ACADEMIA ROMÂNĂ
SCOSAAR

HABILITATION THESIS

Moduli spaces and their fundamental groups: homology, representations and lower central series

Martin Palmer-Anghel

Domeniul fundamental: Matematică și științe ale naturii

Domeniul de abilitare: Matematică

Teză elaborată în vederea obținerii atestatului de abilitare în scopul conducerii lucrărilor de doctorat în domeniul Matematică

BUCUREȘTI, 2024

Abstract

The overarching goal of my research so far has been to understand the topology of moduli spaces through algebraic invariants, primarily homology and fundamental groups. This thesis presents six different results, three concerned with the homology of moduli spaces and three concerned with studying the fundamental groups of moduli spaces (*motion groups* and *mapping class groups*) via their representations and their lower central series.

In the preliminary Chapter O, I first give a brief overview of my main research results since my PhD thesis. Chapters 1–6 then form the core of the thesis and develop six of these results in full detail.

Chapter 1 proves homological stability for two different flavours of asymptotic monopole moduli spaces, namely moduli spaces of *framed Dirac monopoles* and moduli spaces of *ideal monopoles*. The former are Gibbons-Manton torus bundles over configuration spaces whereas the latter are obtained from them by replacing each circle factor of the fibre with a monopole moduli space by the Borel construction. They form boundary hypersurfaces in a partial compactification of the classical monopole moduli spaces. These results follow from a general homological stability result for configuration spaces equipped with non-local data (*non-local configuration spaces*). This chapter corresponds to a joint paper with U. Tillmann [PT23] published in the *Proceedings of the Royal Society A*.

Moduli spaces of manifolds with marked points were proven in [Til16] to be homologically stable as the number of marked points goes to infinity. Chapter 2 generalises this result to *moduli spaces of manifolds with conical singularities*. (Marked points may be thought of as inessential conical singularities, since a disc neighbourhood of a marked point is the cone on its boundary sphere.) This is deduced as a special case of a more general homological stability result for classifying spaces of *symmetric diffeomorphism groups* of manifolds, with respect to *parametric connected sum*, an operation generalising ordinary connected sum and surgery (including Dehn surgery).

The key input for the proof of this result is homological stability for *moduli spaces of submanifolds* as the number of components of the submanifold goes to infinity, which was proven in my PhD thesis and published in [Pal21]. The relation to conical singularities is given by collapsing tubular neighbourhoods of submanifolds to isolated points. The results of Chapter 2 correspond to the preprint [Pal18a], which is submitted for publication.

Chapter 3 is concerned with "big mapping class groups", i.e. mapping class groups of *infinite-type surfaces*, and corresponds to a joint paper with X. Wu [PW22a] accepted for publication in *Documenta Mathematica*. In this chapter we prove that, for any infinite-type surface S, the integral homology of the closure of the compactly-supported mapping class group $\overline{\mathrm{PMap}_c(S)}$ and of the Torelli group $\mathcal{T}(S)$ is uncountable in every positive degree. By our earlier results in [PW22b], and other known computations, such a statement cannot be true for the *full* mapping class group $\mathrm{Map}(S)$ for *all* infinite-type surfaces S. However, we are still able to prove that the integral homology of $\mathrm{Map}(S)$ is uncountable in all positive degrees for a large class of infinite-type surfaces S. The key property of this class of surfaces is, roughly, that the space of ends of the surface S contains a limit point of topologically distinguished points. This result includes in particular all finite-genus surfaces having countable end spaces with a unique point of maximal Cantor-Bendixson rank α, where α is a successor ordinal.

Understanding the *lower central series* of a group is, in general, a difficult task. However, successfully computing the lower central series and the associated Lie algebras of a group or of some of its subgroups can lead to a deep understanding of the underlying structure of that group. The goal of Chapter 4 is to showcase several techniques aimed at carrying out part of this task. In particular, we seek to answer the following question: *when does the lower central series stop?* We introduce a number of tools to answer this question that we then apply to *partitioned surface braid groups* on any surface and with respect to any partition. The path from our general techniques to their application is far from straightforward, and a certain amount of tenacity is required to deal with all of the cases encountered along the way. We finally arrive at an answer to our question for every one of these groups, with the sole exception of one family of partitioned braid groups on the projective plane. In a number of cases, we even compute completely the lower central series. This chapter corresponds to a part of the monograph [DPS22b], joint with J. Darné and A. Soulié, which is accepted for publication in the *Memoirs of the American Mathematical Society*.

Next, turning from lower central series to *representations* of motion groups, in Chapter 5 we give a simple topological construction of the Burau representations of the loop braid groups. There are four versions: defined either on the non-extended or extended loop braid groups, and in each case there is an unreduced and a reduced version. Three are not surprising, and one could easily guess the correct matrices to assign to generators. The fourth is more subtle, and does not seem combinatorially obvious, although it is topologically very natural. This chapter corresponds to a joint paper with A. Soulié [PS22a] published in the *Comptes Rendus Mathématique*.

Chapter 6 is concerned with constructing homological representations of mapping class groups of surfaces, and corresponds to a joint paper with C. Blanchet and A. Shaukat [BPS23] accepted for publication in *Contemporary Mathematics*. In previous work with the same co-authors [BPS21], we constructed twisted representations of mapping class groups of surfaces, depending on a choice of representation V of the Heisenberg group $\mathcal{H}$. For certain V we were able to untwist these mapping class group representations. In this chapter, we study the restrictions of our twisted representations to different subgroups of the mapping class group. Notably, we prove that these representations may be untwisted on the *Torelli group* for any given representation V of $\mathcal{H}$. In the case when V is the Schrödinger representation, we also construct untwisted representations of subgroups defined as kernels of crossed homomorphisms studied by Earle and Morita.

In the final Chapter F, I describe various open problems and questions related to the topics of the thesis, some of which are immediately approachable and others of which are very difficult.

Rezumat

Scopul global al cercetării mele de până acum a fost de a înțelege topologia spațiilor de moduli prin invarianți algebrici, în primul rând omologie și grupuri fundamentale. Această teză prezintă șase rezultate diferite, trei se referă la omologia spațiilor de moduli și trei se referă la studierea grupurilor fundamentale ale spațiilor de moduli (*grupuri de deplasări* și *grupuri mapping class*) prin reprezentările lor și seriile lor centrale descendente.

În capitolul preliminar O, ofer mai întâi o scurtă prezentare generală a principalelor mele rezultate de cercetare de după teza mea de doctorat. Capitolele 1–6 formează apoi partea principală a tezei și dezvoltă șase dintre aceste rezultate în detaliu.

Capitolul 1 demonstrează stabilitatea omologică pentru două variante diferite de spații de moduli ale monopolurilor asimptotice, și anume spațiile de moduli ale *monopolurilor Dirac cu framing* și spațiile de moduli ale *monopolurilor ideale*. Primele sunt fibrați în toruri *Gibbons-Manton* peste spații de configurații, iar cele din urmă sunt obținute din ele prin înlocuirea fiecărui factor circular al fibrei cu un spațiu de moduli ale monopolurilor prin construcția Borel. Ele sunt hipersuprafețe limită într-o compactificare parțială a spațiilor clasice de moduli ale monopolurilor. Aceste rezultate rezultă dintr-un rezultat general de stabilitate omologică pentru spațiile de configurații echipate cu date non-locale (*spații de configurații non-locale*). Acest capitol corespunde unei lucrări comune cu U. Tillmann [PT23] publicată în *Proceedings of the Royal Society A*.

S-a demonstrat în [Til16] că spațiile de moduli ale varietăților cu puncte marcate sunt omologic stabile când numărul de puncte marcate tinde la infinit. Capitolul 2 generalizează acest rezultat la *spații de moduli ale varietăților cu singularități conice*. (Punctele marcate pot fi considerate a fi singularități conice neesențiale, deoarece o vecinătate de disc a unui punct marcat este conul sferei sale de limită.) Acest lucru este dedus ca un caz special al unui rezultat de stabilitate omologică mai general pentru spațiile de clasificare ale *grupurilor de difeomorfisme simetrice* de varietăți, în raport cu *suma conexă parametrică*, o operație care generalizează suma conexă obișnuită și chirurgia (inclusiv chirurgia Dehn).

Cheia pentru demonstrarea acestui rezultat este stabilitatea omologică pentru *spații de moduli ale subvarietăților*, când numărul de componente conexe ale subvarietăților tinde la infinit, ceea ce a fost demonstrat în teza mea de doctorat și publicat în [Pal21]. Relația cu singularitățile conice este dată de colapsarea vecinătăților tubulare ale subvarietăților la puncte izolate. Rezultatele capitolului 2 corespund preprintului [Pal18a], care este trimis spre publicare.

Capitolul 3 se referă la "big mapping class groups", adică grupuri mapping class de *suprafețe de tip infinit* și corespunde unei lucrări comune cu X. Wu [PW22a] acceptat pentru publicare în *Documenta Mathematica*. În acest capitol demonstrăm că, pentru orice suprafață de tip infinit S, omologia integrală a închiderii grupului mapping class cu suport compact $\overline{\mathrm{PMap}_c(S)}$ și a grupului Torelli $\mathcal{T}(S)$ este nenumărabilă în fiecare grad pozitiv. După rezultatele noastre anterioare în [PW22b] și alte calcule cunoscute, o astfel de afirmație nu poate fi adevărată pentru *întregul* grup mapping class $\mathrm{Map}(S)$ pentru *toate* suprafețe S de tip infinite. Cu toate acestea, demonstrăm că omologia integrală a lui $\mathrm{Map}(S)$ este nenumărabilă în toate gradele pozitive pentru o clasă mare de suprafețe de tip infinit S. Proprietatea cheie a acestei clase de suprafețe este, aproximativ, că spațiul capetelor suprafeței S conține un punct limită de puncte distinse topologic. Acest rezultat include în special fiecare suprafață de gen finit al cărei spațiu de capete este numărabil și are un punct unic de rang Cantor-Bendixson maxim α, unde α este un ordinal succesor.

Înțelegerea *seriei centrale descendente* a unui grup este, în general, o problemă dificilă. Pe de altă parte, calcularea seriei centrale descendente și a algebrelor Lie asociate ale unui grup sau ale unora dintre subgrupurile sale poate duce la o înțelegere profundă a structurii de bază a acelui grup. Scopul capitolului 4 este de a prezenta mai multe tehnici care vizează realizarea unei părți a acestei probleme. În special, căutăm să răspundem la următoarea întrebare: *când se oprește seria centrală descendentă?* Introducem mai multe tehnici pentru a răspunde la această întrebare, pe care apoi le aplicăm *grupurilor braid partiționate pe suprafețe* pentru orice suprafață și pentru orice partiție. Calea de la tehnicile noastre generale până la aplicarea lor este departe de a fi simplă și este necesară o anumită tenacitate pentru a rezolva toate cazurile întâlnite pe parcurs. În cele din urmă ajungem la un răspuns la întrebarea noastră pentru fiecare dintre aceste grupuri, cu singura excepție a unei familii de grupuri braid partiționate pe planul proiectiv. Într-un număr de cazuri, chiar calculăm complet seria centrală descendentă. Acest capitol corespunde unei părți a monografiei [DPS22b], împreună cu J. Darné și A. Soulié, care este acceptată pentru publicare în *Memoirs of the American Mathematical Society*.

În continuare, trecând de la seria centrală descendentă la *reprezentări* grupurilor de deplasări, în capitolul 5 oferim o construcție topologică simplă a reprezentărilor Burau ale grupurilor *loop-braid*. Există patru versiuni: definite fie pe grupurile loop-braid neextinse, fie pe grupurile loop-braid extinse, iar în fiecare caz există o versiune neredusă și una redusă. Trei nu sunt surprinzătoare și s-ar putea ghici cu ușurință matricele corecte de atribuit generatoarelor. Al patrulea este mai subtil și nu pare evident din punct de vedere combinatoriu, deși este foarte natural din punct de vedere topologic. Acest capitol corespunde unei lucrări comune cu A. Soulié [PS22a] publicată în *Comptes Rendus Mathématique*.

Capitolul 6 se ocupă de construirea reprezentărilor omologice ale grupurilor mapping class de suprafețe și corespunde unei lucrări comune cu C. Blanchet și A. Shaukat [BPS23] acceptate pentru publicare în *Contemporary Mathematics*. În lucrările anterioare cu aceiași coautori [BPS21], am construit reprezentări twistate ale grupurilor mapping class de suprafețe, în funcție de alegerea unei reprezentări V a grupului Heisenberg $\mathcal{H}$. Pentru anumite V am arătat cum aceste reprezentări ale grupurilor mapping class pot fi de-twistate. În acest capitol, studiem restricțiile reprezentărilor noastre twistate la diferite subgrupuri ale grupului mapping class. Demonstrăm în special că aceste reprezentări pot fi de-twistate pe *grupul Torelli* pentru orice reprezentare V a lui $\mathcal{H}$. În cazul în care V este reprezentarea Schrödinger, construim de asemenea reprezentări ne-twistate ale subgrupurilor definite ca nuclee de homomorfisme încrucișate studiate de Earle și Morita.

În capitolul final F, descriu diverse probleme și întrebări deschise legate de subiectele tezei, dintre care unele sunt imediat abordabile, iar altele sunt foarte dificile.

Contents

Chapter O

Overview

My research interests lie primarily within **Algebraic Topology** and **Geometric Topology**. One fundamental goal in the intersection of these subjects is to understand the topology of *moduli spaces* of geometric objects through algebraic invariants. The concept of a *moduli space* is of central importance in mathematics, parametrising collections of all objects of a given kind, such as manifolds with certain properties or configurations of points, submanifolds or fields in an ambient space.

My research so far divides broadly into two themes, concerned with understanding the topology of moduli spaces through their *homology* and their *fundamental groups*. The kinds of moduli spaces that I have studied include:

- Configuration spaces of points in manifolds. In particular, *non-local* configuration spaces, in which configurations are equipped with some additional "non-local" structure, such as:
  - an ordering modulo even permutations,
  - a "field" defined on the complement of the configuration,
  - non-local data encoding the interactions of asymptotic magnetic monopoles.
- Moduli spaces of higher-dimensional disconnected submanifolds. These are related to moduli spaces of manifolds with Baas-Sullivan singularities.
- Mapping class groups of surfaces, including surfaces of *infinite type*.

