Summary of previous research

Introduction

The concept of a *moduli space* is of central importance in mathematics, parametrising collections of all objects of a given kind – solutions to an equation, manifolds with certain properties, configurations of points, submanifolds or fields, etc. The overarching goal of my research so far has been to understand the topology of different kinds of moduli spaces, through their *homology* (§1) and their *fundamental groups* (§2). The kinds of moduli spaces that I have studied include:

- Configuration spaces of points in manifolds (§1.1).
- *Non-local* configuration spaces, in which configurations are equipped with some kind of additional "non-local" structure, such as:
  - an ordering modulo even permutations (§1.1.1),
  - a "field" defined on the complement of the configuration (§1.1.2),
  - non-local data encoding the interactions of asymptotic magnetic monopoles (§1.1.3).
- Moduli spaces of disconnected submanifolds, where points are replaced with higher-dimensional closed manifolds (§1.2). This is related to moduli spaces of manifolds with Baas-Sullivan singularities, via the operation of collapsing each connected component of a submanifold to a point.
- Mapping class groups of surfaces, including surfaces of *infinite type* (§1.3 and §2.2.5).

Homology of moduli spaces. A ubiquitous phenomenon in the homology of moduli spaces is *homological stability*. For a family of moduli spaces indexed by a parameter n, this is the phenomenon where their homology is independent of n in higher and higher degrees as $n \to \infty$. One very important example of this is for the mapping class groups of orientable surfaces: homological stability with respect to genus was proven by Harer [Har85] and the limiting homology was computed by Madsen and Weiss [MW07], together proving the *Mumford conjecture* [Mum83]. Another important example is an application to analytic number theory: Ellenberg, Venkatesh and Westerland [EVW16] proved an asymptotic version of the *Cohen-Lenstra conjecture* for function fields, the core of their proof being a homological stability result for Hurwitz spaces (which are examples of configuration-mapping spaces; see §1.1.2).

Several of the results of my previous work have established homological stability in new contexts, including:

- Configuration spaces with different kinds of *non-locality*: §1.1.1–§1.1.3.
- In particular, this includes *configuration-section spaces* (§1.1.2), a point in which consists of a configuration in a manifold and a "field" on the complement of the configuration with prescribed "charges" around each particle. These generalise *Hurwitz spaces*, which correspond to when the underlying manifold is the plane and fields take values in the classifying space of a discrete group.
- The homology of configuration spaces with respect to *polynomial twisted coefficient systems*: §1.1.5.
- The *motivic cohomology* of configuration schemes on a given smooth scheme: §1.1.6.
- Moduli spaces of higher-dimensional disconnected submanifolds: §1.2.1.
- Mapping class groups of *infinite-type surfaces*: §1.3.1.

There are also settings where homological stability is *false* and instead there exist more complicated periodic patterns in the homology in a stable range of degrees. This holds for example for configuration spaces on *closed* manifolds: §1.1.4.

In the context of mapping class groups of infinite-type surfaces ("big mapping class groups"), I have used homological stability techniques to prove that certain families of big mapping class groups have trivial homology in all degrees (i.e. they are acyclic). In contrast, another result (§1.3.2) shows that many other families of big mapping class groups have *uncountable* (integral) homology in all degrees.

Fundamental groups of moduli spaces. These groups are interesting objects in their own right and include *motion groups* (such as surface braid groups, loop braid groups, etc.) and *mapping class groups*. They have connections to knot theory and also to physics via topological quantum field theories.

Understanding the *lower central series* of a group (and its associated Lie algebra) is typically a difficult task, but it can lead to a deep understanding of the underlying structure of the group. Another powerful method of understanding a group is to understand its *representations*. In particular, a key question for any group is to know whether it has a faithful representation on a finite-dimensional vector space – in other words, whether it is *linear*. This is known for the classical braid groups [Big01; Kra02] but is wide open for almost all other motion groups and mapping class groups. In the context of representations of mapping class groups, another important question is whether they may be extended to define a *topological quantum field theory* (TQFT) – in other words, whether they may be defined not just on automorphisms of manifolds but on a whole cobordism category of manifolds.

I have worked on understanding the lower central series of many different kinds of motion groups (including surface braid groups, loop braid groups and generalisations thereof), in particular the question of when the lower central series *stops*: §2.1. Motivated by the open question of linearity and by the goal of constructing TQFTs, I have worked on new topological constructions of representations of (surface) braid groups, loop braid groups and mapping class groups: §2.2.

1. The homology of moduli spaces

I have studied the homology (including homological stability phenomena) of various different kinds of configuration spaces (§1.1), moduli spaces of submanifolds and manifolds with singularities (§1.2) and mapping class groups of infinite-type surfaces (§1.3).

1.1. Configuration spaces

For a space M, the nth *ordered configuration space* $\widetilde{C}_n(M)$ is the subspace of M^n consisting of all n-tuples of pairwise distinct points in M. The symmetric group $\mathfrak{S}_n$ acts on this space, and

$$C_n(M) = \widetilde{C}_n(M)/\mathfrak{S}_n$$

is the nth *unordered configuration space* on M. It is a classical result, going back to McDuff [McD75] and Segal [Seg73; Seg79], that the sequence $C_n(M)$, when M is a connected, open manifold, is *homologically stable*. This means that, for each degree i, there are isomorphisms $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ once n is sufficiently large (depending on i).

Closed manifolds. In the results of McDuff and Segal, the condition that M is connected is clearly necessary for homological stability, as one can see already by considering H_0. On the other hand, the condition that M is open (i.e. non-compact or with non-empty boundary) is not obviously necessary, and the situation for closed manifolds M is much more subtle. This part of my previous work is discussed in §1.1.4.

Non-local data. A relatively straightforward generalisation of the result of McDuff and Segal is homological stability for *labelled* configuration spaces $C_n(M,X)$ for a path-connected space X: a point in this space consists of an unordered configuration in M together with an element of X attached to each configuration point. (This is proven in [Ran13], for example.) The additional data associated to a configuration in this setting is *local* in the sense that it is simply a product of several pieces of data, each associated to a single point in the configuration.

Oriented configurations. However, one may also consider moduli spaces of configurations equipped with *non-local* data of different kinds. An example of this is the sequence of *oriented configuration spaces*

$$C_n^+(M) = \widetilde{C}_n(M)/A_n$$

given by quotienting by the action of the alternating group instead of the symmetric group. These are double coverings of the unordered configuration spaces and the additional (binary) piece of data given by an ordering modulo even permutations is clearly not associated to any single point of the configuration. These spaces are discussed in §1.1.1.

Configuration-section spaces. Another kind of non-local data for configurations is a "field" defined on the complement of the configuration: these are *configuration-sections spaces*, discussed in §1.1.2 (homological stability) and §1.1.7 (studying the monodromy action of the fibration given by forgetting the field).

Asymptotic magnetic monopoles. In the case where the underlying manifold is Euclidean 3-space, there is another kind of non-local data, encoding the pairwise interactions of the particles and modelling "asymptotic" magnetic monopoles, which is discussed in §1.1.3.

Twisted coefficients. The homological stability result of McDuff and Segal is for untwisted (integral) homology of configuration spaces. I have proven an extension of their result that applies to "polynomial" systems of twisted coefficients on the spaces $C_n(M)$; see §1.1.5.

Motivic cohomology. If $M = X(\mathbb{C})$ is the manifold of complex points of a smooth scheme X, one may wonder whether stability for the singular homology of $C_n(M)$ may be lifted to stability for the motivic cohomology of the sequence of configuration schemes on X. This is discussed in §1.1.6.

1.1.1. Non-locality: oriented configurations

In contrast to unordered configuration spaces, *ordered* configuration spaces $\widetilde{C}_n(M)$ are not homologically stable: for example, the first homology of $\widetilde{C}_n(\mathbb{R}^2)$ is the abelianisation of the pure braid group, which is $\mathbb{Z}^{\binom{n}{2}}$. This raises the question of whether there is an intermediate covering space between $\widetilde{C}_n(M)$ and $C_n(M)$ for which homological stability still holds. I proved in [Pal13] that the answer is positive for the oriented configuration spaces $C_n^+(M)$, which doubly cover the unordered configuration spaces $C_n(M)$:

Theorem ([Pal13]) *The natural stabilisation map $C_n^+(M) \to C_{n+1}^+(M)$ induces isomorphisms on homology in degrees $* \leqslant \frac{n-5}{3}$ and surjections in degrees $* \leqslant \frac{n-2}{3}$.*

The theorem also holds more generally for *labelled configuration spaces*, where each point is equipped with a label in some fixed, path-connected parameter space X. The so-called *slope* of the stability range here is $\frac{1}{3}$ (and one may calculate explicitly for certain M to see that this is sharp), in contrast to the slope of $\frac{1}{2}$ that holds for the unordered configuration spaces.

Twisted coefficients. The integral homology of $C_n^+(M)$ may be interpreted as the twisted homology group $H_*(C_n(M);\mathbb{Z}[\mathbb{Z}/2])$, where $\pi_1(C_n(M))$ acts on the group ring of $\mathbb{Z}/2$ via the natural projection $\pi_1(C_n(M)) \to \mathfrak{S}_n$ followed by the sign homomorphism. This is an example of an *abelian twisted coefficient system* on $C_n(M)$, and is a precursor, in a special case, of the notion of *abelian homological stability*, which was subsequently developed by Randal-Williams and Wahl [RW17] and Krannich [Kra19]. See §1.1.5 for more twisted homological stability results.

