail, in each dimension  $d$ , we construct a large family of representations of a category  $\mathcal{ML}_d$  whose automorphism groups contain all mapping class groups and motion groups in dimension  $d$ . There are three parameters that one may vary in the construction:

- a submanifold  $Z \subset \mathbb{R}^d$ ,
- two integers  $\ell \geq 2$  and  $i \geq 0$ .

Each of these parameters may be varied to obtain interesting representations. For example, the family of Lawrence-Bigelow representations depends on an integer parameter  $k \geq 1$ ; our construction recovers this in the case when  $Z$  is a 0-dimensional manifold of size  $k$  (and  $d = 2$ ,  $\ell = 2$ ,  $i = k$ ). In dimensions  $d \geq 3$ , it becomes interesting to take higher-dimensional submanifolds of  $\mathbb{R}^d$  for  $Z$ , in particular in the case of the *loop-braid groups* (the group of motions of an  $n$ -component unlink in  $\mathbb{R}^3$ ).

The parameter  $\ell \geq 2$  controls the ground ring over which the representation is defined: it is the group-ring of a group of nilpotency class at most  $\ell - 1$ . There are cases where the output of our construction is independent of  $\ell$  (hence the ground ring is commutative), but there are also many interesting cases where we obtain an *infinite tower of representations* as  $\ell \rightarrow \infty$ . In particular, the family of Lawrence-Krammer-Bigelow representations is the  $\ell = 2$  term of such a tower.

The parameter  $i \geq 0$  controls the degree in which we take homology. In the case of the Lawrence-Bigelow representations, there is only one interesting degree in which we can take homology (the homology in other degrees being trivial). However, more generally there can be many interesting degrees in which to take homology. For example, if we consider the family of loop-braid groups and the “naive” analogue of the Lawrence-Bigelow representations (taking  $Z$  to be a 0-dimensional manifold of size  $k$ ), then there are non-trivial homology groups in all degrees  $k \leq i \leq 2k$ .

There is also an iterative version of our construction, which recovers, in the case of the classical braid groups, the *Long-Moody construction* [Lon94].

As a sample of our construction, applied to the (extended and non-extended) loop-braid groups, we have the following. The family of loop-braid groups  $LB_n$  naturally forms a category  $\mathcal{ML}\beta$ , and similarly the family of extended loop-braid groups  $LB'_n$  forms a category  $\mathcal{ML}\beta'$ . For an integer  $k \geq 1$ , if we take either  $Z = k$  or  $Z = U_k$  (an unlink with  $k$  components), as well as  $\ell = 2$  and  $i = k$ , we obtain:

**Theorem ([PS19])** *Our construction specialises to define representations:*

$$\begin{array}{ll}
 \mathfrak{L}_1(1, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z}]} & \mathfrak{L}_1(1, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z}/2]} \\
 \mathfrak{L}_k(k, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus (\mathbb{Z}/2)]} & \mathfrak{L}_k(k, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^2]} \\
 \mathfrak{L}_1(U_1, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus (\mathbb{Z}/2)^2]} & \mathfrak{L}_1(U_1, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^3]} \\
 \mathfrak{L}_k(U_k, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus (\mathbb{Z}/2)^4]} & \mathfrak{L}_k(U_k, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^5]} \\
 \mathfrak{L}_1^+(U_1, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z}^2]} & \mathfrak{L}_1^+(U_1, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z} \oplus (\mathbb{Z}/2)]} \\
 \mathfrak{L}_k^+(U_k, \mathcal{L}\beta) : \mathcal{ML}\beta \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z}^3 \oplus (\mathbb{Z}/2)]} & \mathfrak{L}_k^+(U_k, \mathcal{L}\beta') : \mathcal{ML}\beta' \longrightarrow \text{Mod}_{\mathbb{Z}[\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2]},
 \end{array}$$

where  $k \geq 2$ .

The representations encoded by the first two functors  $\mathfrak{L}_1(1, \mathcal{L}\beta)$  and  $\mathfrak{L}_1(1, \mathcal{L}\beta')$  are studied in more detail, with explicit matrices, in [PS21]; see §12 for more details.

For example, this means that, if we take  $Z$  to be an unoriented unlink with two components, we obtain representations of all loop-braid groups  $LB_n$ , defined over the group-ring  $\mathbb{Z}[\mathbb{Z} \oplus (\mathbb{Z}/2)^4]$ , which may be thought of as a Laurent polynomial ring in 5 variables  $x_1, x_2, x_3, x_4, x_5$  modulo the ideal generated by  $x_i^2 - 1$  for  $i = 2, 3, 4, 5$ . If we quotient also by  $x_1^2 - 1$ , then these representations extend to the extended loop-braid groups  $LB'_n \supset LB_n$ .