In this preliminary chapter, I give a brief overview of these results: §O.1 on the *homology* and §O.2 on the *fundamental groups* of moduli spaces. Three results of §O.1 are developed in detail in the chapters of Part I and three results of §O.2 are developed in detail in the chapters of Part II.

Overview

O.1 Homology of moduli spaces

O.1.1 Configuration spaces

For a space M, the nth *ordered configuration space* $\widetilde{C}_n(M)$ is the subspace of M^n consisting of all n-tuples of pairwise distinct points in M. The symmetric group $\mathfrak{S}_n$ acts on this space, and

$$C_n(M) \;=\; \widetilde{C}_n(M)/\mathfrak{S}_n$$

is the nth *unordered configuration space* on M.

An important phenomenon that frequently occurs in the homology of moduli spaces is *homological stability*. For a family of moduli spaces indexed by a parameter n, this is the phenomenon where their homology is independent of n in higher and higher degrees as $n \to \infty$. One key example of this is for the mapping class groups of orientable surfaces: in this setting, homological stability with respect to genus was proven by Harer [Har85] and the limiting homology was computed by Madsen and Weiss [MW07], together proving the *Mumford conjecture* [Mum83].

In the setting of configuration spaces, it is a classical result, going back to McDuff [McD75] and Segal [Seg73; Seg79], that the sequence $C_n(M)$ of configuration spaces is homologically stable with respect to the number of points n, whenever M is a connected, open manifold.

A relatively straightforward generalisation of this result is homological stability for *labelled* configuration spaces $C_n(M, X)$ for a path-connected space X: a point in this space consists of an unordered configuration in M together with an element of X attached to each configuration point. (This is proven in [Ran13], for example.) The additional data associated to a configuration in this setting is *local* in the sense that it is simply a product of several pieces of data, each associated to a single point in the configuration.

However, one may also consider moduli spaces of configurations equipped with *non-local* data of different kinds. I have worked on three examples of this: §O.1.1.1–§O.1.1.3.

O.1.1.1 Non-locality: oriented configurations

An example of configuration spaces equipped with non-local data is the sequence of *oriented configuration spaces*

$$C_n^+(M) \;=\; \widetilde{C}_n(M)/A_n \tag{O.1}$$

given by quotienting by the action of the alternating group instead of the symmetric group. These are double coverings of the unordered configuration spaces; the additional (binary) piece of data given by an ordering modulo even permutations is not associated to any single point of the configuration. Homological stability for the sequence (O.1) was proven in my PhD thesis and published in [Pal13]. This result led to the question of whether one can identify the *stable homology* of the sequence (O.1), in other words the colimit

$$\lim_{n\to\infty} H_*(C_n^+(M)), \tag{O.2}$$

in terms of other well-understood spaces. In joint work with Jeremy Miller, we answered this question positively by lifting the classical *scanning map* [Seg73; McD75] to a homology equivalence between appropriate covering spaces.

Theorem A ([MP15b]). *Writing $\dot{T}M \to M$ for the fibrewise one-point compactified tangent bundle of M and denoting its space of degree-0 compactly-supported sections by $\Gamma_c(\dot{T}M \to M)_\circ$, we have:*

$$\lim_{n\to\infty} H_*(C_n^+(M)) \;\cong\; H_*(\widetilde{\Gamma_c}(\dot{T}M \to M)_\circ), \tag{O.3}$$

where $\widetilde{\Gamma_c}(\dot{T}M \to M)_\circ$ is the connected covering of $\Gamma_c = \Gamma_c(\dot{T}M \to M)_\circ$ corresponding to

$$\pi_1(\Gamma_c) \longrightarrow\!\!\!\!\rightarrow H_1(\Gamma_c) \cong H_1(C_2(M)) \longrightarrow\!\!\!\!\rightarrow H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/2,$$

where the isomorphism $H_1(\Gamma_c) \cong H_1(C_2(M))$ arises from homological stability and the identification of the stable homology for unordered configuration spaces and the projection $H_1(C_2(M)) \twoheadrightarrow H_1(C_2(\mathbb{R}^\infty))$ is induced by any embedding $M \hookrightarrow \mathbb{R}^\infty$.

In the course of proving Theorem A, we generalised the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff's *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients; see [MP15a].

O.1.1.2 Non-locality: configuration-section spaces

Configuration-section spaces on a manifold M equipped with a bundle $E \to M$ are "non-local" configuration spaces whose elements consist of a finite configuration in M together with a section of $E \to M$ defined on the complement of the configuration. Such spaces may be thought of physically as spaces of "fields" with point-particle singularities.

One often considers subspaces where the behaviour of the field (i.e. section) is constrained in a neighbourhood of the singularities (particles) – this may be thought of as restricting the allowed "charges" of the particles. More precisely, if the bundle $E \to M$ is trivial with fibre X, we fix a set $c \subseteq [S^{d-1}, X]$ of homotopy classes of maps $S^{d-1} \to X$, where d is the dimension of M. The sections are then required to restrict to an element of c, up to homotopy, in a neighbourhood of each particle (they are undefined at the particles themselves). For non-trivial bundles $E \to M$, the definition is similar but a little more delicate.

Homological stability. When $M = \mathbb{R}^2$ and the fibre X of the bundle is BG for a discrete group G, these are the *Hurwitz spaces*, classifying branched coverings of the 2-disc with deck transformation group G (and prescribed monodromy, if we impose a condition $c \subseteq [S^1, BG] = \mathrm{Conj}(G)$). These spaces have important connections with number theory through recent work of Ellenberg, Venkatesh and Westerland [EVW16], who proved an asymptotic version of the *Cohen-Lenstra conjecture* for function fields via a certain rational homological stability result for Hurwitz spaces.

In joint work with Ulrike Tillmann, we have proven another homological stability result for configuration-section spaces [PT21], which is in a sense both more and less general than that of Ellenberg, Venkatesh and Westerland. It is *more* general in the sense that it holds for any bundle over any connected, open manifold M, but it is also *less* general in the sense that we assume a stronger condition on the allowed "charges" of the particles.

Theorem B ([PT21])**.** *Let M be a connected manifold of dimension $d \geqslant 2$ with basepoint $* \in \partial M$ and let $\xi \colon E \to M$ be a fibre bundle whose fibre over $*$ we denote by X. Assume that*

$$\text{the preimage of } c \text{ under } [S^{d-1}, X] \twoheadleftarrow \pi_{d-1}(X) \text{ is a single element,} \tag{O.4}$$

so c corresponds to a fixed point of the $\pi_1(X)$-action on $\pi_{d-1}(X)$. Then the stabilisation maps of configuration-section spaces

$$\mathrm{C\Gamma}_k^{c,*}(M;\xi) \longrightarrow \mathrm{C\Gamma}_{k+1}^{c,*}(M;\xi) \tag{O.5}$$

induce isomorphisms on $H_i(-;\mathbb{Z})$ in the range $k \geqslant 2i+4$ and surjections in the range $k \geqslant 2i+2$. With field coefficients, these ranges improve to $k \geqslant 2i+2$ and $k \geqslant 2i$ respectively.

Point-pushing actions. Let us now assume that the bundle $E \to M$ is trivial with fibre X; in this case we write $\mathrm{CMap}_k^{c,*}(M;X)$ and call these *configuration-mapping spaces.* They fit into a natural fibre sequence

$$\mathrm{Map}^{c,*}(M \smallsetminus k \text{ points}, X) \longrightarrow \mathrm{CMap}_k^{c,*}(M;X) \longrightarrow C_k(\mathring{M}), \tag{O.6}$$

which is obtained functorially from the universal fibre sequence

$$M \smallsetminus k \text{ points} \longrightarrow U_k(M) \longrightarrow C_k(\mathring{M}). \tag{O.7}$$

In [PT22], we obtain explicit formulas, when the dimension of M is at least 3, for the monodromy action $\pi_1(C_k(M)) \to \pi_0(\mathrm{hAut}(M \smallsetminus k \text{ points}))$ of the universal fibre sequence (O.7), from which we also deduce explicit formulas for the monodromy action of the fibre sequence (O.6). A special case of this is as follows:

Theorem C ([PT22])**.** *If $d = \dim(M) \geqslant 3$ and M satisfies at least one of the following conditions:*

- *M is simply-connected, or*
- *the handle-dimension of M is at most $d-2$;*

then the "point-pushing" action of $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$ on the mapping space $\mathrm{Map}^{c,*}(M \smallsetminus k \text{ points}, X) \simeq \mathrm{Map}_*(M, X) \times (\Omega_c^{d-1}X)^k$ is given as follows:

$$(\alpha_1, \ldots, \alpha_k; \sigma) \cdot (f, g_1, \ldots, g_k) = (f, \bar{g}_1, \ldots, \bar{g}_k), \tag{O.8}$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$, and

- *for an element $\alpha \in \pi_1(M)$ we write $\mathrm{sgn}(\alpha) = +1$ if α lifts to a loop in the orientation double cover of M and $\mathrm{sgn}(\alpha) = -1$ otherwise,*
- *$\pi_1(X)$ acts up to homotopy on $\Omega_c^{d-1}X$ in the natural way,*
- *$\{\pm 1\}$ acts on $\Omega_c^{d-1}X$ through the involution given by precomposition with a reflection of S^{d-1} in a hyperplane containing the basepoint.*

As a corollary, we obtain a precise description of the set of path-components $\pi_0(\mathrm{CMap}_k^{c,*}(M, X))$ of the configuration-mapping space, under the above conditions. In addition, we investigate the question of injectivity of the *point-pushing maps*

$$p_k \colon \pi_1(C_k(M)) \longrightarrow \pi_0(\mathrm{Cat}(M, z))$$
$$p_{k,\partial} \colon \pi_1(C_k(M)) \longrightarrow \pi_0(\mathrm{Cat}_\partial(M, z)),$$

where $\mathrm{Cat} \in \{\mathrm{hAut}, \mathrm{Homeo}, \mathrm{Diff}\}$, z denotes the basepoint configuration in $C_k(M)$ and where $_\partial$ means that the boundary of M is fixed, and prove:

Theorem D ([PT22, §8])**.** *When the dimension of M is at least 3, we have*

$$\ker(p_k) = \Delta(\ker(p_1)),$$

i.e. the kernel of p_k is equal to the diagonal of $\ker(p_1)^k \subseteq \pi_1(M)^k \subseteq \pi_1(C_k(M))$, where we identify $\pi_1(C_k(M))$ with $\pi_1(M)^k \rtimes \Sigma_k$. If $\partial M \neq \varnothing$, then $p_{k,\partial}$ is injective.

O.1.1.3 Non-locality: asymptotic magnetic monopoles

In the special case where $M = \mathbb{R}^3$, there is another kind of non-local data, encoding the pairwise interactions of the particles and modelling "asymptotic" magnetic monopoles.

The topology of the moduli spaces of magnetic monopoles $\mathcal{M}_k$ has been the subject of intensive study for many decades. By a theorem of Donaldson [Don84], they have a model as spaces of rational functions on $\mathbb{C}P^1$. Via this model, their homotopy and homology groups are known to stabilise as $k \to \infty$ by a theorem of Segal [Seg79] and their homology (both stable and unstable) was completely computed by [Coh⁺91] in terms of the homology of the braid groups, which is completely known by [CLM76].

The moduli spaces $\mathcal{M}_k$ are non-compact manifolds. Recently, a partial compactification of $\mathcal{M}_k$ has been constructed by Kottke and Singer [KS22] by adding to $\mathcal{M}_k$ certain codimension-1 boundary hypersurfaces $\mathcal{I}_\lambda$ indexed by partitions $\lambda = (k_1, \ldots, k_r)$ of k. Points in these boundary hypersurfaces are thought of as "ideal" or "asymptotic" monopoles of total charge k, with r "clusters" centred at different points in $\mathbb{R}^3$, with charges $k_1, \ldots, k_r$, which are "widely separated" but interact with each other. The space $\mathcal{I}_\lambda$ has the structure of a fibre bundle

$$\mathcal{I}_\lambda \longrightarrow C_\lambda(\mathbb{R}^3) \tag{O.9}$$

over the partitioned configuration space $C_\lambda(\mathbb{R}^3)$ (the covering space of the unordered configuration space of $k_1 + \cdots + k_r$ points where configurations are equipped with a partition of type λ) with fibre $\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}$. The *non-locality* of the non-local configuration spaces $\mathcal{I}_\lambda$ comes from the non-triviality of the bundle (O.9).

In joint work with Ulrike Tillmann, we have proven a homological stability result for these asymptotic monopole moduli spaces as the number of clusters of a fixed charge $c \geqslant 1$ goes to infinity. Fix a positive integer c and a tuple $\lambda = (k_1, \ldots, k_r)$ of positive integers $k_i \neq c$. Write $\lambda[n]_c = (k_1, \ldots, k_r, c, \ldots, c)$, where c appears n times.

Theorem E ([PT23]; **Chapter 1**). *There are natural stabilisation maps*

$$\mathcal{I}_{\lambda[n]_c} \longrightarrow \mathcal{I}_{\lambda[n+1]_c} \tag{O.10}$$

that induce isomorphisms on homology in all degrees $\leqslant n/2-1$ with $\mathbb{Z}$ coefficients and in all degrees $\leqslant n/2$ with field coefficients.

O.1.1.4 Configurations on closed manifolds

Let us return now to ordinary configuration spaces $C_n(M)$ on a connected manifold M (without boundary). The classical results of McDuff and Segal prove homological stability for these configuration spaces if M is *open*, i.e. non-compact. On the other hand, the situation when M is *closed* is much more subtle.

Indeed, when M is closed, homological stability for the configuration spaces $C_n(M)$ is not true in general — for example, one may calculate that $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2n-2)$, which does not stabilise as $n \to \infty$. Moreover, the classical stabilisation maps used by McDuff and Segal do not exist, since these depend on adding a new configuration point in M "near infinity". In joint work with Federico Cantero, we have proven three main results demonstrating that the homology of configuration spaces on closed manifolds exhibits some more subtle kinds of stability.

(1) When the Euler characteristic of M is zero, we construct *replication maps* $C_n(M) \to C_{\lambda n}(M)$ for any integer $\lambda \geqslant 2$, and prove that they induce homological stability after inverting λ:

Theorem F ([CP15]). *These maps induce isomorphisms on $H_i(-, \mathbb{Z}[\frac{1}{\lambda}])$ in the range $2i \leqslant \lambda$.*

(2) When the manifold M is odd-dimensional, the configuration spaces $C_n(M)$ do in fact satisfy homological stability after inverting 2 in the coefficients (strengthening a result of [BM14]):

Theorem G ([CP15]). *When $\dim(M)$ is odd, there are isomorphisms*

$$H_i(C_n(M); \mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M); \mathbb{Z}[\tfrac{1}{2}]) \qquad \text{and} \qquad H_i(C_n(M); \mathbb{Z}) \cong H_i(C_{n+2}(M); \mathbb{Z})$$

in the range $2i \leqslant n$, induced by a zigzag of maps.