This result leads to the question of whether one can identify the *stable homology* of the sequence $\{C_n^+(M) \mid n \in \mathbb{N}\}$, in other words the colimit

$$\lim_{n\to\infty} H_*(C_n^+(M)),$$

in terms of other well-understood spaces. In joint work with **Jeremy Miller**, we answered this question positively by lifting the classical *scanning map* [Seg73; McD75] to a homology equivalence between appropriate covering spaces.

- First, in our paper [MP15a], we generalised the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff's *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients (more precisely to a setting where we consider homology with respect to all twisted coefficient systems in a fixed class $\mathfrak{C}$ that is closed under pullbacks).
- Using these tools, we proved in [MP15b] a new kind of scanning result, lifting the classical scanning map to covering spaces and showing that it remains a homology equivalence after doing so. This identifies the stable homology of oriented configuration spaces on M with the homology of an explicit double cover of the section space of a certain bundle over M:

Theorem ([MP15b]) *Writing $\dot{T}M \to M$ for the fibrewise one-point compactified tangent bundle of M and $\Gamma_c(\dot{T}M \to M)_\circ$ for its space of compactly-supported sections of degree zero, we have:*

$$\lim_{n\to\infty} H_*(C_n^+(M)) \;\cong\; H_*(\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ), \tag{1}$$

for a certain explicit double cover $\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ \longrightarrow \Gamma_c(\dot{T}M \to M)_\circ$.

This double cover may be defined as the connected covering space of $\Gamma_c = \Gamma_c(\dot{T}M \to M)_\circ$ corresponding to the projection

$$\pi_1(\Gamma_c) \to H_1(\Gamma_c) \cong H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/(2).$$

Here, the isomorphism $H_1(\Gamma_c) \cong H_1(C_2(M))$ arises from homological stability and the identification of the stable homology for *unordered* configuration spaces, and the map $H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty))$ is induced by any embedding $M \hookrightarrow \mathbb{R}^\infty$. For example, when $M = \mathbb{R}^\infty$ the right-hand side of (1) is the universal cover of (one component of) the infinite loopspace $\Omega^\infty S^\infty = Q\mathbb{S}^0$. When $M = \mathbb{R}^2$ it is the unique connected double cover of $\Omega^2 S^3$.

1.1.2. Non-locality: configuration-section spaces

Configuration-section spaces on a manifold M equipped with a bundle $E \to M$ are a kind of "non-local" configuration space (in a difference sense from the non-locality of §1.1.1 above), whose elements consist of a finite configuration in M together with a section of $E \to M$ on the *complement* of the configuration. Such spaces may be thought of physically as spaces of "fields" with singularities.

One often considers subspaces where the behaviour of the field (section) is constrained in a neighbourhood of the singularities (particles) – this may be thought of as restricting the allowed "charges" of the particles. More precisely, if the bundle $E \to M$ is trivial with fibre X, we fix a set $c \subseteq [S^{d-1}, X]$ of homotopy classes of maps $S^{d-1} \to X$, where d is the dimension of M. The sections are then required to restrict to an element of c, up to homotopy, in a neighbourhood of each particle (they are undefined at the particles themselves). [For non-trivial bundles $E \to M$, the definition is similar but a little more delicate.]

When $M = \mathbb{R}^2$ and the fibre X of the bundle is BG for a discrete group G, these are the *Hurwitz spaces*, classifying branched coverings of the 2-disc with deck transformation group G (and prescribed monodromy, if we impose a condition $c \subseteq [S^1, BG] = \mathrm{Conj}(G)$). These spaces have important connections with number theory through recent work of Ellenberg, Venkatesh and Westerland [EVW16], who proved an asymptotic version of the *Cohen-Lenstra conjecture* for function fields via a certain rational homological stability result for Hurwitz spaces.

In joint work with **Ulrike Tillmann**, we have proven another homological stability result for configuration-section spaces [PT21], which is in a sense both more and less general than that of Ellenberg, Venkatesh and Westerland. It is *more* general in the sense that it holds for any bundle over any connected, open manifold M, but it is also *less* general in the sense that we assume a stronger condition on the allowed "charges" of the particles.

Theorem ([PT21]) *Let M be a connected manifold of dimension $d \geqslant 2$ with basepoint $* \in \partial M$ and let $\xi \colon E \to M$ be a fibre bundle whose fibre over $*$ we denote by X. Assume that*

$$\text{the preimage of } c \text{ under } [S^{d-1}, X] \twoheadleftarrow \pi_{d-1}(X) \text{ is a single element,} \tag{2}$$

so c corresponds to a fixed point of the $\pi_1(X)$-action on $\pi_{d-1}(X)$. Then the stabilisation maps of configuration-section spaces

$$C\Gamma_k^{c,*}(M;\xi) \longrightarrow C\Gamma_{k+1}^{c,*}(M;\xi) \tag{3}$$

induce isomorphisms on $H_i(-;\mathbb{Z})$ in the range $k \geqslant 2i+4$ and surjections in the range $k \geqslant 2i+2$. With field coefficients, these ranges may be improved to $k \geqslant 2i+2$ and $k \geqslant 2i$ respectively.

1.1.3. Non-locality: asymptotic magnetic monopoles

The topology of the moduli spaces of magnetic monopoles $\mathscr{M}_k$ has been the subject of intensive study for many decades. By a theorem of Donaldson [Don84], they have a model as spaces of rational functions on $\mathbb{C}P^1$. Via this model, their homotopy and homology groups are known to stabilise as $k \to \infty$ by a theorem of Segal [Seg79] and their homology (both stable and unstable) was completely computed by [Coh+91] in terms of the homology of the braid groups, which is completely known [CLM76].

The moduli spaces $\mathscr{M}_k$ are non-compact manifolds. Recently, a partial compactification of $\mathscr{M}_k$ has been constructed by Kottke and Singer [KS22] by adding to $\mathscr{M}_k$ certain codimension-1 boundary hypersurfaces $\mathscr{I}_\lambda$ indexed by partitions $\lambda = (k_1, \ldots, k_r)$ of k. Points in these boundary hypersurfaces are thought of as "ideal" or "asymptotic" monopoles of total charge k, with r "clusters" centred at different points in $\mathbb{R}^3$, with charges $k_1, \ldots, k_r$, which are "widely separated" but nevertheless interact. The space $\mathscr{I}_\lambda$ has the structure of a fibre bundle

$$\mathscr{I}_\lambda \longrightarrow C_\lambda(\mathbb{R}^3) \tag{4}$$

over the partitioned configuration space $C_\lambda(\mathbb{R}^3)$ (the covering space of the unordered configuration space of $k_1 + \cdots + k_r$ points where configurations are equipped with a partition of type λ) with fibre $\mathscr{M}_{k_1} \times \cdots \times \mathscr{M}_{k_r}$. The *non-locality* of the non-local configuration spaces $\mathscr{I}_\lambda$ comes from the non-triviality of the bundle (4).

In joint work with **Ulrike Tillmann**, we have proven a homological stability result for these asymptotic monopole moduli spaces as the number of clusters of a fixed charge $c \geqslant 1$ goes to infinity. Fix a positive integer c and a tuple $\lambda = (k_1, \ldots, k_r)$ of positive integers $k_i \neq c$. Write $\lambda[n]_c = (k_1, \ldots, k_r, c, \ldots, c)$, where c appears n times.

Theorem ([PT22]) *There are natural stabilisation maps*

$$\mathscr{I}_{\lambda[n]_c} \longrightarrow \mathscr{I}_{\lambda[n+1]_c} \tag{5}$$

that induce isomorphisms on homology in all degrees $\leqslant n/2-1$ with $\mathbb{Z}$ coefficients and in all degrees $\leqslant n/2$ with field coefficients.

1.1.4. Configurations on closed manifolds

When M is closed, homological stability for the unordered configuration spaces $C_n(M)$ is not true in general, for example one may calculate that $H_1(C_n(S^2);\mathbb{Z}) \cong \mathbb{Z}/(2n-2)$, which does not stabilise as $n \to \infty$. Moreover, the classical stabilisation maps used by McDuff and Segal do not exist, since these depend on adding a new configuration point in M "near infinity". In joint work with **Federico Cantero**, we have proven three main results demonstrating that the homology of configuration spaces on closed manifolds exhibits some more subtle kinds of stability despite these complicating factors.

(1) When the Euler characteristic of M is zero, we construct *replication maps* $C_n(M) \to C_{\lambda n}(M)$ for any integer $\lambda \geqslant 2$, and prove that they induce homological stability after inverting λ:

Theorem ([CP15]) *These maps induce isomorphisms on $H_i(-,\mathbb{Z}[\frac{1}{\lambda}])$ in the range $2i \leqslant \lambda$.*

Note. A construction related to our replication maps has also been used in the paper [Ber+06], in which they use something similar to a replication map in §3 to build a crossed simplicial group out of the configuration spaces on any given manifold M that admits a non-vanishing vector field.

(2) When the manifold M is odd-dimensional, the configuration spaces $C_n(M)$ do in fact satisfy homological stability after inverting 2 in the coefficients.

Theorem ([CP15]) *When* $\dim(M)$ *is odd, there are isomorphisms*

$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}]) \qquad \text{and} \qquad H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$

in the range $2i \leqslant n$*, induced by a zigzag of maps.*

This strengthens a result of [BM14].