Moreover, using non-stopping results for lower central series of *partitioned tripartite welded braid groups* from [DPS22] (see §13 for details), we obtain *infinite towers of representations* when we consider all  $\ell \geq 2$  in the cases  $Z \in \{2, U_2, U_3\}$  (that is, a configuration of two points or an unlink with either two or three components).

## 6. Higher crossing diagrams in knot theory

Another theme of my previous work is in knot theory (which one may think of as the study of  $\pi_0$  of the moduli space of 1-dimensional submanifolds of the 3-sphere), and more precisely with the representation of links by diagrams in the plane (or 2-sphere). A classical link diagram is an immersion of a 1-manifold into the plane, which is an embedding except at a finite number of double points, where the 1-manifold must intersect itself transversely, together with additional data at each intersection point specifying which strand passes “over” the other at that point. For a given link, such a diagram is unique up to ambient isotopy and the well-known *Reidemeister moves*.

In joint work with **Colin Adams** and **Jim Hoste** [AHP19], we study instead *triple-crossing diagrams*, which consist of an immersed 1-manifold in the plane, which is an embedding except at a finite number of points, at which *exactly three* strands must intersect transversely (plus additional data at each intersection point specifying which strands pass “over” others). We introduce an analogue of the Reidemeister moves for such diagrams, consisting of:

- Analogues of the (classical) I- and II-moves, which may be thought of as surgeries supported on a small subdisc of a diagram.
- The *trivial pass move*, which consists of cutting a strand and re-attaching it through another part of the diagram without introducing any new crossings. This may be thought of as a surgery on an annular neighbourhood of a diagram.
- Two families of moves, called the *band moves* and *basepoints moves*, which each consist of a surgery supported on a pair of disjoint subdiscs of the diagram.

**Definition** For a link  $L$  in  $S^3$ , a *maximal nonsplit sublink* of  $L$  is a sublink  $L_1$  of  $L$  such that (a) there exists an embedded 2-sphere in  $S^3 \setminus L$  separating  $L_1$  and  $L \setminus L_1$ , and (b) there does *not* exist any embedded 2-sphere in  $S^3 \setminus L_1$  separating  $L_1$  into two smaller sublinks.

A *relative orientation* of  $L$  is an orientation of  $L$  modulo orientation-reversal of each maximal nonsplit sublink. Equivalently, this is a choice, for each maximal nonsplit sublink  $L_1$  of  $L$ , of an orientation of  $L_1$  modulo complete reversal (reversing every component of  $L_1$  simultaneously).

**Theorem** ([AHP19]) *A triple-crossing diagram determines a relatively oriented link. Any two triple-crossing diagrams representing the same relatively oriented link differ by a finite sequence of I-moves, II-moves, trivial pass moves, band moves, basepoints moves and ambient isotopy.*

The notion of triple-crossing diagram may be generalised to any integer  $n \geq 2$ , with  $n = 2$  corresponding to the classical notion of link diagram. The *n-crossing number*  $c_n(L)$  of a link  $L$  is then the smallest number of crossings among all  $n$ -crossing diagrams of that link. One can ask how the sequence  $\{c_n(L) \mid n \in \mathbb{N}\}$  behaves for each  $L$ , and which relations between the crossing numbers hold for all links  $L$  (or for all but finitely many links  $L$ ). For example, it is not hard to show that  $c_n(L) \geq c_{n+2}(L)$  for all  $n$  and  $L$ , and this inequality is known to be strict for  $n = 2$ . We prove that, with a few small exceptions, it is also strict for  $n = 3$ .

**Theorem** ([AHP19]) *Let  $L$  be a non-split link that is neither the unlink nor the Hopf link. Then*

$$c_3(L) > c_5(L). \tag{6}$$

Note that this inequality is clearly false for the unlink and the Hopf link: for these links,  $c_3(L)$  and  $c_5(L)$  are both equal to 0 and both equal to 1, respectively.

Also note that, if  $L$  can be written as the union of maximal nonsplit sublinks  $L_1, \dots, L_k$ , then  $c_n(L) = c_n(L_1) + \dots + c_n(L_k)$ . Thus the strict inequality (6) holds whenever  $L$  has at least one maximal nonsplit sublink that is nontrivial and not the Hopf link.