(3) When the manifold M is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many authors [BCT89; ML88; Chu12; Ran13; BM14; Knu17] that homological stability holds for $C_n(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_n(S^2); \mathbb{F}) \cong \begin{Bmatrix} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{Bmatrix} \quad \text{for } n \geqslant 2.$$

From this example we see that the first homology of $C_n(S^2)$ is not stable, but it is p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p.

Theorem H ([CP15]). *Suppose that $\dim(M)$ is even. For each fixed i, the sequence*

$$H_i(C_n(M); \mathbb{F}) \quad \text{for } n \geqslant 2i \tag{O.11}$$

is p^{a+1}-periodic and takes on at most $a+2$ values. Moreover, if $\chi \equiv 1 \bmod p$ then the above sequence is 1-periodic, i.e. homological stability holds with coefficients in $\mathbb{F}$.

The p^{a+1}-periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of p depending on i rather than on χ. The periodicity part of Theorem H was later improved to p-periodicity in [KM16] (independent of i or χ). Combining this with (a slightly more precise statement of) our result, a corollary is that the sequence (O.11) above takes on only *two* different values.

O.1.1.5 Motivic cohomology

In the case when $M = X(\mathbb{C})$ is the manifold of complex points of a smooth scheme X over a number field K, one may ask whether stability for the singular (co)homology of $C_n(M)$ may be lifted to stability for the *motivic* or *étale motivic* cohomology of the sequence of configuration schemes $C_n(X)$. In joint work with Geoffroy Horel, we have answered this question as follows.

Theorem I ([HP23])**.** *Suppose that X may be written as $Y - D$, where Y is a smooth scheme over K and $D \subset Y$ is a smooth, closed subscheme that has a K-point. Assume that the étale motive of X is mixed Tate and that Y is geometrically connected. Then there are maps of étale motivic cohomology groups*

$$H^{p,q}_{et}(C_{n+1}(X);\Lambda) \longrightarrow H^{p,q}_{et}(C_n(X);\Lambda) \tag{O.12}$$

that are isomorphisms for $p \leqslant n/2$ and under mild conditions on the coefficient ring Λ. In the case when $X = \mathbb{A}^d$ is affine space, there are analogous maps of motivic cohomology groups

$$H^{p,q}(C_{n+1}(\mathbb{A}^d);\Lambda) \longrightarrow H^{p,q}(C_n(\mathbb{A}^d);\Lambda) \tag{O.13}$$

that are isomorphisms for $p \leqslant n/2 - 1$ and any coefficient ring Λ.

A key input for part (O.13) of Theorem I is a stability result for the homology of the symmetric groups $\mathfrak{S}_n$ with certain (polynomial) twisted coefficients, which was proven in my PhD thesis and published as [Pal18b].

O.1.2 Higher-dimensional submanifolds and conical singularities

Instead of configurations of points (closed 0-dimensional submanifolds) in M, one may consider configurations of *higher-dimensional* closed submanifolds of M that are isotopic to the disjoint union of finitely many copies of a fixed ("model") manifold L.

It was proven in my PhD thesis (and published as [Pal21]) that homological stability generalises to this setting, as long as the dimension of L is at most $\frac{1}{2}(\dim(M) - 3)$. In later work [Pal18a], I used this to prove homological stability for:

- *Symmetric diffeomorphism groups*, with respect to *parametric connected sum.* (Given embeddings $L \hookrightarrow M$ and $L \hookrightarrow Q$ with isomorphic normal bundles, their parametric connected sum $M \sharp_L Q$ is the result of cutting out a tubular neighbourhood of each embedding and gluing the resulting boundaries. If L is a point this is the ordinary connected sum. Other examples of this operation are surgery and Dehn surgery of 3-manifolds.) This generalises [Til16], which corresponds to the case $L =$ point, i.e., the usual connected sum operation.
- *Diffeomorphism groups of manifolds with conical singularities*, with respect to the number of singularities.

Let us consider the second point in more detail. Fix an $(m-1)$-dimensional manifold T and let $\mathrm{cone}(T) = (T \times [0,\infty))/(T \times \{0\})$ be the open cone on T. An m-dimensional *manifold with conical T-singularities* is a space M that is locally homeomorphic to $\mathrm{cone}(T)$, together with a smooth atlas on the subset $M_{mfd} \subseteq M$ of locally Euclidean points of M. (This is a special case of a *manifold with Baas-Sullivan singularities.*) A *diffeomorphism* of M is a homeomorphism $M \to M$ that restricts to a diffeomorphism $M_{mfd} \to M_{mfd}$ and is of the form $\mathrm{cone}(\varphi)$ for some diffeomorphism $\varphi \colon T \to T$ near each point of the discrete subset $M \smallsetminus M_{mfd} \subseteq M$. These form a subgroup

$$\mathrm{Diff}^T(M) \leqslant \mathrm{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood $\mathcal{T}(L)$ of any closed submanifold $L \subset M$. The quotient $M_L = M/\mathcal{T}(L)$ is a manifold with a single conical $\partial\mathcal{T}(L)$-singularity. Iterating this by collapsing tubular neighbourhoods of n pairwise disjoint, isotopic copies of L in M, we obtain a manifold with n conical $\partial\mathcal{T}(L)$-singularities, which we denote by $M_{n\cdot L}$.

Theorem J ([Pal18a]; **Chapter 2**). *If M is connected, $\partial M \neq \varnothing$ and $\dim(L) \leqslant \frac{1}{2}(\dim(M) - 3)$, then the sequence of classifying spaces $B\mathrm{Diff}^{\partial\mathcal{T}(L)}(M_{n\cdot L})$ is homologically stable as $n \to \infty$.*

O.1.3 Big mapping class groups

Connected, compact, orientable surfaces are classified by their genus and number of boundary components; in particular there are countably many such surfaces. If we remove the assumption of compactness (but still assume second countability and require the boundary to be compact), surfaces are classified by:

- their genus (which may be a non-negative integer or ∞);
- their number of boundary components (a non-negative integer);
- their *space of ends* E (a space that is homeomorphic to a closed subset of the Cantor set $\mathcal{C}$);
- their space of *non-planar ends* E_{np} (a closed subset of E).

The classification [Ker23; Ric63] says that homeomorphism classes of surfaces are in one-to-one correspondence with choices of the above list of data, with the single restriction that $E_{np} = \varnothing$ if and only if the genus is finite. In particular, there are *uncountably many* such surfaces.

A surface has *finite type* if its fundamental group is finitely generated (this occurs if and only if both its genus and its space of ends are finite); otherwise it has *infinite type*. Examples of infinite type surfaces (without boundary) include:

- the sphere minus a Cantor set (genus zero; $E = \mathcal{C}$; $E_{np} = \varnothing$);
- the colimit of the compact surfaces $\Sigma_{g,1}$ as $g \to \infty$ (infinite genus; $E = E_{np} = \{*\}$);
- the "flute surface" $\mathbb{C} \smallsetminus \mathbb{Z}$ (genus zero; $E = [0, \omega]$, $E_{np} = \varnothing$).

The space of ends in the last example is the closed ordinal space $[0, \omega]$, in other words the ordinal $\omega + 1$ in the order topology, which is homeomorphic to the subspace $\{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{C}$.

The mapping class group $\mathrm{Map}(S)$ of a surface S is countable if and only if S has finite type. Mapping class groups of infinite type surfaces (which are always uncountable discrete groups) are often called *big mapping class groups*; a recent survey is [AV20].

O.1.3.1 Homological stability and acyclicity

The degree-one homology (i.e. abelianisation) of big mapping class groups is known in some cases. For example, if Σ is a *finite* type surface and $\mathcal{C} \subset \Sigma$ is a subspace homeomorphic to the Cantor set, the natural map

$$\mathrm{Map}(\Sigma \smallsetminus \mathcal{C}) \longrightarrow \mathrm{Map}(\Sigma)$$

(given by extending homeomorphisms of $\Sigma \smallsetminus \mathcal{C}$ uniquely to Σ) induces an isomorphism on $H_1(-)$ [CC22]. In the special case of the 2-sphere, it is also known that $H_2(\mathrm{Map}(S^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}/2$ [CC21].

In joint work with Xiaolei Wu, we gave the first complete calculation *in all degrees* of the homology of certain big mapping class groups. To describe this, we first need to introduce some constructions.

Definition. Let Σ be any surface with empty boundary. For $n \geqslant 1$ write $\Sigma^{(n)}$ for the surface obtained by removing the interiors of n pairwise disjoint discs from Σ. Then:

- $\mathfrak{B}(\Sigma)$ is the result of gluing together infinitely many copies of $\Sigma^{(3)}$ in a binary tree pattern;
- $\mathfrak{L}(\Sigma)$ is the result of gluing together infinitely many copies of $\Sigma^{(2)}$ in a linear pattern.

Note that both $\mathfrak{B}(\Sigma)$ and $\mathfrak{L}(\Sigma)$ have one boundary component. Write $\overline{\mathfrak{B}}(\Sigma)$ and $\overline{\mathfrak{L}}(\Sigma)$ respectively to denote the result of filling the boundary component with a disc.

Examples. Some key examples are:

- $\mathfrak{L}(S^2) \cong D^2 \smallsetminus \{0\}$;
- $\mathfrak{B}(S^2) \cong D^2 \smallsetminus \mathcal{C}$;
- $\overline{\mathfrak{L}}(\mathbb{C}) \cong \mathbb{C} \smallsetminus \mathbb{Z}$ (the *flute surface*);
- $\overline{\mathfrak{L}}(T^2)$ is the colimit of the compact surfaces $\Sigma_{g,1}$ as $g \to \infty$ (the *Loch Ness monster surface*).

Theorem K ([PW22b]). *For any surface Σ, the mapping class group $\mathrm{Map}(\mathfrak{B}(\Sigma))$ is acyclic, i.e.*

$$\widetilde{H}_*(\mathrm{Map}(\mathfrak{B}(\Sigma))) = 0.$$

As a consequence, we also have

$$H_i(\mathrm{Map}(\overline{\mathfrak{B}}(\Sigma) \smallsetminus \{*\})) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

In particular, when $\Sigma = S^2$ the second part of this theorem gives a complete calculation of the homology of the mapping class group of the plane minus a Cantor set $\mathbb{C} \smallsetminus \mathcal{C}$.

A key ingredient in the proof is a homological stability result for big mapping class groups:

Theorem L ([PW22b]). *Let A be a connected surface with one boundary component and let Σ be a connected surface with empty boundary. Then the sequence of big mapping class groups*

$$\mathrm{Map}(A \natural \mathfrak{B}(\Sigma)^{\natural n})$$

is homologically stable as $n \to \infty$, where $\natural$ denotes boundary connected sum. The same statement is true for the sequence

$$\mathrm{Map}(A \natural \mathfrak{L}(\Sigma)^{\natural n})$$

under a mild condition on the end space E of Σ.

The condition on the end space E of Σ in the above theorem is the following. First notice that the end space of $\mathfrak{L}(\Sigma)$ is $(E\omega)^+$, where $E\omega$ denotes the disjoint union of countably infinitely many copies of E and $(-)^+$ denotes one-point compactification. The condition is that the point at infinity of $(E\omega)^+$ must be topologically distinguished, i.e. *not* locally homeomorphic to any other point of $(E\omega)^+$. Although this is a mild condition, it is not vacuous: for example, in the case when E is the Cantor set $\mathcal{C}$, the space $(\mathcal{C}\omega)^+ \cong \mathcal{C}$ is homogeneous, so in particular the point at infinity is not topologically distinguished.

O.1.3.2 Uncountability

In a contrasting direction, in further joint work with Xiaolei Wu, we have proven that other families of big mapping class groups have *uncountable* homology in all positive degrees.

Theorem M ([PW22a]; **Chapter 3**). *Let Σ be a connected surface with empty boundary. Assume that Σ has genus zero and that its space of ends has a topologically distinguished point. Then there is an embedding of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longhookrightarrow H_*(\mathrm{Map}(\overline{\mathfrak{L}}(\Sigma))),$$

where $\mathfrak{c}$ denotes the cardinality of the continuum and $\Lambda^*(-)$ denotes the exterior algebra over $\mathbb{Z}$ on an abelian group. In particular, the homology of the mapping class group of $\overline{\mathfrak{L}}(\Sigma)$ is uncountable in each degree $\geqslant 1$.

In the special case when $\Sigma = \mathbb{C}$ this theorem proves uncountability in all positive degrees of the homology of the mapping class group of the flute surface $\mathbb{C} \smallsetminus \mathbb{Z}$.

Remark. The assumption in Theorem M that the space of ends of Σ has a topologically distinguished point is essential. Without this assumption we could set $\Sigma = S^2 \smallsetminus \mathcal{C}$, in which case

$$\overline{\mathfrak{L}}(S^2 \smallsetminus \mathcal{C}) \cong S^2 \smallsetminus \mathcal{C},$$

but it is known by [CC21; CC22] that the homology of $\mathrm{Map}(S^2 \smallsetminus \mathcal{C})$ is *not* uncountable in degrees 1 and 2 (it is isomorphic to 0 and $\mathbb{Z}/2$ respectively).

We also prove similar uncountability results for certain subgroups of big mapping class groups. Specifically, for any infinite-type surface S, we prove uncountability in all positive degrees of the homology of the subgroups $\mathcal{T}(S) \subseteq \overline{\mathrm{PMap}_c(S)} \subseteq \mathrm{Map}(S)$, the *Torelli group* and the *closure of the compactly-supported mapping class group*. See **Chapter 3** for more details.

O.2 Fundamental groups of moduli spaces

Fundamental groups of moduli spaces include the classical braid groups $\mathbf{B}_n = \pi_1(C_n(\mathbb{R}^2))$ and their relatives (loop/welded braid groups $\mathbf{wB}_n = C_{nS^1}(\mathbb{R}^3)$, surface braid groups $\mathbf{B}_n(S) = \pi_1(C_n(S))$, etc.) as well as *mapping class groups* $\mathrm{Map}(M) = \pi_0(\mathrm{Homeo}(M)) = \pi_1(B\mathrm{Homeo}(M))$. These groups are interesting objects in their own right, as well as having connections to knot theory and to physics via topological quantum field theories.