(3) When the manifold M is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many authors [BCT89; ML88; Chu12; Ran13; BM14; Knu17] that homological stability holds for $C_n(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_n(S^2);\mathbb{F}) \cong \begin{Bmatrix} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{Bmatrix} \quad \text{for } n \geqslant 2.$$

From this example we see that the first homology of $C_n(S^2)$ is not stable, but it is p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p.

Theorem ([CP15]) *Suppose that* $\dim(M)$ *is even. For each fixed i, the sequence*

$$H_i(C_n(M);\mathbb{F}) \quad \text{for } n \geqslant 2i \tag{6}$$

is p^{a+1}*-periodic and takes on at most* $a+2$ *values. Moreover, if* $\chi \equiv 1 \bmod p$ *then the above sequence is* 1*-periodic, i.e. homological stability holds with coefficients in* $\mathbb{F}$*.*

The p^{a+1}-periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of p depending on i rather than on χ. This result was later improved by [KM16] to p-periodicity, independent of i or χ. Combining this with (a slightly more precise statement of) our result, a corollary is that in fact the sequence (6) above takes on only two different values.

1.1.5. Polynomial twisted homology

As well as the notion of abelian homological stability mentioned in §1.1.1, another sense in which a sequence of spaces (or groups) can satisfy *twisted homological stability* is with respect to a so-called *polynomial twisted coefficient system*. This consists of a choice of local coefficient system on each space X_n in the sequence, together with additional morphisms of local coefficient systems between them, organised into a functor $\mathscr{C} \to \mathsf{Ab}$, where $\mathscr{C}$ is a certain category whose automorphism groups are the fundamental groups $\pi_1(X_n)$. To be polynomial, the *degree* of this functor (defined using the intrinsic structure of $\mathscr{C}$) is required to be finite.

Many families of groups $G = \{G_n\}$ are known to be homologically stable in this sense (where we set $X_n = BG_n$ in the above paragraph), for appropriately-defined categories $\mathscr{C}_G$, for example the symmetric groups $\mathfrak{S}_n$, braid groups β_n, general linear groups, automorphism groups of free groups $\mathrm{Aut}(F_n)$ and mapping class groups of surfaces and of 3-manifolds.[1] In [Pal18b], I proved the first such result for a sequence of *spaces*, namely the unordered configuration spaces $C_n(M)$ on any connected, open manifold M. Note that, when $\dim(M) \geqslant 3$, these spaces are not aspherical (in contrast to the case of surfaces), so this does not reduce to a statement about the homology of their fundamental groups. When $\dim(M) = 2$, these configuration spaces are aspherical, so the result may be thought of as twisted homological stability for their fundamental groups, the surface braid groups $B_n(M) = \pi_1(C_n(M))$.

Theorem ([Pal18b]) *Let M be an open, connected manifold and let T be a twisted coefficient system for* $\{C_n(M)\}$*. This includes in particular the data of a local coefficient system* T_n *for each* $C_n(M)$*, as well as a homomorphism*

$$H_*(C_n(M);T_n) \longrightarrow H_*(C_{n+1}(M);T_{n+1}).$$

This homomorphism is split-injective in all degrees, and, if T has finite degree d, it is an isomorphism in the range $* \leqslant \frac{n-d}{2}$*.*

This theorem also generalises to configuration spaces with labels in any path-connected space X.

[1] Symmetric groups: [Bet02]; braid groups: [CF13; RW17; Pal18b]; general linear groups: [Dwy80; Kal80]; automorphism groups of free groups: [RW17]; mapping class groups of surfaces: [Iva93; CM09; Bol12; RW17]; mapping class groups of 3-manifolds: [RW17]. Note that these are references for the proofs of *twisted* homological stability; in each case, homological stability with untwisted coefficients was known earlier.

1.1.6. Motivic cohomology

In joint work with **Geoffroy Horel**, we have proven stability for the *motivic and étale cohomology* of configuration schemes on a smooth scheme X. In more detail, if X is a smooth scheme over a number field K, one may ask whether the corresponding unordered configuration schemes $C_n(X)$ exhibit stability in their étale or motivic cohomology – lifting the classical stability for the singular (co)homology of the complex manifolds $C_n(X_{\mathbb{C}})$, where $X_{\mathbb{C}}$ denotes the complex points of X.

Theorem ([HP20]) *Suppose that X may be written as $Y - D$, where Y is a smooth scheme over K and $D \subset Y$ is a smooth, closed subscheme that has a K-point. Assume that the étale motive of X is mixed Tate and that Y is geometrically connected. Then there are maps of étale motivic cohomology groups*

$$H^{p,q}_{et}(C_{n+1}(X);\Lambda) \longrightarrow H^{p,q}_{et}(C_n(X);\Lambda)$$

that are isomorphisms for $p \leqslant n/2$ and under mild conditions on the coefficient ring Λ. In the case when $X = \mathbb{A}^d$ is affine space, there are analogous maps of motivic cohomology groups

$$H^{p,q}(C_{n+1}(\mathbb{A}^d);\Lambda) \longrightarrow H^{p,q}(C_n(\mathbb{A}^d);\Lambda)$$

that are isomorphisms for $p \leqslant n/2 - 1$ and any coefficient ring Λ.

1.1.7. Point-pushing actions

Following on from §1.1.2, where we discussed homological stability for *configuration-section spaces*, let us now assume that the bundle $E \to M$ is trivial with fibre X; in this case we write $\mathrm{CMap}^{c,*}_k(M;X)$ and call these *configuration-mapping spaces*. They fit into a natural fibre sequence

$$\mathrm{Map}^{c,*}(M \smallsetminus k\text{ points}, X) \longrightarrow \mathrm{CMap}^{c,*}_k(M;X) \longrightarrow C_k(\mathring{M}) \tag{7}$$

(the right-hand map is a "*configuration-mapping bundle*"), which is obtained functorially from the universal fibre sequence

$$M \smallsetminus k\text{ points} \longrightarrow U_k(M) \longrightarrow C_k(\mathring{M}). \tag{8}$$

In [PT20], we obtain explicit formulas, when the dimension of M is at least 3, for the monodromy action $\pi_1(C_k(M)) \to \pi_0(\mathrm{hAut}(M \smallsetminus k\text{ points}))$ of the universal fibre sequence (8), from which we also deduce explicit formulas for the monodromy action of the fibre sequence (7). A special case of this is as follows:

Theorem ([PT20]) *If $d = \dim(M) \geqslant 3$ and M satisfies at least one of the following conditions:*

- *M is simply-connected, or*
- *the handle-dimension of M is at most $d-2$;*

then the "point-pushing" action of $\gamma = (\alpha_1,\dots,\alpha_k;\sigma) \in \pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$ on the mapping space $\mathrm{Map}^{c,*}(M \smallsetminus z, X) \simeq \mathrm{Map}_*(M,X) \times (\Omega^{d-1}_c X)^k$ is given as follows:

$$(\alpha_1,\dots,\alpha_k;\sigma)\cdot(f,g_1,\dots,g_k) = (f,\bar{g}_1,\dots,\bar{g}_k), \tag{9}$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$, and

- *for an element $\alpha \in \pi_1(M)$ we write $\mathrm{sgn}(\alpha) = +1$ if α lifts to a loop in the orientation double cover of M and $\mathrm{sgn}(\alpha) = -1$ otherwise,*
- *$\pi_1(X)$ acts up to homotopy on $\Omega^{d-1}_c X$ in the natural way,*
- *$\{\pm 1\}$ acts on $\Omega^{d-1}_c X$ through the involution given by precomposition with a reflection of S^{d-1} in a hyperplane containing the basepoint.*

As a corollary, we obtain a precise description of the set of path-components $\pi_0(\mathrm{CMap}^{c,*}_k(M,X))$ of the configuration-mapping space, under the above conditions.

In addition, we investigate the question of injectivity of the point-pushing maps. We show that, up to isomorphism, the kernel of the point-pushing map is independent of k regardless whether diffeomorphisms, homeomorphisms or homotopy equivalences are considered. Precisely, we consider the point-pushing maps

$$p_k \colon \pi_1(C_k(M)) \longrightarrow \pi_0(\mathrm{Cat}(M,z))$$
$$p_{k,\partial} \colon \pi_1(C_k(M)) \longrightarrow \pi_0(\mathrm{Cat}_\partial(M,z))$$

where $\mathrm{Cat} \in \{\mathrm{hAut}, \mathrm{Homeo}, \mathrm{Diff}\}$, and where $_\partial$ means that the boundary of M is fixed, and prove:

Theorem ([PT20, §8]) *When the dimension of M is at least* 3*, we have*

$$\ker(p_k) = \Delta(\ker(p_1)),$$

i.e. the kernel of p_k is equal to the diagonal of $\ker(p_1)^k \subseteq \pi_1(M)^k \subseteq \pi_1(C_k(M))$*, where we identify $\pi_1(C_k(M))$ with $\pi_1(M)^k \rtimes \Sigma_k$. If $\partial M \neq \varnothing$, then $p_{k,\partial}$ is injective.*

1.2. Higher-dimensional configurations

Instead of configurations of points (closed 0-dimensional submanifolds) in M, one may consider configurations of *higher-dimensional* closed submanifolds of M that are diffeomorphic to the disjoint union of finitely many copies of a fixed ("model") manifold L.