## 7. Configuration-section spaces

*Configuration-section spaces* on a manifold  $M$  equipped with a bundle  $E \rightarrow M$  are a kind of “non-local” configuration space (in a difference sense from the non-locality of §1), whose elements consists of a finite configuration in  $M$  together with a section of  $E \rightarrow M$  on the *complement* of the configuration. Such spaces may be thought of physically as spaces of “fields” with singularities.

One often considers subspaces where the behaviour of the field (section) is constrained in a neighbourhood of the singularities (particles) – this may be thought of as restricting the allowed “charges” of the particles. More precisely, if the bundle  $E \rightarrow M$  is trivial with fibre  $X$ , we fix a set  $c \subseteq [S^{d-1}, X]$  of homotopy classes of maps  $S^{d-1} \rightarrow X$ , where  $d$  is the dimension of  $M$ . The sections are then required to restrict to an element of  $c$ , up to homotopy, in a neighbourhood of each particle (they are undefined at the particles themselves). [For non-trivial bundles  $E \rightarrow M$ , the definition is similar but a little more delicate.]

When  $M = \mathbb{R}^2$  and the fibre  $X$  of the bundle is  $BG$  for a discrete group  $G$ , these are the *Hurwitz spaces*, classifying branched coverings of the 2-disc with deck transformation group  $G$  (and prescribed monodromy, if we impose a condition  $c \subseteq [S^1, BG] = \text{Conj}(G)$ ). These spaces have important connections with number theory through recent work of Ellenberg, Venkatesh and Westerland [EVW16], who proved an asymptotic version of the *Cohen-Lenstra conjecture* for function fields via a certain rational homological stability result for Hurwitz spaces.

In joint work with **Ulrike Tillmann**, we have proven another homological stability result for configuration-section spaces [PT21], which is in a sense both more and less general than that of Ellenberg, Venkatesh and Westerland. It is *more* general in the sense that it holds for any bundle over any connected, open manifold  $M$ , but it is also *less* general in the sense that we assume a stronger condition on the allowed “charges” of the particles.

**Theorem ([PT21])** *Let  $M$  be a connected manifold of dimension  $d \geq 2$  with basepoint  $* \in \partial M$  and let  $\xi : E \rightarrow M$  be a fibre bundle whose fibre over  $*$  we denote by  $X$ . Assume that*

$$\text{the preimage of } c \text{ under } [S^{d-1}, X] \xleftarrow{\pi_{d-1}} \pi_{d-1}(X) \text{ is a single element,} \quad (7)$$

*so  $c$  corresponds to a fixed point of the  $\pi_1(X)$ -action on  $\pi_{d-1}(X)$ . Then the stabilisation maps of configuration-section spaces*

$$\text{CI}_k^{c,*}(M; \xi) \longrightarrow \text{CI}_{k+1}^{c,*}(M; \xi) \quad (8)$$

*induce isomorphisms on  $H_i(-; \mathbb{Z})$  in the range  $k \geq 2i + 4$  and surjections in the range  $k \geq 2i + 2$ . With field coefficients, these ranges may be improved to  $k \geq 2i + 2$  and  $k \geq 2i$  respectively.*

## 8. Point-pushing actions

Following on from the previous section, let us now assume that the bundle  $E \rightarrow M$  is trivial with fibre  $X$ ; in this case we write  $\text{CMap}_k^{c,*}(M; X)$  and call these *configuration-mapping spaces*. They fit into a natural fibre sequence

$$\text{Map}^{c,*}(M \setminus k \text{ points}, X) \longrightarrow \text{CMap}_k^{c,*}(M; X) \longrightarrow C_k(\overset{\circ}{M}) \quad (9)$$

(the right-hand map is a “*configuration-mapping bundle*”), which is obtained functorially from the universal fibre sequence

$$M \setminus k \text{ points} \longrightarrow U_k(M) \longrightarrow C_k(\overset{\circ}{M}). \quad (10)$$

In [PT20], we obtain explicit formulas, when the dimension of  $M$  is at least 3, for the monodromy action  $\pi_1(C_k(M)) \rightarrow \pi_0(\text{hAut}(M \setminus k \text{ points}))$  of the universal fibre sequence (10), from which we also deduce explicit formulas for the monodromy action of the fibre sequence (9). A special case of this is as follows:

**Theorem ([PT20])** *If  $d = \dim(M) \geq 3$  and  $M$  satisfies at least one of the following conditions:*

- $M$  is simply-connected, or
- the handle-dimension of  $M$  is at most  $d - 2$ ;