Understanding the *lower central series* of a group (and its associated Lie algebra) is typically a difficult task, but it can lead to a deep understanding of the underlying structure of the group. Another powerful method of understanding a group is to understand its *representations*. In particular, a key question for any group is to know whether it has a faithful representation on a finite-dimensional vector space – in other words, whether it is *linear*. This is known to be true for the classical braid groups [Big01; Kra02] but it is wide open for almost all other braid-like groups and mapping class groups. In the context of representations of mapping class groups, another important question is whether they may be extended to define a *topological quantum field theory* (TQFT) – in other words, whether they may be defined not just on automorphisms of manifolds but on a whole cobordism category of manifolds.

I have worked on understanding the lower central series of many different braid-like groups (including surface braid groups, loop braid groups and generalisations), especially the question of when the lower central series *stops*; see §O.2.1. Motivated by the open question of linearity and by the goal of constructing TQFTs, I have worked on new topological constructions of representations of (surface) braid groups, loop braid groups and mapping class groups; see §O.2.2.

O.2.1 Lower central series

One of the most basic objects one needs to understand when studying the structure of a group G is its lower central series $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$, defined recursively by

$$\Gamma_{i+1}(G) = [\Gamma_i(G), G] = \{ghg^{-1}h^{-1} \mid g \in \Gamma_i(G), h \in G\}.$$

If G is perfect, its lower central series is completely trivial. On the other hand, if it is nilpotent or residually nilpotent, the filtration $\Gamma_*(G)$ and its associated graded Lie ring $\mathcal{L}_*(G) = \bigoplus_i \Gamma_i(G)/\Gamma_{i+1}(G)$ contain deep information about the structure of G. The lower central series is also deeply connected to the structure of the group ring of G.

The amount of information one can hope to extract from the study of a lower central series depends in the first place on whether or not it *stops*, meaning that there exists an integer $i \geqslant 1$ such that $\Gamma_i(G) = \Gamma_{i+1}(G)$. If there is such an integer, then the smallest such integer is the *length* of the lower central series of G.

In joint work with Jacques Darné and Arthur Soulié, we give a complete answer to the question of the length (finite or infinite) of the lower central series of surface braid groups, virtual braid groups and loop/welded braid groups, as well as *partitioned* versions of all of these groups. The answer depends subtly on the number of strands, how they are partitioned and the topology of the underlying surface. For example:

Theorem N ([DPS22b]; **Chapter 4**). *For $n \geqslant 3$, the lower central series of:*

- *$B_n(S)$ has length 2 if $S \subseteq \mathbb{S}^2$ or S is non-orientable;*
- *$B_n(S)$ has length 3 if $S \nsubseteq \mathbb{S}^2$ and S is orientable;*
- *$B_{(2,n)}(\mathbb{R}^2)$ has length ∞;*
- *$B_{(2,n)}(\mathbb{S}^2)$ has length $\nu_2(n) + 2 + \epsilon$, where $\nu_2(n)$ is the 2-adic valuation of n and $\epsilon \in \{0, \pm 1\}$.*

For a complete answer, see Tables 4.1–4.3 in Chapter 4.

O.2.2 Representations of motion groups and mapping class groups

I have several strands of research concerning (homological) representations of motion groups and mapping class groups.

The *Lawrence-Bigelow representations* are an important family of representations of the braid groups, which were used in the proof [Big01; Kra02] that the braid groups are linear. They come in many different flavours, and in joint work with Cristina Anghel (§O.2.2.1) we have studied the precise relationships between these different flavours, for a general class of homological representations of mapping class groups including the Lawrence-Bigelow representations.

In a sequence of joint work with Arthur Soulié (§O.2.2.2), we have given a unified foundation for the construction of homological representations of motion groups and mapping class groups, using a functorial approach. This has yielded new constructions, such as a pro-nilpotent extension of the Lawrence-Krammer-Bigelow representation and extensions of the Burau representations to (extended) loop braid groups, as well as *polynomiality* properties for families of representations.

In joint work with Christian Blanchet and Awais Shaukat (§O.2.2.3), we have constructed non-commutative analogues of the Lawrence-Bigelow representations for higher-genus mapping class groups using the representation theory of the *Heisenberg group*.

O.2.2.1 Lawrence-Bigelow representations

In joint work with Cristina Anghel, we have investigated the fundamental relationships between the many different flavours of homological representations of mapping class groups – with the *Lawrence-Bigelow representations* of the braid groups being our motivating example. Understanding the relationships between these representations is important for applications in quantum topology (see for example [Ang20]).

Consider a surface Σ equipped with a decomposition of its boundary into a “fixed” part and a “free” part (this specifies which part of its boundary must be fixed by diffeomorphisms). Choose a local system $\mathcal{L}$ on the configuration space $C_k(\Sigma)$ that is preserved by the action of $\mathrm{Map}(\Sigma)$. For each k we then obtain various different homological representations of the mapping class group $\mathrm{Map}(\Sigma)$, depending on:

- whether we require configurations to be disjoint from the *fixed* or *free* part of the boundary;
- the homology theory that we apply: ordinary homology, Borel-Moore homology, homology relative to the boundary of the configuration space, etc.

For example, if $\Sigma = \Sigma_{0,n+1}$ with n of its boundary-components considered as *free* and one considered as *fixed*, we obtain (for an appropriate $\mathcal{L}$) the various different flavours of the Lawrence-Bigelow representations.

In [AP20], we have established general relationships between these different flavours of homological representations, expressed in terms of *non-degenerate pairings*, *embeddings* and *completions*. More concretely, we have also defined free generating sets for each of the representations under consideration, and described the corresponding matrices of these pairings and embeddings.

O.2.2.2 Unified topological construction

In joint work with Arthur Soulié, we have given a unified foundation for the construction of homological representations of motion groups and mapping class groups. Namely, we define *homological representation functors* encoding a large class of homological representations, defined on categories containing all motion groups and mapping class groups in a fixed dimension d. These source categories are defined using a topological enrichment of the Quillen bracket construction applied to categories of decorated manifolds. This unifies many previously-known constructions, including those of Lawrence-Bigelow, and yields many new representations. The construction depends on four parameters:

- a closed submanifold $Z \subset \mathbb{R}^d$ and open subgroup $G \leqslant \mathrm{Diff}(Z)$;
- a functorial quotient of groups Q;
- an integer $i \geqslant 0$.

For example, the family of Lawrence-Bigelow representations depends on an integer $k \geqslant 1$; our construction (for $d = 2$) recovers this in the case when:

- Z is the 0-dimensional manifold $\{1, \ldots, k\}$ and $G = \mathfrak{S}_k$;
- Q is the abelianisation functor;
- $i = k$.

In higher dimensions $d \geqslant 3$, it is especially interesting to consider higher-dimensional submanifolds $Z \subset \mathbb{R}^d$, for example in the case of the loop braid groups.

An important choice of functorial quotient of groups is $Q = \Gamma_\ell$ for an integer $\ell \geqslant 2$: this is the *universal* $(\ell - 1)$*-nilpotent quotient*, generalising the abelianisation functor.

Theorem O ([PS19]). *For a set of parameters* $\{Z, G, Q, i\}$ *as above, the action of motion groups and mapping class groups on the twisted homology of certain embedding spaces determines functors*

$$L_i(\mathcal{F}_{(Z,G,Q)}) \colon \langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle \longrightarrow \mathrm{Mod}^{\mathrm{tw}}_{\mathbb{Z}[\mathcal{Q}]}, \tag{O.14}$$

where $\mathcal{Q}$ *denotes a group built out of the deck transformation groups of the regular covering spaces corresponding to the coefficients in the twisted homology. There is also a universal quotient* $\mathcal{Q}^{\mathrm{unt}}$ *of the group* $\mathcal{Q}$*, together with functors:*

$$L_i(\mathcal{F}^{\mathrm{unt}}_{(Z,G,Q)}) \colon \langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathcal{Q}^{\mathrm{unt}}]}. \tag{O.15}$$

The domain $\langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle$ is a *Quillen bracket category* of decorated d-manifolds. The superscript $^{\mathrm{tw}}$ in (O.14) denotes an enlargement of the module category to a category of *twisted* modules. The functor (O.14) therefore restricts to *twisted* representations of motion groups and mapping class groups, whereas the functor (O.15) restricts to *untwisted* representations.

Pro-nilpotent Lawrence-Krammer-Bigelow representation. By fixing the three parameters $\{Z, G, i\}$ and considering $Q = \Gamma_\ell$ for all $\ell \geqslant 2$ simultaneously, we construct *pro-nilpotent towers of representations* of classical braid groups, surface braid groups and (extended and non-extended) loop braid groups. (Some of the non-stopping results for lower central series from §O.2.1 are crucial for this construction.) In particular, we prove:

Theorem P ([PS22b]). *There is a pro-nilpotent tower of representations of the classical braid groups* $\mathbf{B}_n$ *whose* $\ell = 2$ *layer is the Lawrence-Krammer-Bigelow representation.*

The limit as $l \to \infty$ of this tower of representations is a representation of $\mathbf{B}_n$ defined over the integral group ring $\mathbb{Z}[\mathbf{RB}_2]$, where $\mathbf{RB}_2$ is the two-strand ribbon braid group. Since $\mathbf{RB}_2 = \mathbb{Z}^2 \rtimes \mathbb{Z}$, this gives a non-commutative 3-variable enrichment of the classical Lawrence-Krammer-Bigelow representation. Moreover, in [PS22b], we also compute explicit matrices over $\mathbb{Z}[\mathbf{RB}_2]$ describing the actions of the standard generators σ_i of $\mathbf{B}_n$ for this 3-variable representation.

Burau representations of loop braid groups. As a special case of Theorem O, we obtain a topological construction of *Burau representations* of the loop braid groups $\mathbf{wB}_n$ and the extended loop braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$. These each come in an *unreduced* version and a *reduced* version. In **Chapter 5** we give an explicit, more concrete construction of each of these representations, describe free generating sets for their underlying modules and compute the corresponding matrices of the representations. Three of these four representations are not surprising, and one could easily guess the correct matrices to assign to generators. However, the fourth is more subtle, and does not seem combinatorially obvious, although its topological construction is very natural:

Theorem Q ([PS22a]; **Chapter 5**). *The reduced Burau representation of the extended loop braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$, defined over the ring $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, acts on generators as described in Table 5.1 on page 158.*

Polynomiality. The structure of the domain category $\langle \mathcal{G}_\circ, \mathcal{M}_\circ \rangle$ of the homological representations functors of Theorem O means that there is a notion of *polynomiality* for functors of this form. For a wide range of homological representation functors produced by Theorem O — including those encoding the Lawrence-Bigelow representations of the braid groups — we prove in [PS23] that they are polynomial. This has applications to twisted homological stability as well as to understanding the structure of the representation theory of these families of groups.

O.2.2.3 Heisenberg homology and representations of mapping class groups

In joint work with Christian Blanchet and Awais Shaukat, we have constructed analogues of the Lawrence-Bigelow representations (of the braid groups) for the mapping class groups $\mathrm{Map}(\Sigma_{g,1})$ for $g \geqslant 1$. To do this, we first show that for each $n \geqslant 2$ there is a natural quotient

$$\phi \colon \mathbf{B}_n(\Sigma_{g,1}) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{H}(\Sigma_{g,1}), \tag{O.16}$$

where the *Heisenberg group* $\mathcal{H}(\Sigma_{g,1})$ is the central extension of $H_1(\Sigma_{g,1})$ classified by the intersection form. This determines a local system $\mathcal{L}$, defined over the group ring $\mathbb{Z}[\mathcal{H}(\Sigma_{g,1})]$, on the configuration space $C_n(\Sigma_{g,1})$. We also show that the kernel of (O.16) is preserved by the action of $\mathrm{Map}(\Sigma_{g,1})$, which implies that there is a well-defined action $\Phi \colon \mathrm{Map}(\Sigma_{g,1}) \to \mathrm{Aut}(\mathcal{H}(\Sigma_{g,1}))$.

Notation. For any representation $\rho \colon \mathcal{H}(\Sigma_{g,1}) \to \mathrm{Aut}_R(V)$ and automorphism $\tau \in \mathrm{Aut}(\mathcal{H}(\Sigma_{g,1}))$ we denote by ${}_\tau V$ the τ-twisted representation $\rho \circ \tau$. Write $\Sigma'_{g,1}$ for the non-compact surface given by removing one point (equivalently, a closed interval) from the boundary of $\Sigma_{g,1}$.

Theorem R ([BPS21]). *For any $n \geqslant 2$ and representation V of $\mathcal{H}(\Sigma_{g,1})$ over R, there is a twisted representation of $\mathrm{Map}(\Sigma_{g,1})$ on the collection of Borel-Moore homology R-modules*

$$\mathcal{V}_n({}_\bullet V) \;=\; \big\{ \mathcal{V}_n({}_\tau V) := H_n^{BM}(C_n(\Sigma'_{g,1}); \mathcal{L} \otimes {}_\tau V) \quad \textit{for} \quad \tau \in \mathrm{Aut}(\mathcal{H}(\Sigma_{g,1})) \big\} \tag{O.17}$$

where each $f \in \mathrm{Map}(\Sigma_{g,1})$ acts by $\mathcal{V}_n({}_{\tau\circ\Phi(f)}V) \to \mathcal{V}_n({}_\tau V)$.

This is a *twisted* representation since it involves a collection (O.17) of R-modules, rather than a single R-module. Although the modules in this collection are all mutually isomorphic, they are not *canonically* isomorphic. Upgrading (O.17) to an *untwisted* (genuine) representation requires a consistent choice of identifications

$${}_{\tau\circ\Phi(f)}V \;\cong\; {}_\tau V \tag{O.18}$$

of coefficients for all f and τ. For certain choices of V this is possible:

Theorem S ([BPS21]). *In the special case when V is the Schrödinger representation V_{Sch}, if we pass to the stably universal central extension of $\mathrm{Map}(\Sigma_{g,1})$, then there are consistent identifications (O.18) and hence, for each $n \geqslant 2$, an untwisted unitary representation*

$$\widetilde{\mathrm{Map}}(\Sigma_{g,1}) \longrightarrow U(\mathcal{V}_n(V_{\mathrm{Sch}})).$$

When V is the regular representation $\mathbb{Z}[\mathcal{H}(\Sigma_{g,1})]$ we prove an upper bound on the kernels of the twisted representations (O.17) of Theorem R.