In this setting, I proved (see §1.2.1) that *moduli spaces of disconnected submanifolds* of this kind are homologically stable as the number of components goes to infinity, just as in the classical setting of points [Seg73; McD75; Seg79] – under a restriction on the relative dimensions of the manifolds involved. As a corollary, I proved (see §1.2.2) a homological stability result for moduli spaces of manifolds with Baas-Sullivan singularities.

1.2.1. Moduli spaces of disconnected submanifolds

We first define the moduli spaces under consideration precisely. Let $\bar{M}$ be a connected manifold with non-empty boundary and of dimension at least 2, and denote its interior by M. Also fix a closed manifold L and an embedding $\iota_0 \colon L \hookrightarrow \partial\bar{M}$. Choose a self-embedding $e \colon \bar{M} \hookrightarrow \bar{M}$ that is isotopic to the identity and such that $e(\iota_0(L))$ is contained in the interior $M \subset \bar{M}$. We then obtain a sequence of pairwise-disjoint embeddings of L into M by defining $\iota_n := e^n \circ \iota_0$ for $n \geqslant 0$. Intuitively, this is just a sequence of "parallel" copies of $\iota_0(L)$ in a tubular neighbourhood of the boundary of $\bar{M}$.

Let $nL = \{1,\dots,n\} \times L$ and write $\iota_{1,\dots,n} \colon nL \hookrightarrow M$ for the embedding $(i,x) \mapsto \iota_i(x)$.

Definition Define $C_{nL}(M)$ to be the path-component of

$$\mathrm{Emb}(nL, M)/\mathrm{Diff}(nL)$$

containing $[\iota_{1,\dots,n}]$. Here, the embedding space is given the Whitney topology and $C_{nL}(M)$ the quotient topology. There is a natural *stabilisation map*

$$C_{nL}(M) \longrightarrow C_{(n+1)L}(M) \qquad (10)$$

defined by

- adjoining the embedding ι_0 to a given embedding $nL \hookrightarrow M$, to obtain a new embedding $(n+1)L \hookrightarrow \bar{M}$,
- composing with the self-embedding e to push this into the interior M of $\bar{M}$.

In symbols, this is $[\phi] \mapsto [\phi^+]$, where $\phi^+(i,x) = e \circ \phi(i,x)$ for $1 \leqslant i \leqslant n$ and $\phi^+(n+1,x) = \iota_1(x)$.

Theorem ([Pal21]) *Assume that* $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$*. Then* (10) *induces isomorphisms on homology in degrees $* \leqslant \frac{n-2}{2}$ and surjections in degrees $* \leqslant \frac{n}{2}$.*

The theorem proven in [Pal21] is in fact a more general version of this theorem, for moduli spaces of submanifolds where each copy of $L \subset M$ in a configuration:

- is parametrised modulo a fixed subgroup of $\mathrm{Diff}(L)$;
- comes equipped with labels in a fixed bundle over the embedding space $\mathrm{Emb}(L, M)$.

1.2.2. Manifolds with Baas-Sullivan singularities

In the sequel [Pal18a], I used homological stability for the moduli spaces $C_{nL}(M)$ (and their more refined versions mentioned just after the theorem above) to prove homological stability for:

- *Symmetric diffeomorphism groups*, with respect to *parametric connected sum*. (Given embeddings $L \hookrightarrow M$ and $L \hookrightarrow Q$ with isomorphic normal bundles, their parametric connected sum $M \sharp_L Q$ is the result of cutting out a tubular neighbourhood of each embedding and gluing the resulting boundaries. If L is a point this is the ordinary connected sum. Other examples of this operation are surgery and Dehn surgery of 3-manifolds.) This generalises a result of Tillmann [Til16], which corresponds to the case $L =$ point (i.e. the usual connected sum operation).

- *Diffeomorphism groups of manifolds with conical singularities*, with respect to the number of singularities.

In more detail for the second point, let us fix an $(m-1)$-dimensional manifold T. Let $\mathrm{cone}(T) = (T \times [0,\infty))/(T \times \{0\})$ be the open cone on T. An m-dimensional *manifold with conical T-singularities* is a space M that is locally homeomorphic to $\mathrm{cone}(T)$, together with a smooth atlas on the subset $M_{mfd} \subseteq M$ of locally Euclidean points of M. (This is a special case of a *manifold with Baas-Sullivan singularities*.) A *diffeomorphism* of M is a homeomorphism $M \to M$ that restricts to a diffeomorphism $M_{mfd} \to M_{mfd}$ and is of the form $\mathrm{cone}(\varphi)$ for some diffeomorphism $\varphi \colon T \to T$ near each point of the discrete subset $M \smallsetminus M_{mfd} \subseteq M$. These form a subgroup

$$\mathrm{Diff}^T(M) \leqslant \mathrm{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood $\mathscr{T}(L)$ of any submanifold $L \subset M$. The quotient $M_L = M/\mathscr{T}(L)$ is then a manifold with a single conical $\partial\mathscr{T}(L)$-singularity. In the setting of §1.2.1, we may collapse a tubular neighbourhood of each submanifold $\iota_i(L) \subset M$ for $1 \leqslant i \leqslant n$, to obtain a manifold $M_{n\cdot L}$ with precisely n conical $\partial\mathscr{T}(L)$-singularities.

Theorem ([Pal18a]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence of classifying spaces $B\mathrm{Diff}^{\partial\mathscr{T}(L)}(M_{n\cdot L})$ is homologically stable as $n \to \infty$.*

1.3. Big mapping class groups

Connected, compact, orientable surfaces are classified by their genus and number of boundary components; in particular there are countably many such surfaces. If we remove the assumption of compactness (but still assume second countability and require the boundary to be compact), surfaces are classified by:

- their genus (which may be a non-negative integer or ∞);
- their number of boundary components (a non-negative integer);
- their *space of ends* E (a space that is homeomorphic to a closed subset of the Cantor set $\mathscr{C}$);
- their space of *non-planar ends* E_{np} (a closed subset of E).

The classification [Ker23; Ric63] says that homeomorphism classes of surfaces are in one-to-one correspondence with choices of the above list of data, with the single restriction that $E_{np} = \varnothing$ if and only if the genus is finite. In particular, there are *uncountably many* such surfaces.

A surface has *finite type* if its fundamental group is finitely generated (this occurs if and only if both its genus and its space of ends are finite); otherwise it has *infinite type*. Examples of infinite type surfaces include:

- the sphere minus a Cantor set (genus zero; no boundary; $E = \mathscr{C}$; $E_{np} = \varnothing$);
- the colimit of the compact surfaces $\Sigma_{g,1}$ as $g \to \infty$ (infinite genus; no boundary; $E = E_{np} = \{*\}$);
- the "flute surface" $\mathbb{C} \smallsetminus \mathbb{Z}$ (genus zero; no boundary; $E = [0,\omega]$, $E_{np} = \varnothing$).

The space of ends in the last example is the closed ordinal space $[0,\omega]$, in other words the ordinal $\omega+1$ in the order topology, which is homeomorphic to the subspace $\{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{C}$.

The mapping class group $\mathrm{Map}(S)$ of a surface S is countable if and only if S has finite type. Mapping class groups of infinite type surfaces (which are thus always uncountable discrete groups) are often called *big mapping class groups*. See [AV20] for a recent survey.

1.3.1. Homological stability and acyclicity

The degree-one homology (i.e. abelianisation) of big mapping class groups is known in some cases. For example, if Σ is a *finite* type surface and $\mathscr{C} \subset \Sigma$ is a subspace homeomorphic to the Cantor set, the natural map

$$\mathrm{Map}(\Sigma \smallsetminus \mathscr{C}) \longrightarrow \mathrm{Map}(\Sigma)$$

(given by extending homeomorphisms of $\Sigma \smallsetminus \mathscr{C}$ to Σ in the unique possible way) induces an isomorphism on $H_1(-)$ [CC22]. In the special case of the 2-sphere, it is also known that $H_2(\mathrm{Map}(S^2 \smallsetminus \mathscr{C})) \cong \mathbb{Z}/2$ [CC21].

In joint work with **Xiaolei Wu**, we gave the first *complete calculation of the homology of big mapping class groups in all degrees*. To state our results, we first need to introduce some notation and constructions.

Definition Let Σ be any surface with empty boundary. For $n \geqslant 1$ write $\Sigma^{(n)}$ for the surface obtained by removing the interiors of n pairwise disjoint discs from Σ. Then:

- $\mathfrak{B}(\Sigma)$ is the result of gluing together infinitely many copies of $\Sigma^{(3)}$ in a binary tree pattern;

- $\mathfrak{L}(\Sigma)$ is the result of gluing together infinitely many copies of $\Sigma^{(2)}$ in a linear pattern.

Note that both $\mathfrak{B}(\Sigma)$ and $\mathfrak{L}(\Sigma)$ have one boundary component. Write $\overline{\mathfrak{B}}(\Sigma)$ and $\overline{\mathfrak{L}}(\Sigma)$ respectively to denote the result of filling the boundary component with a disc.

Examples Some key examples are:
- $\mathfrak{L}(S^2) \cong D^2 \smallsetminus \{0\}$;
- $\mathfrak{B}(S^2) \cong D^2 \smallsetminus \mathscr{C}$;
- $\overline{\mathfrak{L}}(\mathbb{C}) \cong \mathbb{C} \smallsetminus \mathbb{Z}$ (the *flute surface*);
- $\overline{\mathfrak{L}}(T^2)$ is the colimit of the compact surfaces $\Sigma_{g,1}$ as $g \to \infty$ (the *Loch Ness monster surface*).