*then the “point-pushing” action of  $\gamma = (\alpha_1, \dots, \alpha_k; \sigma) \in \pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$  on the mapping space  $\text{Map}^{c,*}(M \setminus z, X) \simeq \text{Map}_*(M, X) \times (\Omega_c^{d-1}X)^k$  is given as follows:*

$$(\alpha_1, \dots, \alpha_k; \sigma) \cdot (f, g_1, \dots, g_k) = (f, \bar{g}_1, \dots, \bar{g}_k), \quad (11)$$

where  $\bar{g}_i = f_*(\alpha_i) \cdot g_{\sigma(i)} \cdot \text{sgn}(\alpha_i)$ , and

- for an element  $\alpha \in \pi_1(M)$  we write  $\text{sgn}(\alpha) = +1$  if  $\alpha$  lifts to a loop in the orientation double cover of  $M$  and  $\text{sgn}(\alpha) = -1$  otherwise,
- $\pi_1(X)$  acts up to homotopy on  $\Omega_c^{d-1}X$  in the natural way,
- $\{\pm 1\}$  acts on  $\Omega_c^{d-1}X$  through the involution given by precomposition with a reflection of  $S^{d-1}$  in a hyperplane containing the basepoint.

As a corollary, we obtain a precise description of the set of path-components  $\pi_0(\text{CMap}_k^{c,*}(M, X))$  of the configuration-mapping space, under the above conditions.

In addition, we investigate the question of injectivity of the point-pushing maps. We show that, up to isomorphism, the kernel of the point-pushing map is independent of  $k$  regardless whether diffeomorphisms, homeomorphisms or homotopy equivalences are considered. Precisely, we consider the point-pushing maps

$$\begin{aligned} p_k &: \pi_1(C_k(M)) \longrightarrow \pi_0(\text{Cat}(M, z)) \\ p_{k, \partial} &: \pi_1(C_k(M)) \longrightarrow \pi_0(\text{Cat}_{\partial}(M, z)) \end{aligned}$$

where  $\text{Cat} \in \{\text{hAut}, \text{Homeo}, \text{Diff}\}$ , and where  $\partial$  means that the boundary of  $M$  is fixed, and prove:

**Theorem ([PT20, §8])** *When the dimension of  $M$  is at least 3, we have*

$$\ker(p_k) = \Delta(\ker(p_1)),$$

*i.e. the kernel of  $p_k$  is equal to the diagonal of  $\ker(p_1)^k \subseteq \pi_1(M)^k \subseteq \pi_1(C_k(M))$ , where we identify  $\pi_1(C_k(M))$  with  $\pi_1(M)^k \rtimes \Sigma_k$ . If  $\partial M \neq \emptyset$ , then  $p_{k, \partial}$  is injective.*

## 9. Motivic homological stability

In joint work with **Geoffroy Horel** [HP20], we have proven that the *motivic and étale cohomology* of configuration spaces on smooth algebraic varieties (or schemes) is stable.

In more detail, if  $X$  is a smooth scheme over a number field  $K$ , one may ask whether the corresponding unordered configuration schemes  $C_n(X)$  exhibit stability in their étale or motivic cohomology – lifting the classical stability for the singular (co)homology of the complex manifolds  $C_n(X_{\mathbb{C}})$ , where  $X_{\mathbb{C}}$  denotes the complex points of  $X$ .

**Theorem ([HP20])** *Suppose that  $X$  may be written as  $Y - D$ , where  $Y$  is a smooth scheme and  $D$  is a non-empty, closed, smooth subscheme admitting a  $K$ -point. If the étale motive of  $X$  is mixed Tate and the complex manifold  $X_{\mathbb{C}}$  is connected, then there are maps of étale motivic cohomology groups*

$$H_{\text{ét}}^{p,q}(C_{n+1}(X); \Lambda) \longrightarrow H_{\text{ét}}^{p,q}(C_n(X); \Lambda)$$

*that are isomorphisms for  $p \leq n/2$  and under mild conditions on the coefficient ring  $\Lambda$ . In the case when  $X = \mathbb{A}^d$  is affine space, there are analogous maps of motivic cohomology groups*

$$H^{p,q}(\tilde{C}_{n+1}(\mathbb{A}^d); \Lambda) \longrightarrow H^{p,q}(\tilde{C}_n(\mathbb{A}^d); \Lambda)$$

*that are isomorphisms for  $p \leq n/2$  and any coefficient ring  $\Lambda$ . (Here,  $\tilde{C}_n$  is a stacky version of the unordered configuration scheme  $C_n$ .)*

## 10. Lawrence-Bigelow representations

In joint work with **Cristina Anghel** [AP20], we have investigated the fundamental relationships (in terms of non-degenerate pairings, embeddings and isomorphisms) between the many different flavours of homological representations of mapping class groups – with the *Lawrence-Bigelow representations* of the braid groups being our motivating example. Understanding the relationships between these representations is important for applications in quantum topology (see for example [Ang20]).