Theorem T ([BPS21]). *The kernel of the twisted representation $\mathcal{V}_n({}_\bullet\mathbb{Z}[\mathcal{H}(\Sigma_{g,1})])$ of $\mathrm{Map}(\Sigma_{g,1})$ is contained in the intersection of the* ***Magnus kernel*** *and the nth term of the* ***Johnson filtration****.*

In the special case when $g = 1$ and $n = 2$, we also compute explicit matrices for the action of the twisted representation $\mathcal{V}_2({}_\bullet\mathbb{Z}[\mathcal{H}(\Sigma_{1,1})])$ of $\mathrm{Map}(\Sigma_{1,1}) \cong \mathbf{B}_3$ over the non-commutative ground ring $\mathbb{Z}[\mathcal{H}(\Sigma_{1,1})] \cong \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle/(ab = u^2 ba)$.

Untwisting on the Torelli group. In more recent work, we have proven a general untwisting result for the twisted representations (O.17) of Theorem R after restricting to the Torelli group:

Theorem U ([BPS23]; **Chapter 6**). *For any representation V of the Heisenberg group $\mathcal{H}(\Sigma_{g,1})$, if we restrict to the Torelli group $\mathfrak{T}(\Sigma_{g,1}) \subseteq \mathrm{Map}(\Sigma_{g,1})$, the twisted representation (O.17) may be untwisted to obtain, for each $n \geqslant 2$, a genuine, untwisted representation*

$$\mathfrak{T}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_R(\mathcal{V}_n(V)).$$

Part I

Homology of moduli spaces

Chapter 1

Homology stability for asymptotic monopole moduli spaces

The results of this chapter have been published as [PT23] in joint work with Ulrike Tillmann.

Introduction

The topology of the moduli spaces of magnetic monopoles $\mathcal{M}_k$ has been the subject of intensive study for many decades. By a theorem of Donaldson [Don84], they have a model as spaces of rational functions on $\mathbb{C}P^1$. Via this model, their homotopy and homology groups are known to stabilise as $k \to \infty$ by a theorem of Segal [Seg79] and their homology (both stable and unstable) was completely computed by [Coh+91] in terms of the homology of the braid groups, which is completely known [CLM76].

The moduli spaces $\mathcal{M}_k$ are non-compact manifolds. Recently, a partial compactification of $\mathcal{M}_k$ has been constructed by Kottke and Singer [KS22] by adding certain boundary hypersurfaces $\mathcal{I}_\lambda$ to $\mathcal{M}_k$ indexed by partitions $\lambda = (k_1, \dots, k_r)$ of k.

Points in these boundary hypersurfaces are thought of as "ideal" monopoles of total charge k, with r "clusters" centred at different points in $\mathbb{R}^3$, with charges $k_1, \dots, k_r$, which are "widely separated" but nevertheless interact.

Our main theorem proves a homology stability result for these ideal monopole moduli spaces as the number of clusters of a fixed charge $c \geqslant 1$ goes to infinity:

Theorem 1.A. *Fix a positive integer c and a tuple $\lambda = (k_1, \dots, k_r)$, of fixed length r, of positive integers $k_i \neq c$. Write $\lambda[n]_c = (k_1, \dots, k_r, c, \dots, c)$, where c appears n times. There are natural stabilisation maps*

$$\mathcal{I}_{\lambda[n]_c} \longrightarrow \mathcal{I}_{\lambda[n+1]_c} \tag{1.1}$$

that induce isomorphisms on homology in all degrees $\leqslant n/2-1$ with $\mathbb{Z}$ coefficients and in all degrees $\leqslant n/2$ with field coefficients.

We also prove an analogous result for *moduli spaces of framed Dirac monopoles* (in other words *Gibbons-Manton torus bundles*; see §1.1.2 for the definitions) and, more generally, Gibbons-Manton $\mathbf{Z}$-bundles for any sequence $\mathbf{Z}$ of path-connected S^1-spaces; see Theorems 1.3.1 and 1.3.9.

These results follow from a general homology stability result (Proposition 1.2.3) for unordered configuration spaces with *non-local parameters*. Homology stability for configuration spaces whose points are labelled by elements of a fixed space X is well-known; these are configuration spaces with *local* parameters. However, the ideal monopole moduli spaces $\mathcal{I}_\lambda$ are *non-local*. The key observation in §1.2 is that homology stability only requires the parameters associated to a configuration to satisfy much weaker properties, which allows us to consider interesting non-local parameters. In [PT21], we recently proved a different homology stability result for non-local configuration spaces, namely for *configuration-section spaces*; this encouraged us to try to prove homology stability also in the context of ideal monopole moduli spaces. Proposition 1.2.3 is the abstract general result that applies in our situation in the present chapter. Though similar in nature, it neither is implied by nor implies the homology stability result in [PT21].

Outline. We first recall some background on moduli spaces of magnetic monopoles in §1.1: first on the moduli spaces themselves in §1.1.1 and then on their partial compactifications introduced by [KS22] in §1.1.2, whose boundary hypersurfaces are the ideal monopole moduli spaces. In §1.2 we then prove a general homology stability result for configuration spaces equipped with "non-local" data, deducing it from twisted homological stability for configuration spaces [Pal18b] (see also [Kra19]). In §1.3 we apply it to prove our main theorem, homology stability for ideal monopole moduli spaces, as well as an extension (Theorem 1.3.9) to Gibbons-Manton $\mathbf{Z}$-bundles more generally.

1.1 Monopole moduli space and boundary hypersurfaces

1.1.1 Monopole moduli space

We briefly recall from [AH88] some different monopole moduli spaces and the relations between them.

A *magnetic monopole* on $\mathbb{R}^3$ is a pair consisting of a connection A on the trivial principal $SU(2)$-bundle on $\mathbb{R}^3$ together with a field ϕ taking values in the associated Lie algebra $\mathfrak{su}(2)$. Fixing a framing, these may be viewed, respectively, as a smooth 1-form and a smooth function on $\mathbb{R}^3$ taking values in $\mathfrak{su}(2)$, which we may identify topologically as $\mathfrak{su}(2) \cong \mathbb{R}^3$. These data A and ϕ must satisfy the *Bogomolny equations* and a certain finiteness condition; see [AH88, pp. 14–15] for details. This finiteness condition implies that $\phi(x) \neq 0$ for $|x|$ sufficiently large, so the restriction of ϕ to $\mathbb{R}^3 \smallsetminus B_R(0)$ takes values in $\mathfrak{su}(2) \smallsetminus \{0\}$ for $R \gg 0$. The degree of this map is the charge of the monopole, and is always positive. The set of all magnetic monopoles of charge $k \geqslant 1$, up to gauge equivalence (automorphisms of the trivial bundle $\mathbb{R}^3 \times \mathfrak{su}(2) \to \mathbb{R}^3$), suitably topologised, is the *monopole moduli space* $\mathcal{N}_k$. A slight variation of the construction, quotienting by a smaller gauge group, yields a different space $\mathcal{M}_k$ related to $\mathcal{N}_k$ by a principal S^1-bundle

$$\mathcal{M}_k \longrightarrow \mathcal{N}_k = \mathcal{M}_k/S^1. \tag{1.2}$$

Translation of solutions to the Bogomolny equations in $\mathbb{R}^3$ also defines a principal $\mathbb{R}^3$-bundle

$$\mathcal{N}_k \longrightarrow \mathcal{M}_k^0 = \mathcal{N}_k/\mathbb{R}^3. \tag{1.3}$$

The spaces $\mathcal{M}_k$ and $\mathcal{M}_k^0$ admit the structure of hyperKähler manifolds of dimensions $4k$ and $4k-4$ respectively. For charge $k=1$ we have $\mathcal{M}_1^0 = pt$ (and $\mathcal{M}_1 \cong S^1 \times \mathbb{R}^3$) and for $k=2$, the 4-manifold $\mathcal{M}_2^0$ is known as the *Atiyah-Hitchin manifold* and has been studied in detail in [AH88].

By [Don84], $\mathcal{M}_k$ is homeomorphic to the space R_k of degree-k rational self-maps of $\mathbb{C}P^1$ that send ∞ to 0. Thus, it is also homeomorphic to the space R'_k of degree-k rational self-maps of $\mathbb{C}P^1$ that send ∞ to 1. The points of the space R'_k may conveniently be described as pairs (p,q) of coprime monic polynomials with coefficients in $\mathbb{C}$, both of degree k. Identifying these polynomials with their sets of roots, we obtain a natural embedding

$$R'_k \hookrightarrow SP^k(\mathbb{C}) \times SP^k(\mathbb{C})$$

whose image consists of all pairs (A,B) of multi-subsets of $\mathbb{C}$ that are disjoint. On the other hand, the space R_k is convenient in that the circle action is easy to see: under the isomorphism $\mathcal{M}_k \cong R_k$, the circle action is given simply by multiplying rational self-maps of $\mathbb{C}P^1$ by $e^{i\theta}$.

The fundamental group of $\mathcal{M}_k$ is $\mathbb{Z}$, by [Seg79, Proposition 6.4]. Also, by [AH88, chapter 2], the fundamental group of $\mathcal{N}_k$ is $\mathbb{Z}/k$ and the projection map (1.2) induces the reduction-mod-k map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/k$. It follows from the long exact sequence that (1.2) induces isomorphisms on π_i for all $i \geqslant 2$, so $\mathcal{M}_k$ and $\mathcal{N}_k$ have the same universal cover, up to homotopy equivalence, which is denoted by $\mathcal{X}_k$.

There are stabilisation maps $\mathcal{M}_k \to \mathcal{M}_{k+1}$, which may be defined under the isomorphism $\mathcal{M}_k \cong R_k$ by adding to a given rational self-map a new zero and a new pole "far away" from the origin. (This is not invariant under the circle action, so it does not descend to a stabilisation map on the moduli spaces $\mathcal{N}_k$.) The stabilisation maps $\mathcal{M}_k \to \mathcal{M}_{k+1}$ induce isomorphisms on homotopy groups (and hence also homology groups) in a stable range, by [Seg79]. Lifting to universal covers, it follows that there are also stabilisation maps $\mathcal{X}_k \to \mathcal{X}_{k+1}$ that induce isomorphisms on homotopy (and homology) groups in a stable range.

By the main theorem of [Seg79], the homotopy colimit of the stabilisation maps $\mathcal{M}_k \to \mathcal{M}_{k+1} \to \cdots$ is weakly equivalent to $\Omega_0^2 S^2$. Thus the stable homology of $\mathcal{M}_k$ is the homology of $\Omega_0^2 S^2$ and the stable homology of $\mathcal{X}_k$ is the homology of the universal cover of $\Omega_0^2 S^2$. Moreover, the *unstable* homology of $\mathcal{M}_k$ (i.e. its homology outside of the stable range) is also known: by the main result of [Coh+91; Coh+93], the homology of $\mathcal{M}_k$ is isomorphic to the group homology of the braid group B_{2k}, which is completely computed [CLM76]. The *rational* unstable homology of $\mathcal{X}_k$ has also been computed by [SS96], and is significantly more complicated than the rational unstable homology of $\mathcal{M}_k$: the rational homology $H_*(\mathcal{M}_k;\mathbb{Q})$ is the same as that of the circle, so it has total dimension 2, whereas [SS96] shows that the rational homology $H_*(\mathcal{X}_k;\mathbb{Q})$ has total dimension k, concentrated in degrees of the form $2(k-d)$ where d is a divisor of k.

Notation 1.1.1. The principal bundles (1.2) and (1.3) arise from a principal (in particular free) action of the product $S^1 \times \mathbb{R}^3$ on on $\mathcal{M}_k$. If we first quotient by $\mathbb{R}^3$ (Euclidean translations) we obtain a principal $\mathbb{R}^3$-bundle

$$\mathcal{M}_k \longrightarrow \mathcal{M}_k^c = \mathcal{M}_k/\mathbb{R}^3. \tag{1.4}$$

In particular, we have a homotopy equivalence $\mathcal{M}_k^c \simeq \mathcal{M}_k$. (The superscript c stands for *centred* monopoles.) The quotient $\mathcal{M}_k^c$ is a $(4k-3)$-dimensional manifold and there is a principal S^1-bundle

$$\mathcal{M}_k^c \longrightarrow \mathcal{M}_k^0 = \mathcal{M}_k^c/S^1 = \mathcal{N}_k/\mathbb{R}^3. \tag{1.5}$$

1.1.2 Boundary hypersurfaces

Kottke and Singer [KS22] have constructed a partial compactification of $\mathcal{M}_k^c \simeq \mathcal{M}_k$ of the form

$$\overline{\mathcal{M}}_k^c = \bigsqcup_\lambda \mathcal{I}_\lambda^c \tag{1.6}$$

with strata indexed by sequences $\lambda = (k_1, \ldots, k_r)$ of positive integers that sum to k. The stratum $\mathcal{I}_{(k)}^c$ is the interior $\mathcal{M}_k^c$ of $\overline{\mathcal{M}}_k^c$ and the union of all strata $\mathcal{I}_\lambda^c$ for $\lambda \neq (k)$ is the boundary of $\overline{\mathcal{M}}_k^c$. Points in $\mathcal{I}_\lambda^c$ are called *centred ideal* monopoles associated to the partition λ.

We will not recall here the construction of $\mathcal{I}_\lambda^c$ in [KS22]; instead we will take an alternative characterisation of $\mathcal{I}_\lambda^c$ to be its definition (see Definitions 1.1.6 and 1.1.12 and Remark 1.1.13). To begin with, we recall the definitions of ordered and unordered configuration spaces.

Definition 1.1.2. For any space M, let us write $F_r(M) = \{(v_1, \ldots, v_r) \in M^r \mid v_i \neq v_j \text{ for } i \neq j\}$ for the *ordered configuration space* of r points in M, topologised as a subspace of the product M^r. We also write $C_r(M) = F_r(M)/\Sigma_r$ for the *unordered configuration space* of r points in M.