Theorem ([PW22b]) *For any surface Σ, the mapping class group $\mathrm{Map}(\mathfrak{B}(\Sigma))$ is acyclic, i.e.*

$$\widetilde{H}_*(\mathrm{Map}(\mathfrak{B}(\Sigma))) = 0.$$

As a consequence, we also have

$$H_i(\mathrm{Map}(\overline{\mathfrak{B}}(\Sigma) \smallsetminus \{*\})) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases}$$

In particular, when $\Sigma = S^2$ the second part of this theorem gives a complete calculation of the homology of the mapping class group of the plane minus a Cantor set $\mathbb{C} \smallsetminus \mathscr{C}$.

A key ingredient in the proof is a homological stability result for big mapping class groups:

Theorem ([PW22b]) *Let A be a connected surface with one boundary component and let Σ be a connected surface with empty boundary. Then the sequence of big mapping class groups*

$$\mathrm{Map}(A \natural \mathfrak{B}(\Sigma)^{\natural n})$$

is homologically stable as $n \to \infty$, where $\natural$ denotes boundary connected sum. The same statement is true for the sequence

$$\mathrm{Map}(A \natural \mathfrak{L}(\Sigma)^{\natural n})$$

under a mild condition on the end space E of Σ.

The condition on the end space E of Σ in the above theorem is the following. First notice that the end space of $\mathfrak{L}(\Sigma)$ is $(E\omega)^+$, where $E\omega$ denotes the disjoint union of countably infinitely many copies of E and $(-)^+$ denotes one-point compactification. The condition is that the point at infinity of $(E\omega)^+$ must be topologically distinguished, i.e. *not* locally homeomorphic to any other point of $(E\omega)^+$.

1.3.2. Uncountability

In a contrasting direction, in further joint work with **Xiaolei Wu**, we have proven that other families of big mapping class groups have *uncountable* homology in all positive degrees.

Theorem ([PW22a]) *Let Σ be a connected surface with empty boundary. Assume that Σ has genus zero and that its space of ends has a topologically distinguished point. Then there is an embedding of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(\overline{\mathfrak{L}}(\Sigma))),$$

where $\mathfrak{c}$ is the cardinality of the continuum and $\Lambda^*(-)$ denotes the exterior algebra on an abelian group. In particular, the homology of the mapping class group of $\overline{\mathfrak{L}}(\Sigma)$ is uncountable in all positive degrees.

In particular, when $\Sigma = \mathbb{C}$ this theorem proves uncountability in all degrees of the homology of the mapping class group of the flute surface $\mathbb{C} \smallsetminus \mathbb{Z}$.

We note that the assumption in the above theorem that the space of ends of Σ has a topologically distinguished point is essential. Without this assumption we could set $\Sigma = S^2 \smallsetminus \mathscr{C}$, in which case

$$\overline{\mathfrak{L}}(S^2 \smallsetminus \mathscr{C}) \cong S^2 \smallsetminus \mathscr{C},$$

but it is known by [CC21; CC22] that the homology of $\mathrm{Map}(S^2 \smallsetminus \mathscr{C})$ is *not* uncountable in degrees 1 and 2 (is is 0 and $\mathbb{Z}/2$ respectively).

Moreover, we also prove in [PW22a] a similar uncountability result for *pure* mapping class groups. Any self-homeomorphism φ of a surface Σ induces a self-homeomorphism of its space of ends E, which depends on φ only up to isotopy, so we have a homomorphism

$$\mathrm{Map}(\Sigma) \longrightarrow \mathrm{Homeo}(E), \tag{11}$$

whose image is the subgroup of all homeomorphisms of E that send $E_{np} \subseteq E$ onto itself.

Definition The pure mapping class group $\mathrm{PMap}(\Sigma)$ is the kernel of (11).

Theorem ([PW22a]) *Let Σ be any infinite type surface. Then the homology*

$$H_*(\mathrm{PMap}(\Sigma))$$

is uncountable in all positive degrees. In fact, it contains an embedded copy of either

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \qquad \text{or} \qquad \Lambda^*(\mathbb{Z}^{\mathbb{N}}).$$

Pure vs. full mapping class groups. Notice that the theorem above applies to *all* pure mapping class groups $\mathrm{PMap}(S)$ of infinite type surfaces, whereas the full mapping class groups $\mathrm{Map}(S)$ of infinite type surfaces exhibit very different behaviours depending on the surface S: they are acyclic when $S = \mathfrak{B}(\Sigma)$ (§1.3.1) but their homology is uncountable in all positive degrees when $S = \overline{\mathfrak{L}}(\Sigma)$ for a genus-zero surface Σ with a topologically-distinguished end.

The proofs of these two theorems ([PW22a]) build on work of [APV20], [Dom20] and [MT21].

2. Fundamental groups of moduli spaces

Examples of fundamental groups of moduli spaces include the classical braid groups $\mathbf{B}_n = \pi_1(C_n(\mathbb{R}^2))$ and their relatives (loop/welded braid groups $\mathbf{wB}_n = C_{nS^1}(\mathbb{R}^3)$, surface braid groups $\mathbf{B}_n(S) = \pi_1(C_n(S))$, etc.) as well as *mapping class groups* $\mathrm{Map}(M) = \pi_0(\mathrm{Homeo}(M)) = \pi_1(B\mathrm{Homeo}(M))$. I have studied these groups from the point of view of their lower central series (§2.1) and their representation theory (§2.2).

2.1. Lower central series

One of the most basic objects one needs to understand when studying the structure of a group G is its lower central series $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$, defined recursively by

$$\Gamma_{i+1}(G) = [\Gamma_i(G), G] = \{ghg^{-1}h^{-1} \mid g \in \Gamma_i(G), h \in G\}.$$

If G is perfect, its lower central series is completely trivial. On the other hand, if it is nilpotent or residually nilpotent, the filtration $\Gamma_*(G)$ and its associated graded Lie ring $\mathscr{L}_*(G) = \bigoplus_i \Gamma_i(G)/\Gamma_{i+1}(G)$ contain deep information about the structure of G. The lower central series is also deeply connected to the structure of the group ring of G.

The amount of information one can hope to extract from the study of a lower central series depends in the first place on whether or not it *stops*, meaning that there exists an integer $i \geqslant 1$ such that $\Gamma_i(G) = \Gamma_{i+1}(G)$. If there is such an integer, then the smallest such integer is the *length* of the lower central series of G.

In joint work with **Jacques Darné** and **Arthur Soulié**, we give a complete answer to the question of the length (finite or infinite) of the lower central series of surface braid groups, virtual braid groups and loop/welded braid groups, as well as *partitioned* versions of all of these groups. The answer depends subtly on the number of strands, how they are partitioned and the topology of the underlying surface. For example:

Theorem ([DPS22]) *For $n \geqslant 3$, the lower central series of:*

- *$B_n(S)$ has length 2 if S is planar or non-orientable;*
- *$B_n(S)$ has length 3 if S is non-planar and orientable;*
- *$B_{(2,n)}(\mathbb{R}^2)$ has length ∞;*
- *$B_{(2,n)}(\mathbb{S}^2)$ has length $\nu_2(n) + 2 + \varepsilon$, where $\nu_2(n)$ is the 2-adic valuation of n and $\varepsilon \in \{0, \pm 1\}$.*

For a complete answer, see the tables on pages xi–xiii of [DPS22].

2.2. Representation theory

The representation theory of motion groups (braid groups and their relatives) and mapping class groups is known to be "wild" (roughly speaking, this means that there is no classification schema with finitely many parameters of their irreducible representations). Although this means that there is no chance of understanding the totality of their representations, it is of great interest to construct and understand the behaviour of as wide a range of their representations as possible. Since the origin of these groups – as fundamental groups of moduli spaces – is topological, the most natural strategy is to use topology to construct representations, so that one may then use topological tools to study them.

Linearity. A unifying question in the representation theory of groups is whether a given group G is *linear*, i.e. whether it admits a faithful representation on a finite-dimensional vector space. Thus one motivation for constructing new representations of motion groups and mapping class groups is the linearity question: the search for faithful representations on finite-dimensional vector spaces. The answer to this question is known to be *yes* for the classical braid groups, by a celebrated result of Bigelow [Big01] and Krammer [Kra02], but for almost all other surface braid groups, loop braid groups and mapping class groups of surfaces, the question is wide open.

TQFTS. In the case of mapping class groups, their representation theory is closely connected to *topological quantum field theories* (TQFTs). The data of an n-dimensional TQFT includes representations of mapping class groups of $(n-1)$-dimensional manifolds, so constructing representations of mapping class groups of surfaces may be viewed as a first step towards constructing 3-dimensional TQFTs.

Motivated by these questions, I have worked on understanding the structure of and relations between the different versions of the Lawrence-Bigelow representations of the classical braid groups (§2.2.1), as well as developing new, topological constructions of representations of motion groups and mapping class groups (§2.2.2–§2.2.5).

2.2.1. Lawrence-Bigelow representations

In joint work with **Cristina Anghel**, we have investigated the fundamental relationships (in terms of non-degenerate pairings, embeddings and isomorphisms) between the many different flavours of homological representations of mapping class groups – with the *Lawrence-Bigelow representations* of the braid groups being our motivating example. Understanding the relationships between these representations is important for applications in quantum topology (see for example [Ang20]).