For any surface  $\Sigma$  equipped with a decomposition  $\partial_{\text{in}}\Sigma \cup \partial_{\text{out}}\Sigma$  of its boundary (a *surface triad*), we think of  $\partial_{\text{in}}\Sigma$  as its *inner* “free” boundary and  $\partial_{\text{out}}\Sigma$  as its *outer* “fixed” boundary, and consider the mapping class group

$$\Gamma(\Sigma) = \pi_0(\text{Diff}(\Sigma, \partial_{\text{out}}\Sigma)).$$

For example, if  $\Sigma = \Sigma_{0,n+1}$  with  $n$  of its boundary-components considered as *inner* and one considered as *outer*, this is isomorphic to the braid group on  $n$  strands. Choosing a local system  $\mathcal{L}$  (defined over a ring  $R$ ) on the configuration space  $C_k(\Sigma)$  that is preserved by the action of  $\Gamma(\Sigma)$ , we obtain homological  $\Gamma(\Sigma)$ -representations

$$H^\bullet(C_\circ),$$

where  $C_\circ$  is either  $C_{\text{out}} = C_k(\Sigma \setminus \partial_{\text{in}}\Sigma)$  or  $C_{\text{in}} = C_k(\Sigma \setminus \partial_{\text{out}}\Sigma)$  and  $H^\bullet$  denotes one of

- ordinary homology  $H$ , twisted by  $\mathcal{L}$ ,
- homology relative to the boundary  $H^\partial$ , twisted by  $\mathcal{L}$ ,
- locally-finite (Borel-Moore) homology  $H^{\text{lf}}$ , twisted by  $\mathcal{L}$ ,
- locally-finite homology of the associated covering space  $\tilde{C}_k(\Sigma) \rightarrow C_k(\Sigma)$  (in the case where  $\mathcal{L}$  arises from such a covering), denoted  $H^{\text{lf},\sim}$ .

For example, when  $\Sigma = \Sigma_{0,n+1}$  we may take  $\mathcal{L}$  to be the local system arising from the covering corresponding to the kernel of the homomorphism

$$\varphi: \pi_1(C_k(\Sigma)) \longrightarrow \begin{cases} \mathbb{Z}\{c\} & k = 1 \\ \mathbb{Z}\{c, x\} & k \geq 2 \end{cases}$$

given by  $\varphi(\gamma) = ic + jx$ , where  $i$  is the total winding number of  $\gamma$  around the  $n$  inner boundary components and  $j$  is the *writhe* of the braid obtained from  $\gamma$  by filling in the  $n$  inner boundary components with discs. In this example the  $B_n$ -representations above are 8 different flavours of the *Lawrence-Bigelow representations*.

**Theorem** ([AP20]) (i) For  $\circ \in \{\text{in}, \text{out}\}$ , there is a non-degenerate pairing

$$H^{\text{lf}}(C_\circ) \otimes H^\partial(C_\circ) \longrightarrow R.$$

Moreover, these representations are free as  $R$ -modules. We describe explicit bases such that the pairing above is given by the identity matrix.

(ii) Under a mild condition on the local system  $\mathcal{L}$ , there are embeddings of representations

$$\begin{aligned} H^\partial(C_{\text{in}}) &\longrightarrow H^{\text{lf}}(C_{\text{out}}) \\ H^\partial(C_{\text{out}}) &\longrightarrow H^{\text{lf}}(C_{\text{in}}). \end{aligned}$$

When  $k \geq 2$  this implies that  $H^{\text{lf}}(C_{\text{out}})$  and  $H^{\text{lf}}(C_{\text{in}})$  are reducible. In fact, with respect to the explicit bases that we describe, the matrices of these embeddings are diagonal, and their diagonal entries are products of quantum factorials in  $R$ .