Recall (see for example [FH01, Theorem V.1.1]) that the degree-$(d-1)$ cohomology of $F_r(\mathbb{R}^d)$ is given by:

$$H^{d-1}(F_r(\mathbb{R}^d);\mathbb{Z}) \;\cong\; \mathbb{Z}\{\alpha_{ij} \mid 1 \leqslant i < j \leqslant r\}, \tag{1.7}$$

where α_{ij} is the pullback of a generator of $H^{d-1}(S^{d-1};\mathbb{Z})$ along the map $\iota_{ij}\colon F_r(\mathbb{R}^d) \to S^{d-1}$ given by the formula

$$\mathbf{x} = (x_1, \ldots, x_r) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}.$$

Since principal S^1-bundles over a space X are classified by $H^2(X;\mathbb{Z})$, this means that principal S^1-bundles over $F_r(\mathbb{R}^3)$ are classified by integer linear combinations of the α_{ij}. (One dimension lower, the same data classifies principal $\mathbb{Z}$-bundles over $F_r(\mathbb{R}^2)$, in other words regular coverings of $F_r(\mathbb{R}^2)$ with infinite cyclic deck transformation group.)

Definition 1.1.3 ([KS22, Definition 4.6 and the paragraph preceding it])**.** For a sequence of integers $\lambda = (k_1, \ldots, k_r)$, the corresponding *Gibbons-Manton circle factors* are the principal S^1-bundles

$$S_{\lambda,j} \longrightarrow F_r(\mathbb{R}^3),$$

for $j \in \{1, \ldots, r\}$, corresponding to the element $\sum_{i\in\{1,\ldots,r\}, i\neq j} k_i.\alpha_{ij}$, where we define $\alpha_{ij} = -\alpha_{ji}$ if $i > j$. The *Gibbons-Manton torus bundle* weighted by λ is the principal T^r-bundle

$$\widetilde{\mathcal{T}}_\lambda = \bigoplus_{j=1}^{r} S_{\lambda,j} \longrightarrow F_r(\mathbb{R}^3). \tag{1.8}$$

A point in $S_{\lambda,j}$ may be thought of as an ordered configuration together with a non-local circle parameter encoding the interaction of the jth particle with all other particles, weighted by λ. A point in $\widetilde{\mathcal{T}}_\lambda$ may similarly be thought of as an ordered configuration together with r non-local circle parameters, each encoding the interaction of one of the particles with all of the others (again, weighted by λ).

Definition 1.1.4. The symmetric group Σ_r acts on $F_r(\mathbb{R}^3)$ by permuting the particles. Let $\Sigma_\lambda \leqslant \Sigma_r$ be the stabiliser of $\lambda = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ under the obvious permutation action of Σ_r on $\mathbb{Z}^r$. Then the action of Σ_λ on $F_r(\mathbb{R}^3)$ lifts to a well-defined action on $\widetilde{\mathcal{T}}_\lambda$. The *Gibbons-Manton configuration space* is the quotient space $\mathcal{T}_\lambda = \widetilde{\mathcal{T}}_\lambda/\Sigma_\lambda$. Note that there is a principal T^r-bundle

$$\mathcal{T}_\lambda \longrightarrow F_r(\mathbb{R}^3)/\Sigma_\lambda. \tag{1.9}$$

In particular, when $k_1 = k_2 = \cdots = k_r$, we have $\Sigma_\lambda = \Sigma_r$ and $\mathcal{T}_\lambda$ is a principal T^r-bundle over the unordered configuration space $C_r(\mathbb{R}^3)$.

Remark 1.1.5. One may make analogous definitions for Euclidean spaces $\mathbb{R}^d$ in general, replacing $S^1 = K(\mathbb{Z},1)$ with $K(\mathbb{Z}, d-2)$, so that $\mathcal{T}_\lambda$ is a principal $K(\mathbb{Z}, d-2)^r$-bundle over $F_r(\mathbb{R}^d)$. For example, when $d = 2$, it is a regular covering space with deck transformation group isomorphic to $\mathbb{Z}^r$. In particular, for $d = 2$ and $\lambda = (1, 1, \ldots, 1)$, it is the regular covering space corresponding to the homomorphism

$$\varphi_r \colon \pi_1(F_r(\mathbb{R}^2)) = PB_r \longrightarrow \mathbb{Z}^r$$

that records, for each $1 \leqslant i \leqslant r$, the total winding number of the ith strand of a given pure braid around the other $r-1$ strands. This is a disconnected covering with components indexed by $\mathrm{coker}(\varphi_r)$; each connected component is a classifying space for the subgroup $\ker(\varphi_r) \leqslant PB_r$ consisting of those pure braids b where each strand of b has zero total winding number around the other $r-1$ strands:

$$\bigsqcup_{\mathrm{coker}(\varphi_r)} B(\ker(\varphi_r)) \longrightarrow F_r(\mathbb{R}^2).$$

Definition 1.1.6. The *moduli space of ideal monopoles* of weight λ is defined as follows. Recall that the monopole moduli space $\mathcal{M}_k$ is equipped with a circle action. The product $\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}$ is therefore equipped with an action of the torus T^r. We define $\widetilde{\mathcal{I}}_\lambda$ to be the total space of the fibre bundle associated to the principal T^r-bundle $\widetilde{\mathcal{T}}_\lambda$ by changing the fibre to $\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}$. In other words, it is the Borel construction

$$\widetilde{\mathcal{I}}_\lambda = \widetilde{\mathcal{T}}_\lambda \times_{T^r} \left(\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}\right) \longrightarrow F_r(\mathbb{R}^3).$$

We then define $\mathcal{I}_\lambda = \widetilde{\mathcal{I}}_\lambda/\Sigma_\lambda$, where Σ_λ acts diagonally on $\widetilde{\mathcal{T}}_\lambda$ (see Definition 1.1.4) and on the product $\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}$. The *moduli space of ideal monopoles* of weight λ is this space $\mathcal{I}_\lambda$. It is the total space of a fibre bundle

$$\pi \colon \mathcal{I}_\lambda \longrightarrow F_r(\mathbb{R}^3)/\Sigma_\lambda \tag{1.10}$$

with fibre $\mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_r}$.

Remark 1.1.7. This is not yet the boundary stratum $\mathcal{I}^c_\lambda$ constructed by [KS22] in their partial compactification of $\mathcal{M}^c_k$, since it has the wrong dimension. Recall that the dimension of $\mathcal{M}^c_k$ is $4k-3$, so its boundary strata must have dimension $4k-4$, whereas the dimension of $\mathcal{I}_\lambda$ is $4k+3r$. The definition of $\mathcal{I}^c_\lambda$ is similar to that of $\mathcal{I}_\lambda$ (and these two spaces are homotopy equivalent; see Remark 1.1.10), using the centred moduli spaces $\mathcal{M}^c_{k_i}$ instead of $\mathcal{M}_{k_i}$ and using a centred version of the configuration space, which we define next.

Definition 1.1.8. The ordered centred configuration space $F^c_r(\mathbb{R}^3) \subseteq F_r(\mathbb{R}^3)$ is defined to be the space of all ordered configurations $(x_1, \ldots, x_r)$ in $F_r(\mathbb{R}^3)$ such that

$$\sum_{i=1}^{r} x_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{r} |x_i|^2 = 1 \tag{1.11}$$

and has dimension $3r-4$. The unordered version $C^c_r(\mathbb{R}^3) \subseteq C_r(\mathbb{R}^3)$ is defined similarly and we have $C^c_r(\mathbb{R}^3) = F^c_r(\mathbb{R}^3)/\Sigma_r$.

Definition 1.1.9. The *moduli space of centred ideal monopoles* of weight λ is defined as follows. Analogously to Definition 1.1.6, consider the Borel construction

$$\widetilde{\mathcal{I}}^c_\lambda = \widetilde{\mathcal{T}}^c_\lambda \times_{T^r} \left(\mathcal{M}^c_{k_1} \times \cdots \times \mathcal{M}^c_{k_r}\right) \longrightarrow F^c_r(\mathbb{R}^3),$$

where $\widetilde{\mathcal{T}}^c_\lambda$ is the restriction of $\widetilde{\mathcal{T}}_\lambda \to F_r(\mathbb{R}^3)$ to $F^c_r(\mathbb{R}^3) \subseteq F_r(\mathbb{R}^3)$. We then define $\mathcal{I}^c_\lambda = \widetilde{\mathcal{I}}^c_\lambda/\Sigma_\lambda$, which is the total space of a fibre bundle

$$\pi\colon \mathcal{I}^c_\lambda \longrightarrow F^c_r(\mathbb{R}^3)/\Sigma_\lambda \tag{1.12}$$

with fibre $\mathcal{M}^c_{k_1} \times \cdots \times \mathcal{M}^c_{k_r}$.

Remark 1.1.10. Since the inclusion $F^c_r(\mathbb{R}^3) \subseteq F_r(\mathbb{R}^3)$ and the projection (1.4) are homotopy equivalences, we also have

$$\mathcal{I}^c_\lambda \simeq \mathcal{I}_\lambda.$$

They are therefore interchangeable when studying their homotopical properties individually. However, they are not homeomorphic, and $\mathcal{I}^c_\lambda$ (rather than $\mathcal{I}_\lambda$) is the boundary stratum corresponding to λ in the partial compactification of [KS22]. Note that the space $\mathcal{I}^c_\lambda$ has the correct dimension, namely $(3r-4) + \sum_{i=1}^{r}(4k_i - 3) = 3r - 4 + 4k - 3r = 4k - 4$.

However, since we focus in this chapter on the homological properties of $\mathcal{I}_\lambda$, the difference between $\mathcal{I}_\lambda$ and $\mathcal{I}^c_\lambda$ will not be relevant to us.

Terminology 1.1.11. When $\lambda = (1, 1, \ldots, 1)$, the moduli space $\mathcal{I}_\lambda$ is called the *moduli space of widely separated magnetic monopoles*. This terminology follows the intuition that points $x \in \mathcal{I}_\lambda$ should be thought of as monopoles of total charge k, with r different "clusters" centred at the points $\pi(x)$, with charges k_i, which are "widely separated" but nevertheless interact: these interactions are encoded in the structure group T^r of the bundle (1.10).

Definition 1.1.12. The *moduli space of framed Dirac monopoles* of weight λ is the Gibbons-Manton configuration space $\mathcal{T}_\lambda$ of Definition 1.1.4, which has the total space of the Gibbons-Manton torus bundle (1.8) as a finite covering.

Remark 1.1.13 (*On definitions.*)**.** Definitions 1.1.6 and 1.1.12 are not precisely the definitions given in [KS22]. By [KS22, Theorem 4.9], the moduli space of ideal monopoles of weight λ – according to their definition – is equivalent to the space denoted by $\widetilde{\mathcal{I}}_\lambda$ in Definition 1.1.6. However, as pointed out in [KS22] (see the Remark on page 53), this is not the correct space to form the boundary hypersurfaces of the compactification $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$, and one should instead pass to the quotient space $\mathcal{I}_\lambda = \widetilde{\mathcal{I}}_\lambda/\Sigma_\lambda$. We have therefore made this replacement in Definition 1.1.6. (The difference between $\mathcal{I}_\lambda$ and its finite covering space $\widetilde{\mathcal{I}}_\lambda$ is not significant in [KS22] since they are interested primarily in studying the geometry of these spaces *locally*.) Similarly, by [KS22, Proposition 4.8], the moduli space of framed Dirac monopoles of weight λ – according to their definition – is equivalent to the total space $\widetilde{\mathcal{T}}_\lambda$ of the Gibbons-Manton torus bundle (1.8). For the same reasons as above, we instead consider the moduli space of framed Dirac monopoles to be the quotient space $\mathcal{T}_\lambda = \widetilde{\mathcal{T}}_\lambda/\Sigma_\lambda$ (Definition 1.1.12). Henceforth, we treat Definitions 1.1.6 and 1.1.12 as the *definitions* of the ideal and framed Dirac monopole moduli spaces respectively.

Remark 1.1.14. Another small difference between our definition and that of [KS22] concerns the action of the symmetric group Σ_λ. In [KS22], the ordered centred configuration spaces (cf. Definition 1.1.8) are defined in a slightly asymmetric way, which does not allow for taking a quotient by Σ_λ (as we do above), since they single out one point of the configuration to lie at $0 \in \mathbb{R}^3$. We have modified the definition to be more symmetric by instead requiring the centre of mass to lie at 0. This does not change the homeomorphism type of the centred ordered configuration space and it has the advantage of having a natural action of the full symmetric group Σ_r, not just Σ_{r-1}.

Remark 1.1.15. When $k = 1$, the monopole moduli space $\mathcal{M}_1^c$, as an S^1-space, is simply S^1 itself. Thus, according to Definition 1.1.6, we have $\widetilde{\mathcal{I}}_{(1,\ldots,1)} = \widetilde{\mathcal{T}}_{(1,\ldots,1)}$. The moduli space of widely separated magnetic monopoles $\mathcal{I}_{(1,\ldots,1)}$ (cf. Terminology 1.1.11) is therefore the quotient of the total space of the Gibbons-Manton torus bundle $\widetilde{\mathcal{T}}_{(1,\ldots,1)}$ by the symmetric group Σ_r.

Remark 1.1.16 (*Higher codimension boundary strata.*)**.** The space (1.6) is only a partial compactification of $\mathcal{M}_k$: it is a manifold with boundary whose interior is $\mathcal{M}_k$, but it is still non-compact. In a recent preprint [FKS18], a full compactification of $\mathcal{M}_k$ is proposed,[1] which is a smooth manifold with corners that recovers the partial compactification $\overline{\mathcal{M}}_k$ if one discards corners of codimension greater than 1. It would be interesting to extend our study of the homology of $\mathcal{I}_\lambda$ to the deeper boundary strata of this full compactification.

1.2 Homology stability for configurations with non-local data

The goal of this section is to prove Proposition 1.2.3, which gives sufficient conditions that imply homology stability for configuration spaces equipped with additional (possibly “non-local”) parameters.

Labelled configuration spaces, where each separate point of a configuration is equipped with a label taking values in a fixed space, are the most obvious examples of this setting – we refer to these as configuration spaces with *local* data, since the labels are associated to individual points of the configuration. However, the key observation of this section is that the proof of homology stability requires only weaker properties of the parameters, which are satisfied also in other interesting, *non-local* settings.

In particular, in §1.3 we will apply this to our key motivating example of non-local configuration spaces, Gibbons-Manton torus bundles and moduli spaces of ideal monopoles, where the parameters are genuinely non-local, encoding the pairwise interactions of the points of the configuration.