For any surface Σ equipped with a decomposition $\partial_{\mathrm{in}}\Sigma \cup \partial_{\mathrm{out}}\Sigma$ of its boundary (a *surface triad*), we think of $\partial_{\mathrm{in}}\Sigma$ as its *inner* "free" boundary and $\partial_{\mathrm{out}}\Sigma$ as its *outer* "fixed" boundary, and consider the mapping class group

$$\mathrm{Map}(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma, \partial_{\mathrm{out}}\Sigma)).$$

For example, if $\Sigma = \Sigma_{0,n+1}$ with n of its boundary-components considered as *inner* and one considered as *outer*, this is isomorphic to the braid group on n strands. Choosing a local system $\mathscr{L}$ (defined over a ring R) on the configuration space $C_k(\Sigma)$ that is preserved by the action of $\mathrm{Map}(\Sigma)$, we obtain homological $\mathrm{Map}(\Sigma)$-representations

$$H^\bullet(C_\circ),$$

where $C_\circ$ is either $C_{\mathrm{out}} = C_k(\Sigma \smallsetminus \partial_{in}\Sigma)$ or $C_{\mathrm{in}} = C_k(\Sigma \smallsetminus \partial_{out}\Sigma)$ and $H^\bullet$ denotes one of

- ordinary homology H, twisted by $\mathscr{L}$,
- homology relative to the boundary H^∂, twisted by $\mathscr{L}$,
- locally-finite (Borel-Moore) homology H^{lf}, twisted by $\mathscr{L}$,
- locally-finite homology of the associated covering space $\widetilde{C}_k(\Sigma) \to C_k(\Sigma)$ (in the case where $\mathscr{L}$ arises from such a covering), denoted $H^{lf,\sim}$.

For example, when $\Sigma = \Sigma_{0,n+1}$ we may take $\mathscr{L}$ to be the local system arising from the covering corresponding to the kernel of the homomorphism

$$\varphi\colon \pi_1(C_k(\Sigma)) \longrightarrow \begin{cases} \mathbb{Z}\{c\} & k=1 \\ \mathbb{Z}\{c,x\} & k \geqslant 2 \end{cases}$$

given by $\varphi(\gamma) = ic + jx$, where i is the total winding number of γ around the n inner boundary components and j is the *writhe* of the braid obtained from γ by filling in the n inner boundary components with discs. In this example the B_n-representations above are 8 different flavours of the *Lawrence-Bigelow representations*.

Theorem ([AP20]) (i) *For* $\circ \in \{\mathrm{in}, \mathrm{out}\}$, *there is a non-degenerate pairing*

$$H^{lf}(C_\circ) \otimes H^{\partial}(C_\circ) \longrightarrow R.$$

Moreover, these representations are free as R-modules. We describe explicit bases such that the pairing above is given by the identity matrix.

(ii) *Under a mild condition on the local system* $\mathscr{L}$, *there are embeddings of representations*

$$H^{\partial}(C_{\mathrm{in}}) \longrightarrow H^{lf}(C_{\mathrm{out}})$$
$$H^{\partial}(C_{\mathrm{out}}) \longrightarrow H^{lf}(C_{\mathrm{in}}).$$

When $k \geqslant 2$ *this implies that* $H^{lf}(C_{\mathrm{out}})$ *and* $H^{lf}(C_{\mathrm{in}})$ *are reducible. In fact, with respect to the explicit bases that we describe, the matrices of these embeddings are diagonal, and their diagonal entries are products of quantum factorials in R.*

(iii) *For* $\circ \in \{\mathrm{in}, \mathrm{out}\}$, *there is a natural injective map*

$$H^{lf}(C_\circ) \longrightarrow H^{lf,\sim}(C_\circ).$$

If $\mathscr{B}$ *denotes a free basis for* $H^{lf}(C_\circ)$ *as a module over* $R = k[G]$, *then* $H^{lf,\sim}(C_\circ)$ *is a direct sum over* $\mathscr{B}$ *of copies of the completion* $k[[G]] = \prod_G k$ *of* $k[G] = \bigoplus_G k$.

2.2.2. Unified topological construction

In joint work with **Arthur Soulié**, we give a unified topological construction of representations of motion groups and mapping class groups. In more detail, in each dimension d, we construct a large family of representations of a category $\mathfrak{U}\mathscr{D}_d$ whose automorphism groups contain all mapping class groups and motion groups in dimension d. There are three parameters that one may vary in the construction:

- a submanifold $Z \subset \mathbb{R}^d$,
- two integers $\ell \geqslant 2$ and $i \geqslant 0$.

Each of these parameters may be varied to obtain interesting representations. For example, the family of Lawrence-Bigelow representations depends on an integer parameter $k \geqslant 1$; our construction recovers this in the case when Z is a 0-dimensional manifold of size k (and $d = 2$, $\ell = 2$, $i = k$). In dimensions $d \geqslant 3$, it becomes interesting to take higher-dimensional submanifolds of $\mathbb{R}^d$ for Z, in particular in the case of the *loop braid groups* (the group of motions of an n-component unlink in $\mathbb{R}^3$).

The parameter $\ell \geqslant 2$ controls the ground ring over which the representation is defined: it is the group-ring of a group of nilpotency class at most $\ell - 1$. There are cases where the output of our construction is independent of ℓ (hence the ground ring is commutative), but there are also many interesting cases where we obtain an *infinite tower of representations* as $\ell \to \infty$. In particular, the family of Lawrence-Krammer-Bigelow representations is the $\ell = 2$ term of such a tower; see §2.2.3 for more details.

The parameter $i \geqslant 0$ controls the degree in which we take homology. In the case of the Lawrence-Bigelow representations, there is only one interesting degree in which we can take homology (the homology in other degrees being trivial). However, more generally there can be many interesting degrees in which to take homology. For example, if we consider the family of loop braid groups and the "naive" analogue of the Lawrence-Bigelow representations (taking Z to be a 0-dimensional manifold of size k), then there are non-trivial homology groups in all degrees $k \leqslant i \leqslant 2k$.

There is also an iterative version of our construction, which recovers, in the case of the classical braid groups, the *Long-Moody construction* [Lon94].

As a sample of our construction, applied to the loop (="welded") braid groups $\mathbf{wB}_n$ and extended loop braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$, we have the following. The family of loop braid groups $\mathbf{wB}_n$ naturally forms a category $\mathfrak{U}\mathbf{w}\beta$, and similarly the family of extended loop braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$ forms a category $\mathfrak{U}\widetilde{\mathbf{w}}\beta$. For an integer $k \geqslant 1$, if we take either $Z = k$ or $Z = U_k$ (an unlink with k components), as well as $\ell = 2$ and $i = k$, we obtain:

Theorem ([PS19]) *Let $k \geqslant 2$. Our construction specialises to define representations:*

$$
\begin{aligned}
\mathfrak{L}_1(1,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}]} & \mathfrak{L}_1(1,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}/2]} \\
\mathfrak{L}_k(k,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]} & \mathfrak{L}_k(k,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^2]} \\
\mathfrak{L}_1(U_1,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^2]} & \mathfrak{L}_1(U_1,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^3]} \\
\mathfrak{L}_k(U_k,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]} & \mathfrak{L}_k(U_k,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^5]} \\
\mathfrak{L}_1^+(U_1,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2]} & \mathfrak{L}_1^+(U_1,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]} \\
\mathfrak{L}_k^+(U_k,\mathscr{L}\beta)\colon \mathfrak{Uw}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^3\oplus(\mathbb{Z}/2)]} & \mathfrak{L}_k^+(U_k,\mathscr{L}\beta')\colon \mathfrak{U}\widetilde{\mathbf{w}}\beta &\longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2\oplus(\mathbb{Z}/2)^2]}
\end{aligned}
$$

For example, this means that, if we take Z to be an unoriented unlink with two components, we obtain representations of all loop braid groups $\mathbf{wB}_n$ defined over the group-ring $\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]$, which may be thought of as a Laurent polynomial ring in 5 variables x_1,x_2,x_3,x_4,x_5 modulo the ideal generated by x_i^2-1 for $i=2,3,4,5$. If we quotient also by x_1^2-1, then these representations extend to the extended loop-braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n \supset \mathbf{wB}_n$.

The representations encoded by the first two functors $\mathfrak{L}_1(1,\mathscr{L}\beta)$ and $\mathfrak{L}_1(1,\mathscr{L}\beta')$ are studied in more detail, with explicit matrices, in [PS22a]; see §2.2.4 for more details.