(iii) For  $\circ \in \{\text{in}, \text{out}\}$ , there is a natural injective map

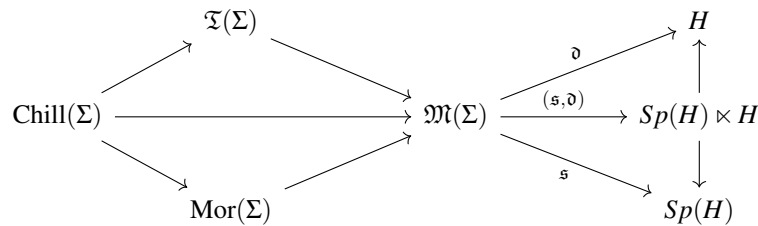
$$H^{\text{lf}}(C_\circ) \longrightarrow H^{\text{lf},\sim}(C_\circ).$$

If  $\mathcal{B}$  denotes a free basis for  $H^{\text{lf}}(C_\circ)$  as a module over  $R = k[G]$ , then  $H^{\text{lf},\sim}(C_\circ)$  is a direct sum over  $\mathcal{B}$  of copies of the completion  $k[[G]] = \prod_G k$  of  $k[G] = \bigoplus_G k$ .

### 11. Mapping class group representations via Heisenberg homology

As mentioned in §5, the *Lawrence-Bigelow representations* of the braid groups are very important in the theory of braid groups. In particular, they were used by Bigelow [Big01] and Krammer [Kra02] to prove that the braid groups are *linear*, i.e., they embed into general linear groups over a field. For the mapping class group of the surface  $\Sigma_{g,1}$ , however, the question of linearity is wide open, except in very low genus.

Inspired by this question, in joint work [BPS21] with **Christian Blanchet** and **Awais Shaukat**, we have constructed a wide family of analogues of the Lawrence-Bigelow representations for the mapping class group  $\mathfrak{M}(\Sigma)$  of the surface  $\Sigma = \Sigma_{g,1}$ , as well as three of its important subgroups, the Torelli group  $\mathfrak{T}(\Sigma)$ , the Chillingworth subgroup  $\text{Chill}(\Sigma)$  and the Morita subgroup  $\text{Mor}(\Sigma)$ , which fit together in the following diagram, where  $H = H_1(\Sigma; \mathbb{Z})$  and  $\mathfrak{d}$  is the *Morita crossed homomorphism*.

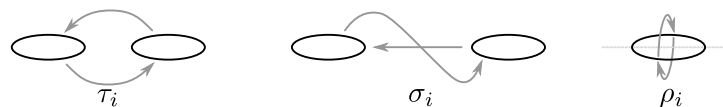


More precisely, for any integer  $m \geq 2$  and representation  $V$  of the discrete Heisenberg group  $\mathcal{H}(\Sigma)$ , we construct representations of  $\text{Chill}(\Sigma)$  and of  $\mathfrak{T}(\Sigma)$ . When  $V$  is the *Schrödinger representation* of  $\mathcal{H}(\Sigma)$ , we moreover construct a unitary representation of  $\text{Mor}(\Sigma)$ , which extends to a unitary representation of the stably universal central extension of  $\mathfrak{M}(\Sigma)$ .

The kernels of each of these representations are contained in the intersection of the Magnus kernel of  $\mathfrak{M}(\Sigma)$  and the  $m$ -th term of the *Johnson filtration* of  $\mathfrak{M}(\Sigma)$ . We perform some explicit computations of these representations when  $m = 2$  and  $V = \mathbb{Z}[\mathcal{H}(\Sigma)]$  is the regular representation of  $\mathcal{H}(\Sigma)$ , and discover that they are highly non-trivial already for genus  $g = 1$ .

### 12. Burau representations of loop braid groups

Loop braid groups appear in many guises in topology and group theory. They may be seen geometrically as fundamental groups of trivial links in  $\mathbb{R}^3$ , diagrammatically as equivalence classes of *welded braids* (closely related to virtual braids and virtual knot theory), algebraically as subgroups of automorphism groups of free groups or combinatorially via explicit group presentations. They are also related to physics via *exotic string statistics* [BWC07]. Generators for these groups are of the form:



In joint work with **Arthur Soulié** [PS21], we give a topological construction of the *Burau representations* of the loop braid groups, which are higher-dimensional analogues of the classical Burau representations of the braid groups. There are four versions: defined either on the non-extended or extended loop braid groups, and in each case there is an unreduced and a reduced version. Three are not surprising, and one could easily guess the correct matrices to assign to generators. However, the fourth is more subtle, and does not seem combinatorially obvious, although it is topologically very natural.