For the general setting of non-local configuration spaces, let us consider a connected manifold $\overline{M}$ with non-empty boundary and denote its interior by M. We first recall the definition of the *stabilisation maps* between the ordered and unordered configuration spaces $F_n(M)$ and $C_n(M)$ (see Definition 1.1.2).

Definition 1.2.1. Choose a collar neighbourhood of $\overline{M}$, in order words an open neighbourhood U of $\partial\overline{M}$ and an identification $\varphi\colon U \cong \partial\overline{M} \times [0,1)$ that restricts to $\varphi(p) = (p,0)$ for $p \in \partial\overline{M} \subset U$. (This exists by [Bro62].) Let $\widehat{M}$ be the result of thickening the collar neighbourhood, i.e. the union of $\overline{M}$ and $\partial\overline{M} \times (-1,1)$ along the identification φ. Also, choose a diffeomorphism $(-1,1) \cong (0,1)$ that restricts to the identity on $(1-\epsilon,1)$ for some $\epsilon > 0$. Taking the product with the identity on $\partial\overline{M}$ and extending by the identity on $M \smallsetminus U$, this determines a diffeomorphism $\theta\colon \widehat{M} \cong M$. Finally, choose a basepoint $* \in \partial\overline{M}$. These choices determine a *stabilisation map*

$$F_n(M) \longrightarrow F_{n+1}(M) \tag{1.13}$$

between ordered configuration spaces on M by adjoining the point $(*, -\frac{1}{2}) \in \widehat{M}$ to a configuration in M and then applying the diffeomorphism θ to each point, i.e. the configuration $(p_1,\ldots,p_n)$ is

[1] Although full details of its (recursive) construction are deferred to forthcoming work of the same authors.

sent to $(\theta(p_1), \ldots, \theta(p_n), \theta((*, -\frac{1}{2})))$. This evidently respects the actions of the symmetric groups on $F_n(M)$ and on $F_{n+1}(M)$, so it also descends to a stabilisation map at the level of unordered configuration spaces:

$$C_n(M) \longrightarrow C_{n+1}(M), \tag{1.14}$$

as well as intermediate quotients between ordered and unordered configuration spaces, namely:

$$F_n(M)/G \longrightarrow F_{n+1}(M)/H \tag{1.15}$$

for any subgroups $G \subseteq \Sigma_n$ and $H \subseteq \Sigma_{n+1}$ such that the natural inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$ takes G into H.

Remark 1.2.2. Up to homotopy, the stabilisation maps (1.13) and (1.14) depend only on the choice of boundary-component of $\bar{M}$ containing the basepoint $*$. These maps (or maps homotopic to them) were introduced in [McD75, §4] and [Seg79, Appendix]; see also [Ran13, §4] or [Pal18b, §2.2].

Let us now consider the sequence

$$\cdots \to C_n(M) \longrightarrow C_{n+1}(M) \to \cdots \tag{1.16}$$

given by the stabilisation maps (1.14) and let

$$\cdots \to E_n \longrightarrow E_{n+1} \to \cdots \tag{1.17}$$

be another sequence of spaces and maps, equipped with fibrations

$$f_n \colon E_n \longrightarrow C_n(M) \tag{1.18}$$

making the evident squares commute. Also choose basepoints $c_n \in C_n(M)$ compatible with the stabilisation maps (1.16).

Proposition 1.2.3. *Fix path-connected spaces Y and Z and suppose that $f_n^{-1}(c_n) = Z^n \times Y$ for all n. Fix a basepoint $* \in Z$. Moreover, we assume also that*

- *the monodromy $\pi_1(C_n(M)) \to \mathrm{hAut}(Z^n \times Y)$ of (1.18) is the projection onto the symmetric group followed by the obvious permutation action on the factors of the product Z^n;*
- *the restriction $Z^n \times Y \to Z^{n+1} \times Y$ of the lifted stabilisation map (1.17) to fibres over basepoints is the natural inclusion $(z_1, \ldots, z_n, y) \mapsto (*, z_1, \ldots, z_n, y)$.*

Then the sequence (1.17) is homologically stable: the map $E_n \to E_{n+1}$ induces isomorphisms on homology in all degrees $\leqslant n/2 - 1$ with $\mathbb{Z}$ coefficients and in all degrees $\leqslant n/2$ with field coefficients.

Example 1.2.4. One source of examples of fibrations (1.18) over configuration spaces $C_n(M)$ equipped with lifted stabilisation maps (1.17) that satisfy the two conditions of Proposition 1.2.3 is *configuration spaces with local data*. This means that we choose a fibration $f \colon E \to \bar{M}$ with path-connected fibres, where $M = \mathrm{int}(\bar{M})$, trivialised over a disc $D \subset \partial\bar{M}$. Then we set

$$E_n = \bigl\{ \{y_1, \ldots, y_n\} \in C_n(E) \bigm| f(y_i) \neq f(y_j) \text{ for } i \neq j \bigr\},$$

the space of unordered configurations in M where each point x of the configuration is equipped with a label $y \in f^{-1}(x)$. In this setting, the space Z is the fibre of f over $* \in D$. The data in this example is "local" in the sense that each label is associated to a single point in the configuration.

However, there also exist labelling data (1.18) and (1.17), satisfying the two conditions of Proposition 1.2.3, that do *not* arise in this way. We will call these "non-local" data:

Definition 1.2.5. A *system of configuration spaces equipped with non-local data* is a choice of (1.18) and (1.17) that do not arise as described in Example 1.2.4 above.

Remark 1.2.6. Proposition 1.2.3, in the setting of configuration spaces with *local* data, is well-known: see [KM18, Appendix A] or [CP15, Appendix B]. The point of this section is to observe that it also holds in a more general setting, requiring just the two assumptions of Proposition 1.2.3, which includes also configuration spaces with *non-local* data. We will see in §1.3 that asymptotic monopole moduli spaces are examples of configuration spaces with non-local data: this is our key motivating example. We note, on the other hand, that configuration-mapping spaces, considered in [PT21], are in general not examples of configuration spaces with non-local data in the sense of Proposition 1.2.3, as the associated monodromy action does not in general factor through the symmetric group. See [PT22, §9] for a detailed study of the monodromy action for configuration-mapping spaces.

In order to prove Proposition 1.2.3, we first need to recapitulate some definitions and results from [Pal18b]. Recall that we are considering a connected manifold $\overline{M}$ with non-empty boundary whose interior we denote by M. Associated to this manifold there is a certain category $\mathcal{B}(M)$, the *partial braid category on* M, whose objects are non-negative integers $\{0, 1, 2, \ldots\}$ and whose morphisms are "partial braids" in $M \times [0, 1]$; the precise definition is given in [Pal18b, §2.3].[2] This category comes equipped with an endofunctor S that acts by $+1$ on objects as well as a natural transformation $\iota\colon \mathrm{id} \Rightarrow S$.

Definition 1.2.7 ([Pal18b, Definitions 2.2 and 3.1])**.** A *twisted coefficient system* for the sequence (1.16) of unordered configuration spaces on M, defined over a ring R, is a functor $T\colon \mathcal{B}(M) \to R\text{-Mod}$. The *degree* of a twisted coefficient system T, taking values in $\{-1, 0, 1, 2, 3, \ldots\} \cup \{\infty\}$, is defined recursively by setting $\deg(0) = -1$ and declaring that $\deg(T) \leqslant d$ if and only if $\deg(\Delta T) \leqslant d - 1$, where ΔT is the cokernel of the natural transformation $T\iota\colon T \Rightarrow TS$.

Remark 1.2.8. In [Pal18b], the ground ring R is always assumed to be $\mathbb{Z}$, but everything generalises directly to an arbitrary ground ring R.

For any twisted coefficient system T, the morphisms $\iota_n\colon n \to Sn = n+1$, which between them constitute the natural transformation ι, induce homomorphisms $T(n) \to T(n+1)$. Together with the stabilisation maps (1.16), these induce homomorphisms

$$H_*(C_n(M); T(n)) \longrightarrow H_*(C_{n+1}(M); T(n+1)) \tag{1.19}$$

of twisted homology groups. The main result of [Pal18b] is the following.

Theorem 1.2.9 ([Pal18b, Theorem A])**.** *If T is a twisted coefficient system for (1.16) of degree d, then the map of twisted homology groups (1.19) is an isomorphism in the range of degrees $* \leqslant \frac{1}{2}(n - d)$.*

An important family of examples of finite-degree twisted coefficient systems are defined on the category $\mathrm{FI}\sharp$,[3] which is the category whose objects are non-negative integers and whose morphisms from m to n are the partially-defined injections from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. For any manifold M, there is a canonical functor $f_M\colon \mathcal{B}(M) \to \mathrm{FI}\sharp$, so any functor $\mathrm{FI}\sharp \to R\text{-Mod}$ determines a twisted coefficient system for any manifold M.

Construction 1.2.10 (A generalisation of [Pal18b, Example 4.1])**.** Choose path-connected spaces Y, Z and a basepoint $* \in Z$. Also choose an integer $q \geqslant 0$ and a field K. There is a functor

$$T_{Z,Y,q,F}\colon \mathrm{FI}\sharp \longrightarrow K\text{-Mod} \tag{1.20}$$

that acts on objects by $n \mapsto H_q(Z^n \times Y; K)$ and, on morphisms, sends each partially-defined injection $j\colon \{1, \ldots, m\} \dashrightarrow \{1, \ldots, n\}$ to the map on homology induced by the map $Z^m \times Y \to Z^n \times Y$ defined by $(z_1, \ldots, z_m, y) \mapsto (z_{j^{-1}(1)}, \ldots, z_{j^{-1}(n)}, y)$. Notice that $j^{-1}(i)$ is either a single element or empty; for the latter case, we interpret $z_\varnothing$ to mean the basepoint $*$ of Z.

Lemma 1.2.11. *For any manifold M, the twisted coefficient system*

$$T_{Z,Y,q,F} \circ f_M\colon \mathcal{B}(M) \longrightarrow K\text{-Mod} \tag{1.21}$$

given by composing (1.20) with the canonical functor $f_M\colon \mathcal{B}(M) \to \mathrm{FI}\sharp$ has degree at most q.

[2] In [Pal18b], the theory is developed more generally for configuration spaces with (local) labels in a space X. We will not need this level of generality here, so we will suppress it (equivalently, we take X to be the one-point space).

[3] This is denoted Σ in [Pal18b], but we use the more common notation $\mathrm{FI}\sharp$.

Proof. When Y is the one-point space, this is [Pal18b, Lemma 4.2]. The extra factor of Y in the product does not affect the proof at all (as long as Y is path-connected), so the proof of the general case is identical to that of [Pal18b, Lemma 4.2]. □

This completes our recapitulation of the necessary definitions and results of [Pal18b], and we may now complete the proof of Proposition 1.2.3.

Proof of Proposition 1.2.3. We will take field coefficients and prove homological stability up to degree $n/2$. This will automatically imply homological stability up to degree $n/2-1$ with integral coefficients (and hence any untwisted coefficients), via the short exact sequences of coefficients

$$1 \to \mathbb{Z}/(p^n) \longrightarrow \mathbb{Z}/(p^{n+1}) \longrightarrow \mathbb{Z}/(p) \to 1 \qquad \text{and} \qquad 1 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 1$$

and the fact that $\mathbb{Q}/\mathbb{Z}$ decomposes into the direct sum of $\mathrm{colim}_n(\mathbb{Z}/(p^n))$ over all primes p.

We therefore consider the Serre spectral sequence, with coefficients in a field K, associated to the fibration (1.18) and the map of Serre spectral sequences induced by the stabilisation maps downstairs (1.16) and upstairs (1.17). The map of E^2 pages is of the form

$$H_p(C_n(M); H_q(Z^n \times Y; K)) \longrightarrow H_p(C_{n+1}(M); H_q(Z^{n+1} \times Y; K)). \tag{1.22}$$

The first assumption of the proposition implies that the local coefficients appearing in the source and target of (1.22) are precisely those arising from the twisted coefficient system (1.21). The second assumption implies that the map (1.22) is precisely the one induced by the stabilisation maps (1.16) together with the morphisms $+1\colon n \to n+1$ of FI♯; thus it is the map (1.19) for $T = $ (1.21). By Lemma 1.2.11, this twisted coefficient system has degree at most q. Hence Theorem 1.2.9 implies that (1.22) is an isomorphism for all $p \leqslant \frac{1}{2}(n-q)$, in particular for all $p+q \leqslant n/2$. A spectral sequence comparison argument then implies that the map on $H_*(-;K)$ induced by $E_n \to E_{n+1}$ is an isomorphism in degrees $* \leqslant n/2$. □

Remark 1.2.12. One may prove Proposition 1.2.3 using the twisted homological stability result [Kra19, Theorem D] instead of the twisted homological stability result [Pal18b, Theorem A], although this results in a range of degrees one smaller, namely $n/2-1$ for field coefficients and $n/2-2$ for integral coefficients.

Remark 1.2.13. The map (1.22) of E^2 pages of Serre spectral sequences is *split-injective in all degrees* by [Pal18b, Theorem A]. However, this does not in general imply split-injectivity in the limit, so we cannot deduce from this that $E_n \to E_{n+1}$ induces split-injections on homology. Anticipating Remark 1.3.7, there are obstructions to proving split-injectivity on homology for configurations with non-local data, in contrast to the case of ordinary configurations and twisted homology.

1.3 Homology stability for asymptotic monopole moduli spaces

Fix a positive integer c and a tuple $\lambda = (k_1, \ldots, k_r)$ of positive integers that sum to k. Denote by $\lambda[n]_c$ the tuple $(k_1, \ldots, k_r, c, \ldots, c)$, where there are n appearances of c. For simplicity we will assume that $k_i \neq c$ for each i (if this is not the case we may simply remove these entries from λ and increase n appropriately). Our main theorem is the following.

Theorem 1.3.1. *There are natural stabilisation maps*

$$\mathcal{T}_{\lambda[n]_c} \longrightarrow \mathcal{T}_{\lambda[n+1]_c} \qquad \text{and} \qquad \mathcal{I}_{\lambda[n]_c} \longrightarrow \mathcal{I}_{\lambda[n+1]_c} \tag{1.23}$$

that induce isomorphisms on homology in all degrees $\leqslant n/2-1$ with $\mathbb{Z}$ coefficients and in all degrees $\leqslant n/2$ with field coefficients.