2.2.3. Pro-nilpotent Lawrence-Krammer-Bigelow representation

Continuing on from §2.2.2, in further joint work with **Arthur Soulié**, we construct *pro-nilpotent towers of representations* of classical braid groups, surface braid groups and (extended and non-extended) loop braid groups. This involves considering all values $\ell \geqslant 2$ of the parameter ℓ of §2.2.2 simultaneously and applying some of the non-stopping results for lower central series from [DPS22] (see §2.1). In particular:

Theorem ([PS22b]) *There is a pro-nilpotent tower of representations of the classical braid groups $\mathbf{B}_n$ whose bottom layer is the Lawrence-Krammer-Bigelow representation.*

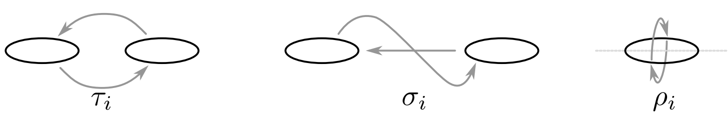
The limit as $l \to \infty$ of this tower of representations is a representation of $\mathbf{B}_n$ defined over the integral group ring $\mathbb{Z}[\mathbf{RB}_2]$, where $\mathbf{RB}_2$ is the two-strand ribbon braid group. This is a special case of a more general construction:

Theorem ([PS22b]) *For each $k \geqslant 2$, there is a representation of $\mathbf{B}_n$ defined over the integral group ring $\mathbb{Z}[\mathbf{RB}_k]$ that recovers the kth Lawrence-Bigelow representation, defined over $\mathbb{Z}[\mathbb{Z}^2]$, when reduced along the abelianisation $\mathbf{RB}_k \twoheadrightarrow (\mathbf{RB}_k)^{ab} = \mathbb{Z}^2$.*

In the case $k=2$, we compute explicit matrices, over the group ring $\mathbb{Z}[\mathbf{RB}_2]$, describing the action of the standard generators σ_i of $\mathbf{B}_n$. Since $\mathbf{RB}_2 = \mathbb{Z}^2 \rtimes \mathbb{Z}$, these give a non-commutative 3-variable enrichment of the classical Lawrence-Krammer-Bigelow representation.

2.2.4. Burau representations of loop braid groups

Loop (="welded") braid groups $\mathbf{wB}_n$ and $\widetilde{\mathbf{w}}\mathbf{B}_n$ appear in many guises in topology and group theory. They may be seen geometrically as fundamental groups of trivial links in $\mathbb{R}^3$, diagrammatically as equivalence classes of *welded braids* (closely related to virtual braids and virtual knot theory), algebraically as subgroups of automorphism groups of free groups or combinatorially via explicit group presentations. They are also related to physics via *exotic string statistics* [BWC07]. Generators for these groups are of the form:



Continuing on from §2.2.2, in further joint work with **Arthur Soulié**, we give a topological construction of the *Burau representations* of the loop braid groups, which are higher-dimensional analogues of the Burau representations of the classical braid groups. There are four versions: defined either on the non-extended loop braid groups $\mathbf{wB}_n$ or the extended loop braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$, and in each case there is an *unreduced* and a *reduced* version. Three are not surprising, and one could easily guess the correct matrices to assign to generators. However, the fourth is more subtle, and does not seem combinatorially obvious, although it its topological construction is very natural:

Theorem ([PS22a]) *The reduced Burau representation of the extended loop braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$, defined over the ring $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, acts on generators as described in Table 1.*

	$i = 1$	$2 \leqslant i \leqslant n-2$	$i = n-1$
τ_i	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$	$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -\delta \\ 0 & 0 & 1 \end{bmatrix}$
σ_i	$\begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$	$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & -\delta \\ 0 & 0 & 1 \end{bmatrix}$
	$i = 1$	$2 \leqslant i \leqslant n-1$	$i = n$
ρ_i	$\begin{bmatrix} -t & 0 & \cdots & 0 \\ -\delta & & & \\ \vdots & & \mathrm{I}_{n-1} & \\ -\delta & & & \\ 1 & & & \end{bmatrix}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \delta & -t & 0 & \cdots & 0 \\ \delta & -\delta & & & \\ \vdots & \vdots & & \mathrm{I}_{n-i} & \\ \delta & -\delta & & & \\ -1 & 1 & & & \end{bmatrix}$	$\mathrm{I}_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

Table 1 Matrices for the reduced Burau representation of the extended loop braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$. Notation: $\delta = 1 + t$. All entries lie in $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, except for the bottom row, where they lie in $S/(t-1) \cong \mathbb{Z}$ (in other words, we set $t = 1$ on the bottom row).

2.2.5. Heisenberg homology and representations of mapping class groups

In joint work with **Christian Blanchet** and **Awais Shaukat**, we have constructed analogues of the Lawrence-Bigelow representations (of the classical braid groups $\mathbf{B}_n$) for the mapping class groups $\mathrm{Map}(\Sigma_{g,1})$ for $g \geqslant 1$.

More precisely, we first show that for each $k \geqslant 2$ there is a natural quotient

$$\phi \colon \mathbf{B}_k(\Sigma_{g,1}) \longrightarrow\!\!\!\!\!\rightarrow \mathscr{H}(\Sigma_{g,1}), \tag{12}$$

where the *Heisenberg group* $\mathscr{H}(\Sigma_{g,1})$ is the central extension of $H_1(\Sigma_{g,1})$ classified by the intersection form. This determines a local system $\mathscr{L}$, defined over the group ring $\mathbb{Z}[\mathscr{H}(\Sigma_{g,1})]$, on the configuration space $C_k(\Sigma_{g,1})$. We also show that the kernel of (12) is preserved by the action of $\mathrm{Map}(\Sigma_{g,1})$, which implies that there is an action

$$\Phi \colon \mathrm{Map}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}(\mathscr{H}(\Sigma_{g,1})).$$

Notation. For any representation $\rho \colon \mathscr{H}(\Sigma_{g,1}) \to \mathrm{Aut}_R(V)$ and automorphism $\tau \in \mathrm{Aut}(\mathscr{H}(\Sigma_{g,1}))$ we denote by ${}_\tau V$ the τ-twisted representation $\rho \circ \tau$. Write $\Sigma'_{g,1}$ for the non-compact surface given by removing one point (equivalently, a closed interval) from the boundary of $\Sigma_{g,1}$. Borel-Moore homology is denoted by $H_*^{BM}(-)$.

Theorem ([BPS21]) *For any $k \geqslant 2$ and representation V of $\mathscr{H}(\Sigma_{g,1})$ over R, there is a twisted representation of $\mathrm{Map}(\Sigma_{g,1})$ on the collection of R-modules*

$${}_\tau W_k(V) := H_k^{BM}(C_k(\Sigma'_{g,1}); \mathscr{L} \otimes {}_\tau V) \qquad \text{for} \qquad \tau \in \mathrm{Aut}(\mathscr{H}(\Sigma_{g,1}))$$

where the action of $f \in \mathrm{Map}(\Sigma_{g,1})$ is

$${}_{\tau \circ \Phi(f)} W_k(V) \longrightarrow {}_\tau W_k(V).$$

This is a *twisted* representation in the sense that it involves a collection $\{ {}_\tau W_k(V) \mid \tau \in \mathrm{Aut}(\mathscr{H}(\Sigma_{g,1})) \}$ of R-modules that are mutually isomorphic – but not *canonically* isomorphic. Upgrading this to an *untwisted* representation (i.e. a representation in the usual sense) requires a consistent choice of identifications

$${}_{\tau \circ \Phi(f)} V \;\cong\; {}_\tau V$$

of coefficients for all f and τ. Depending on the representation V, this may sometimes be done directly and sometimes after passing to a central extension of the mapping class group.

Notation. Note that the left translation action of $\mathscr{H}(\Sigma_{g,1})$ on itself is affine with respect to its canonical structure as an affine space over $\mathbb{Z}$. Linearising this, we obtain a $\mathbb{Z}$-linear action on $L := \mathscr{H}(\Sigma_{g,1}) \oplus \mathbb{Z}$. The other specific representation of $\mathscr{H}(\Sigma_{g,1})$ that we consider is the *Schrödinger representation* V_{Sch}, which is a unitary representation over $\mathbb{C}$.

Theorem ([BPS21]) *When $V = L$ the twisted representation above may be untwisted to obtain a genuine representation*

$$\mathrm{Map}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(W_k(L)).$$

When $V = V_{\mathrm{Sch}}$ and we pass to the (stably) universal central extension $\widetilde{\mathrm{Map}}(\Sigma_{g,1})$ of $\mathrm{Map}(\Sigma_{g,1})$, the twisted representation above may be untwisted to obtain a genuine unitary representation

$$\widetilde{\mathrm{Map}}(\Sigma_{g,1}) \longrightarrow U(W_k(V_{\mathrm{Sch}})).$$

For any representation V, if we restrict to the Torelli group $\mathfrak{T}(\Sigma_{g,1}) \subseteq \mathrm{Map}(\Sigma_{g,1})$, the twisted representation above may be untwisted to obtain a genuine representation

$$\mathfrak{T}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_R(W_k(V)).$$

When $V = \mathbb{Z}[\mathscr{H}(\Sigma_{g,1})]$ we prove an upper bound on the kernel of our twisted representations:

Theorem ([BPS21]) *The kernel of the twisted representation $W_k(\mathbb{Z}[\mathscr{H}(\Sigma_{g,1})])$ of $\mathrm{Map}(\Sigma_{g,1})$ is contained in the intersection of the **Magnus kernel** and the kth term of the **Johnson filtration**.*

We also compute explicit matrices for the action of our twisted representation $W_k(V)$ when $g = 1$, $k = 2$ and

$$V = \mathbb{Z}[\mathscr{H}(\Sigma_{1,1})] \cong \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle / (ab = u^2 ba).$$

In particular, the Dehn twist around the boundary $T_\partial \in \mathrm{Map}(\Sigma_{1,1})$ acts by the 3×3 matrix in Figure 1.

$$\begin{pmatrix}
\begin{matrix} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{matrix} &
\begin{matrix} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{matrix} &
\begin{matrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+ \\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+ \\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+ \\ (-u^{-3}+u^{-4})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{matrix} &
\begin{matrix} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{matrix} &
\begin{matrix} (-u^{-6}+u^{-7})a^{-2}b+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+ \\ (-u^{-5}+u^{-6})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{matrix} &
\begin{matrix} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{matrix} &
\begin{matrix} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+ \\ (-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{matrix}
\end{pmatrix}$$

Figure 1 The action of $T_\partial \in \mathrm{Map}(\Sigma_{1,1})$ on $W_2(\mathbb{Z}[\mathscr{H}(\Sigma_{1,1})])$.