### 13. Lower central series of surface, virtual and welded braid groups

One of the most basic objects one needs to understand when studying the structure of a group  $G$  is its lower central series  $G = \mathcal{L}_1(G) \supseteq \mathcal{L}_2(G) \supseteq \dots$ . If  $G$  is perfect, its lower central series is completely trivial; on the other hand, if  $G$  is nilpotent or residually nilpotent,  $\mathcal{L}_*(G)$  contains deep information about the structure of  $G$ . The lower central series is also deeply connected to the structure of the group ring of  $G$ .

The amount of information one can hope to extract from the study of a lower central series depends in the first place on whether or not it *stops*, meaning that there exists an integer  $i \geq 1$  such that  $\mathcal{L}_i(G) = \mathcal{L}_{i+1}(G)$ . If there is such an integer, then the smallest such integer is the *length* of the lower central series of  $G$ .

In joint work with **Jacques Darné** and **Arthur Soulié** [DPS22], we give a complete answer to the question of the length (finite or infinite) of the lower central series of surface braid groups, virtual braid groups and welded (=loop) braid groups, as well as partitioned versions of all of these groups. The answer depends subtly on the number of strands, how they are partitioned and the topology of the underlying surface. For example, for  $n \geq 3$ , the lower central series of:

- $B_n(S)$  has length 2 if  $S$  is planar or non-orientable;
- $B_n(S)$  has length 3 if  $S$  is non-planar and orientable;
- $B_{(2,n)}(\mathbb{R}^2)$  has length  $\infty$ ;
- $B_{(2,n)}(\mathbb{S}^2)$  has length  $v_2(n) + 1$  or  $v_2(n) + 2$ , where  $v_2(n)$  is the 2-adic valuation of  $n$ .

For a complete answer, see the tables in the introduction of [DPS22].

## References

- [AHP19] C. Adams, J. Hoste and M. Palmer. *Triple-crossing number and moves on triple-crossing link diagrams*. *Journal of Knot Theory and Its Ramifications* 28.11 (2019), p. 1940001 ([↑ 10](#)).
- [Ang20] C. A.-M. Anghel. *Coloured Jones and Alexander polynomials as topological intersections of cycles in configuration spaces*. ArXiv:2002.09390. 2020 ([↑ 13](#)).
- [AP20] C. A.-M. Anghel and M. Palmer. *Lawrence-Bigelow representations, bases and duality*. ArXiv:2011.02388. 2020 ([↑ 3, 13](#)).
- [BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123 ([↑ 6](#)).
- [Ber<sup>+</sup>06] A. J. Berrick, F. R. Cohen, Y. L. Wong and J. Wu. *Configurations, braids, and homotopy groups*. *J. Amer. Math. Soc.* 19.2 (2006), pp. 265–326 ([↑ 6](#)).
- [Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) ([↑ 5](#)).
- [Big01] S. J. Bigelow. *Braid groups are linear*. *J. Amer. Math. Soc.* 14.2 (2001), 471–486 (electronic) ([↑ 9, 14](#)).
- [Big04] S. Bigelow. *Homological representations of the Iwahori-Hecke algebra*. *Geom. Topol. Monogr.* 7 (2004), pp. 493–507 ([↑ 9](#)).
- [BM14] M. Bendersky and J. Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815 ([↑ 6](#)).
- [Bol12] S. K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. *Math. Z.* 270.1-2 (2012), pp. 297–329 ([↑ 5](#)).
- [BPS21] C. Blanchet, M. Palmer and A. Shaukat. *Heisenberg homology on surface configurations*. ArXiv:2109.00515. 2021 ([↑ 3, 14](#)).
- [BWC07] J. C. Baez, D. K. Wise and A. S. Crans. *Exotic statistics for strings in 4D BF theory*. *Adv. Theor. Math. Phys.* 11.5 (2007), pp. 707–749 ([↑ 14](#)).
- [CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314 ([↑ 5](#)).
- [Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504 ([↑ 6](#)).
- [CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76 ([↑ 5](#)).
- [CP15] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. *Doc. Math.* 20 (2015), pp. 753–805 ([↑ 2, 5, 6](#)).
- [DPS22] J. Darné, M. Palmer and A. Soulié. *When the lower central series stops*. ArXiv:2201.03542. 2022 ([↑ 3, 10, 15](#)).
- [Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. *Ann. of Math. (2)* 111.2 (1980), pp. 239–251 ([↑ 5](#)).
- [EVW16] J. S. Ellenberg, A. Venkatesh and C. Westerland. *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*. *Ann. of Math. (2)* 183.3 (2016), pp. 729–786 ([↑ 1, 2, 11](#)).
- [HP20] G. Horel and M. Palmer. *Motivic homological stability of configuration spaces*. ArXiv:2007.13718. 2020 ([↑ 2, 12](#)).