References

[Ang20] C. A.-M. Anghel. *Coloured Jones and Alexander polynomials as topological intersections of cycles in configuration spaces*. ArXiv:2002.09390. 2020 (↑ 12).

[AP20] C. A.-M. Anghel and M. Palmer. *Lawrence-Bigelow representations, bases and duality*. ArXiv:2011.02388. 2020 (↑ 13).

[APV20] J. Aramayona, P. Patel and N. G. Vlamis. *The first integral cohomology of pure mapping class groups*. *Int. Math. Res. Not. IMRN* 2020.22 (2020), pp. 8973–8996 (↑ 11).

[AV20] J. Aramayona and N. G. Vlamis. *Big mapping class groups: an overview*. *In the Tradition of Thurston: Geometry and Topology*. Ed. by K. Ohshika and A. Papadopoulos. Springer, 2020, pp. 459–496 (↑ 9).

[BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123 (↑ 6).

[Ber+06] A. J. Berrick, F. R. Cohen, Y. L. Wong and J. Wu. *Configurations, braids, and homotopy groups*. *J. Amer. Math. Soc.* 19.2 (2006), pp. 265–326 (↑ 5).

[Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) (↑ 6).

[Big01] S. J. Bigelow. *Braid groups are linear*. *J. Amer. Math. Soc.* 14.2 (2001), 471–486 (electronic) (↑ 2, 12).

[BM14] M. Bendersky and J. Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815 (↑ 6).

[Bol12] S. K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. *Math. Z.* 270.1-2 (2012), pp. 297–329 (↑ 6).

[BPS21] C. Blanchet, M. Palmer and A. Shaukat. *Heisenberg homology on surface configurations*. ArXiv:2109.00515. 2021 (↑ 15, 16).

[BWC07] J. C. Baez, D. K. Wise and A. S. Crans. *Exotic statistics for strings in 4D BF theory*. *Adv. Theor. Math. Phys.* 11.5 (2007), pp. 707–749 (↑ 14).

[CC21] D. Calegari and L. Chen. *Big mapping class groups and rigidity of the simple circle*. *Ergodic Theory Dynam. Systems* 41.7 (2021), pp. 1961–1987 (↑ 9, 11).

[CC22] D. Calegari and L. Chen. *Normal subgroups of big mapping class groups*. *Trans. Amer. Math. Soc. Ser. B* 9 (2022), pp. 957–976 (↑ 9, 11).

[CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314 (↑ 6).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504 (↑ 6).

[CLM76] F. R. Cohen, T. J. Lada and J. P. May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976, pp. vii+490 (↑ 5).

[CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76 (↑ 6).

[Coh+91] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram. *The topology of rational functions and divisors of surfaces*. *Acta Math.* 166.3-4 (1991), pp. 163–221 (↑ 5).

[CP15] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. *Doc. Math.* 20 (2015), pp. 753–805 (↑ 5, 6).

[Dom20] G. Domat. *Big pure mapping class groups are never perfect*. ArXiv:2007.14929v2. With an appendix by Ryan Dickmann and George Domat. To appear in Math. Res. Lett. 2020 (↑ 11).

[Don84] S. K. Donaldson. *Nahm's equations and the classification of monopoles*. *Comm. Math. Phys.* 96.3 (1984), pp. 387–407 (↑ 5).

[DPS22] J. Darné, M. Palmer and A. Soulié. *When the lower central series stops: a comprehensive study for braid groups and their relatives*. ArXiv:2201.03542. To appear in Memoirs of the AMS. 2022 (↑ 11, 14).

[Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. *Ann. of Math. (2)* 111.2 (1980), pp. 239–251 (↑ 6).

[EVW16] J. S. Ellenberg, A. Venkatesh and C. Westerland. *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*. *Ann. of Math. (2)* 183.3 (2016), pp. 729–786 (↑ 1, 4).

[Har85] J. L. Harer. *Stability of the homology of the mapping class groups of orientable surfaces*. *Ann. of Math. (2)* 121.2 (1985), pp. 215–249 (↑ 1).

[HP20] G. Horel and M. Palmer. *Motivic homological stability of configuration spaces*. ArXiv:2007.13718. To appear in the Bulletin of the London Mathematical Society. 2020 (↑ 7).

[Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (↑ 6).

[Kal80] W. van der Kallen. *Homology stability for linear groups*. *Invent. Math.* 60.3 (1980), pp. 269–295 (↑ 6).

[Ker23] B. von Kerékjártó. *Vorlesungen über Topologie I*. Vol. 8. Springer, Berlin, 1923 (↑ 9).

[KM16] A. Kupers and J. Miller. *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*. *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468 (↑ 6).

[Knu17] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. *Algebr. Geom. Topol.* 17.5 (2017), pp. 3137–3187 (↑ 6).

[Kra02] D. Krammer. *Braid groups are linear*. *Ann. of Math. (2)* 155.1 (2002), pp. 131–156 (↑ 2, 12).

[Kra19] M. Krannich. *Homological stability of topological moduli spaces*. *Geom. Topol.* 23.5 (2019), pp. 2397–2474 (↑ 3).

[KS22] C. Kottke and M. Singer. *Partial compactification of monopoles and metric asymptotics*. *Mem. Amer. Math. Soc.* 280.1383 (2022), pp. vii+110 (↑ 5).

[Lon94] D. D. Long. *Constructing representations of braid groups*. *Comm. Anal. Geom.* 2.2 (1994), pp. 217–238 (↑ 13).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ 2, 4, 8).

[ML88] R. J. Milgram and P. Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424 (↑ 6).

[MP15a] J. Miller and M. Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. *Q. J. Math.* 66.1 (2015), pp. 265–284 (↑ 4).

[MP15b] J. Miller and M. Palmer. *Scanning for oriented configuration spaces*. *Homology, Homotopy and Applications* 17.1 (2015), pp. 35–66 (↑ 4).

[MS76] D. McDuff and G. Segal. *Homology fibrations and the "group-completion" theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284 (↑ 4).

[MT21] J. Malestein and J. Tao. *Self-similar surfaces: involutions and perfection*. ArXiv:2106.03681. 2021 (↑ 11).

[Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328 (↑ 1).

[MW07] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford's conjecture*. *Ann. of Math. (2)* 165.3 (2007), pp. 843–941 (↑ 1).

[Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups*. ArXiv:1505.04294. 2015 (↑ 6).

[Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711 (↑ 3).

[Pal18a] M. Palmer. *Stability for moduli spaces of manifolds with conical singularities*. ArXiv:1807.07558. 2018 (↑ 8, 9).

[Pal18b] M. Palmer. *Twisted homological stability for configuration spaces*. *Homology, Homotopy and Applications* 20.2 (2018), pp. 145–178 (↑ 6).

[Pal21] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. *Algebr. Geom. Topol.* 21.3 (2021), pp. 1371–1444 (↑ 8).

[PS19] M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction*. ArXiv:1910.13423. 2019 (↑ 14).

[PS22a] M. Palmer and A. Soulié. *The Burau representations of loop braid groups*. *C. R. Math. Acad. Sci. Paris* 360 (2022), pp. 781–797 (↑ 14, 15).

[PS22b] M. Palmer and A. Soulié. *The pro-nilpotent Lawrence-Krammer-Bigelow representation*. ArXiv:2211.01855. 2022 (↑ 14).

[PT20] M. Palmer and U. Tillmann. *Point-pushing actions for manifolds with boundary*. ArXiv:2007.11613. 2020. To appear in Groups, Geometry, and Dynamics (↑ 7, 8).

[PT21] M. Palmer and U. Tillmann. *Configuration-mapping spaces and homology stability*. *Res. Math. Sci.* 8.3 (2021), Paper No. 38 (↑ 4, 5).

[PT22] M. Palmer and U. Tillmann. *Homology stability for asymptotic monopole moduli spaces*. ArXiv:2212.11799. 2022 (↑ 5).

[PW22a] M. Palmer and X. Wu. *Big mapping class groups with uncountable integral homology*. ArXiv:2212.11942. 2022 (↑ 10, 11).

[PW22b] M. Palmer and X. Wu. *On the homology of big mapping class groups*. ArXiv:2211.07470. 2022 (↑ 10).

[Ran13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326 (↑ 3, 6).

[Ric63] I. Richards. *On the classification of noncompact surfaces*. *Trans. Amer. Math. Soc.* 106 (1963), pp. 259–269 (↑ 9).

[RW17] O. Randal-Williams and N. Wahl. *Homological stability for automorphism groups*. *Adv. Math.* 318 (2017), pp. 534–626 (↑ 3, 6).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ 2, 4, 8).

[Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (↑ 2, 5, 8).

[Til16] U. Tillmann. *Homology stability for symmetric diffeomorphism and mapping class groups*. *Math. Proc. Cambridge Philos. Soc.* 160.1 (2016), pp. 121–139 (↑ 8).