- [Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (↑ 5).
- [Kal80] W. van der Kallen. *Homology stability for linear groups*. *Invent. Math.* 60.3 (1980), pp. 269–295 (↑ 5).
- [KM16] A. Kupers and J. Miller. *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*. *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468 (↑ 6).
- [Knu17] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. *Algebr. Geom. Topol.* 17.5 (2017), pp. 3137–3187 (↑ 6).
- [Kra02] D. Krammer. *Braid groups are linear*. *Ann. of Math. (2)* 155.1 (2002), pp. 131–156 (↑ 9, 14).
- [Kra19] M. Krannich. *Homological stability of topological moduli spaces*. *Geom. Topol.* 23.5 (2019), pp. 2397–2474 (↑ 4).
- [Kup20] A. Kupers. *Homological stability for unlinked circles in a 3-manifold*. *Q. J. Math.* 71.1 (2020), pp. 335–358 (↑ 8).
- [Law90] R. J. Lawrence. *Homological representations of the Hecke algebra*. *Comm. Math. Phys.* 135.1 (1990), pp. 141–191 (↑ 9).
- [Lon94] D. D. Long. *Constructing representations of braid groups*. *Comm. Anal. Geom.* 2.2 (1994), pp. 217–238 (↑ 9).
- [McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ 2, 4, 6).
- [ML88] R. J. Milgram and P. Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424 (↑ 6).
- [MP15a] J. Miller and M. Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. *Q. J. Math.* 66.1 (2015), pp. 265–284 (↑ 2, 4).
- [MP15b] J. Miller and M. Palmer. *Scanning for oriented configuration spaces*. *Homology, Homotopy and Applications* 17.1 (2015), pp. 35–66 (↑ 2, 4).
- [MS76] D. McDuff and G. Segal. *Homology fibrations and the “group-completion” theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284 (↑ 2, 4).
- [Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328 (↑ 1).
- [MW07] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford’s conjecture*. *Ann. of Math. (2)* 165.3 (2007), pp. 843–941 (↑ 1).
- [Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups*. ArXiv:1505.04294. 2015 (↑ 6).
- [Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711 (↑ 2, 4).
- [Pal17] M. Palmer. *A comparison of twisted coefficient systems*. ArXiv:1712.06310. 2017 (↑ 5).
- [Pal18a] M. Palmer. *Stability for moduli spaces of manifolds with conical singularities*. ArXiv:1807.07558. 2018 (↑ 2, 7, 8).
- [Pal18b] M. Palmer. *Twisted homological stability for configuration spaces*. *Homology, Homotopy and Applications* 20.2 (2018), pp. 145–178 (↑ 2, 5).
- [Pal21] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. *Algebr. Geom. Topol.* 21.3 (2021), pp. 1371–1444 (↑ 2, 6, 7).
- [PS19] M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction*. ArXiv:1910.13423. 2019 (↑ 2, 9).
- [PS21] M. Palmer and A. Soulié. *The Burau representations of loop braid groups*. ArXiv:2109.11468. 2021 (↑ 3, 10, 14).
- [PT14] M. Palmer and T. Tran. *Homological stability for subgroups of surface braid groups*. ArXiv:1410.0923. 2014. To appear in *Homology, Homotopy and Applications* (↑ 8).
- [PT20] M. Palmer and U. Tillmann. *Point-pushing actions for manifolds with boundary*. ArXiv:2007.11613. 2020. To appear in *Groups, Geometry, and Dynamics* (↑ 2, 11, 12).
- [PT21] M. Palmer and U. Tillmann. *Configuration-mapping spaces and homology stability*. *Res. Math. Sci.* 8.3 (2021), Paper No. 38 (↑ 2, 11).
- [Ran13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326 (↑ 6).
- [RW17] O. Randal-Williams and N. Wahl. *Homological stability for automorphism groups*. *Adv. Math.* 318 (2017), pp. 534–626 (↑ 4, 5).
- [Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ 2, 4, 6).
- [Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (↑ 2, 4, 6).
- [Til16] U. Tillmann. *Homology stability for symmetric diffeomorphism and mapping class groups*. *Math. Proc. Cambridge Philos. Soc.* 160.1 (2016), pp. 121–139 (↑ 8).