We first prove Theorem 1.3.1 for the Gibbons-Manton configuration spaces $\mathcal{T}_{\lambda[n]_c}$ in §1.3.1. We then show in §1.3.2 that homological stability is preserved in general when replacing each circle factor in the torus fibre of $\mathcal{T}_{\lambda[n]_c}$ with another space that is equipped with a circle action. In particular, we deduce the second part of Theorem 1.3.1, since moduli spaces of ideal monopoles $\mathcal{I}_{\lambda[n]_c}$ are special cases of this construction.

1.3.1 Gibbons-Manton torus bundles

Recall that the Gibbons-Manton torus bundle $\mathcal{T}_{\lambda[n]_c}$ has base space $F_{r+n}(\mathbb{R}^3)/\Sigma_{\lambda[n]_c}$, where $\Sigma_{\lambda[n]_c} = \Sigma_\lambda \times \Sigma_n$. By abuse of notation, we will write

$$F_{r+n}(\mathbb{R}^3)/\Sigma_{\lambda[n]_c} =: C_{\lambda,n}(\mathbb{R}^3).$$

A point in this space consists of two disjoint configurations in $\mathbb{R}^3$: one λ-partitioned configuration of r points and one unordered configuration of n points.

Our first goal in this section is to lift the classical stabilisation maps of configuration spaces

$$C_{\lambda,n}(\mathbb{R}^3) \longrightarrow C_{\lambda,n+1}(\mathbb{R}^3) \tag{1.24}$$

(see Definition 1.2.1) to the Gibbons-Manton torus bundles:

$$\begin{array}{ccc} \mathcal{T}_{\lambda[n]_c} & \dashrightarrow & \mathcal{T}_{\lambda[n+1]_c} \\ \downarrow & & \downarrow \\ C_{\lambda,n}(\mathbb{R}^3) & \longrightarrow & C_{\lambda,n+1}(\mathbb{R}^3). \end{array} \tag{1.25}$$

Our second goal is to show that these lifted stabilisation maps satisfy the two hypotheses of Proposition 1.2.3. This will imply homological stability for Gibbons-Manton torus bundles, i.e. the first part of Theorem 1.3.1.

We begin with a lemma about pullbacks of Gibbons-Manton circle factors. To prepare for this, we first choose an explicit concrete model for the stabilisation maps (1.24); i.e. we make explicit some of the choices involved in Definition 1.2.1 in the case $M = \mathbb{R}^3$. Up to homotopy, this does not make any difference, but it will be convenient for the proof of Lemma 1.3.3 below to choose a specific representative of this homotopy class of maps.

Definition 1.3.2. We will in fact replace $\mathbb{R}^3$ with the open upper half-space $M = \mathbb{R}^2 \times (0,\infty)$. We may then take $\overline{M} = \mathbb{R}^2 \times [0,\infty)$ with the obvious collar neighbourhood, so $\widehat{M} = \mathbb{R}^2 \times (-1,\infty)$. Take $* = (0,0,0) \in \partial\overline{M} = \mathbb{R}^2 \times \{0\}$ as basepoint. With these choices (and identification of $\mathbb{R}^3$ with $\mathbb{R}^2 \times (0,\infty)$), the stabilisation map

$$F_{r-1}(\mathbb{R}^3) \longrightarrow F_r(\mathbb{R}^3) \tag{1.26}$$

of Definition 1.2.1 acts as follows. To a configuration $(x_1,\ldots,x_{r-1})$ in $\mathbb{R}^2 \times (0,\infty)$, we adjoin the new point $(0,0,-\frac{1}{2})$ and then "push upwards" the resulting configuration in $\mathbb{R}^2 \times (-1,\infty)$, i.e., we keep the first two coordinates of all points fixed and modify their third coordinates according to a chosen diffeomorphism $(-1,\infty) \cong (0,\infty)$.

Lemma 1.3.3. *Let $\lambda = (k_1,\ldots,k_r)$ for positive integers k_i and write $\lambda' = (k_1,\ldots,k_{r-1})$. Then the pullback of the circle bundle $S_{\lambda,j} \to F_r(\mathbb{R}^3)$ along the stabilisation map (1.26) is $S_{\lambda',j} \to F_{r-1}(\mathbb{R}^3)$ if $j \leqslant r-1$ and a trivial bundle if $j = r$.*

Proof. Recall that the bundle $S_{\lambda,j} \to F_r(\mathbb{R}^3)$ is the pullback of the universal S^1-bundle on $\mathbb{C}P^\infty$ along the map $F_r(\mathbb{R}^3) \to \mathbb{C}P^\infty$ given by the sum $\sum_{i=1,i\neq j}^{r} k_i.\iota_{ij}$ where $\iota_{ij}\colon F_r(\mathbb{R}^3) \to S^2 \subset \mathbb{C}P^\infty$ is given by

$$\mathbf{x} = (x_1,\ldots,x_r) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}. \tag{1.27}$$

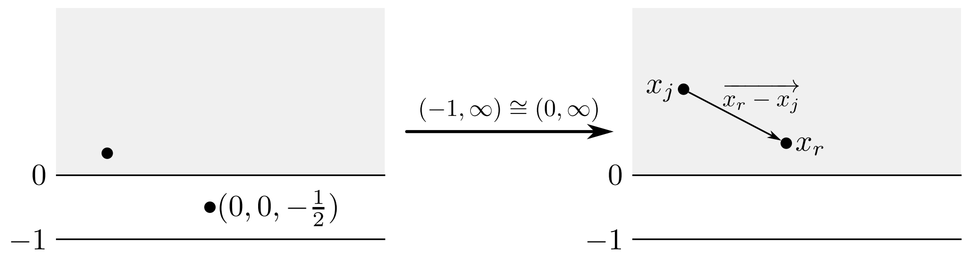


Figure 1.1 Any configuration in the image of the stabilisation map (1.26) has the form depicted on the right-hand side above (only the points x_j and x_r are actually depicted). Namely, x_j is the image, after applying the chosen diffeomorphism $(-1,\infty) \cong (0,\infty)$ to vertical coordinates, of an arbitrary point in $\mathbb{R}^2 \times (0,\infty)$, whereas x_r is the image of $(0,0,-\frac{1}{2})$. Since the diffeomorphism $(-1,\infty) \cong (0,\infty)$ is order-preserving, the vertical coordinate of x_j is higher than the vertical coordinate of x_r. Hence the (normalised) vector from x_j to x_r lies in the bottom hemisphere of S^2.

(Recall from Definition 1.3.2 that we have implicitly replaced $\mathbb{R}^3$ with $\mathbb{R}^2 \times (0,\infty)$; the formula above remains true after this replacement.) Its pullback to $F_{r-1}(\mathbb{R}^3)$ along the stabilisation map (1.26) is therefore given by the same formula, restricting ι_{ij} to $F_{r-1}(\mathbb{R}^3)$ along (1.26).

The key observation is the following. When $i = r$ and we restrict ι_{rj} to $F_{r-1}(\mathbb{R}^3)$ along (1.26), the vertical (third) coordinate of the point x_r will always be smaller than the vertical coordinate of the point x_j, due to the choices made in the construction of (1.26) in Definition 1.3.2; see Figure 1.1 for a detailed explanation. Thus the right-hand side of (1.27) always takes values in the bottom hemisphere of $S^2 \subset \mathbb{C}P^\infty$, and hence ι_{rj} restricted along (1.26) is nullhomotopic. By exactly analogous reasoning, when $j = r$ the map ι_{ir} restricted along (1.26) takes values in the top hemisphere of $S^2 \subset \mathbb{C}P^\infty$ and hence is also nullhomotopic.

Putting this all together, we deduce that the map $F_{r-1}(\mathbb{R}^3) \to \mathbb{C}P^\infty$ classifying the pullback of $S_{\lambda,r}$ is nullhomotopic, so this pullback is trivial. It also implies that the map $F_{r-1}(\mathbb{R}^3) \to \mathbb{C}P^\infty$ classifying the pullback of $S_{\lambda,j}$, for $j \leqslant r-1$, is the sum $\sum_{i=1,i\neq j}^{r-1} k_i.\iota_{ij}$, which is by definition the map that classifies $S_{\lambda',j}$. □

Remark 1.3.4. Recalling that we denote by α_{ij} the pullback of a fixed generator of $H^2(S^2;\mathbb{Z})$ along the map $\iota_{ij} \colon F_r(\mathbb{R}^3) \to S^2$, the discussion in the proof above implies that the stabilisation map $F_{r-1}(\mathbb{R}^3) \to F_r(\mathbb{R}^3)$ acts on $H^2(-;\mathbb{Z})$, in the basis (1.7), by $\alpha_{ij} \mapsto \alpha_{ij}$ if $j \leqslant r-1$ and $\alpha_{ir} \mapsto 0$. It is also easy to see that the automorphism $\sigma_* \colon F_r(\mathbb{R}^3) \to F_r(\mathbb{R}^3)$ induced by a permutation $\sigma \in \Sigma_r$ acts on generators of $H^2(F_r(\mathbb{R}^3);\mathbb{Z})$ by $\alpha_{ij} \mapsto \alpha_{\sigma^{-1}(i),\sigma^{-1}(j)}$. It follows from this that the pullback of the circle bundle $S_{\lambda,j}$ along σ_* is the circle bundle $S_{\sigma^{-1}(\lambda),\sigma^{-1}(j)}$.

Corollary 1.3.5. *The stabilisation map* (1.24) *lifts to* (1.25).

Proof. Let us write $\mu = \lambda[n+1]_c$ and $\mu' = \lambda[n]_c$. Lemma 1.3.3 then implies that the pullback of the Gibbons-Manton torus bundle $\widetilde{\mathcal{T}}_\mu = \bigoplus_{j=1}^{r+n+1} S_{\mu,j} \to F_{r+n+1}(\mathbb{R}^3)$ along the stabilisation map $F_{r+n}(\mathbb{R}^3) \to F_{r+n+1}(\mathbb{R}^3)$ is

$$\bigoplus_{j=1}^{r+n} S_{\mu',j} \oplus \mathrm{tr} = \widetilde{\mathcal{T}}_{\mu'} \oplus \mathrm{tr} \longrightarrow F_{r+n}(\mathbb{R}^3),$$

where tr denotes the trivial S^1-bundle. We therefore have bundle maps

$$\begin{array}{ccccc} \widetilde{\mathcal{T}}_{\lambda[n]_c} & \longrightarrow & \widetilde{\mathcal{T}}_{\lambda[n]_c} \oplus \mathrm{tr} & \longrightarrow & \widetilde{\mathcal{T}}_{\lambda[n+1]_c} \\ \downarrow & & \downarrow & & \downarrow \\ F_{r+n}(\mathbb{R}^3) & \xrightarrow{\mathrm{id}} & F_{r+n}(\mathbb{R}^3) & \longrightarrow & F_{r+n+1}(\mathbb{R}^3), \end{array} \tag{1.28}$$

where the left-hand square is an inclusion of a direct summand and the right-hand square is a pullback. This is equivariant with respect to the actions of $\Sigma_\lambda \times \Sigma_n$ and $\Sigma_\lambda \times \Sigma_{n+1}$. Quotienting by these actions, we obtain the lifted stabilisation map (1.25). □

In order to apply Proposition 1.2.3 to prove the first part of Theorem 1.3.1, we recall the following general fact about mondromy actions of fibrations.

Lemma 1.3.6. *Let $p\colon E \to B$ be a fibration over a based, path-connected space B admitting a universal covering $\pi\colon \widetilde{B} \to B$. Write $\widetilde{p}\colon \widetilde{E} \to \widetilde{B}$ for the pullback of p along π. Let F denote the fibre of p over the basepoint $b_0 \in B$ and note that the fibre of $\widetilde{p}$ over each point in $\pi^{-1}(b_0) \subset \widetilde{B}$ is also canonically identified with F. Then the monodromy action $\pi_1(B) \to \mathrm{hAut}(F)$ of p is equal to*

$$\pi_1(B) \cong \mathrm{Aut}(\pi\colon \widetilde{B} \to B) \longrightarrow \mathrm{hAut}(F),$$

where the left-hand isomorphism is the action by deck transformations and the right-hand map is given by the action on $\widetilde{E} \to \widetilde{B}$ by pullback.

Proof of Theorem 1.3.1 for $\mathcal{T}_{\lambda[n]_c}$. We first assume that $\lambda = ()$ and $r = 0$, so that $\lambda[n]_c$ is the tuple $(c, c, \ldots, c)$ of n copies of $c \geqslant 1$. We are now in the setting of Proposition 1.2.3 with (1.16) = (1.24), (1.17) = (1.25), (1.18) = (1.9) and $Z = S^1$.[4]

To complete the proof under this assumption, it suffices to check the two hypotheses of Proposition 1.2.3. The first hypothesis says that the monodromy $\pi_1(C_n(\mathbb{R}^3)) \to \mathrm{hAut}(T^n)$ of the Gibbons-Manton torus bundle (1.9) is the obvious permutation action on the circle factors of the torus T^n. To check this property, we use Lemma 1.3.6. In our setting, the universal covering of $C_n(\mathbb{R}^3)$ is $F_n(\mathbb{R}^3)$ and the pullback of $\mathcal{T}_{\lambda[n]_c} \to C_n(\mathbb{R}^3)$ is $\widetilde{\mathcal{T}}_{\lambda[n]_c} \to F_n(\mathbb{R}^3)$. The deck transformation action of $\pi_1(C_n(\mathbb{R}^3)) \cong \Sigma_n$ sends a loop (permutation) σ to the obvious automorphism σ_* of the ordered configuration space $F_n(\mathbb{R}^3)$. By Remark 1.3.4, the action of σ_* by pullback on Gibbons-Manton circle factors sends $S_{\lambda[n]_c,j}$ to $S_{\lambda[n]_c,\sigma^{-1}(j)}$ (here we use the fact that $\lambda[n]_c = (c, c, \ldots, c)$, so $\sigma^{-1}(\lambda[n]_c) = \lambda[n]_c$). Hence σ_* simply permutes the different circle factors in the Gibbons-Manton torus bundle; in particular its action on the torus fibre simply permutes the different copies of S^1, as required.

The second hypothesis of Proposition 1.2.3 says that the restriction of the lifted stabilisation map (1.25) to the fibres over the basepoints is the natural inclusion $T^n \to T^{n+1}$. This is immediate by construction of the lifted stabilisation map: it is given (before quotienting by symmetric groups and therefore also afterwards) by including into a direct sum with a (trivial) circle bundle and then a pullback of bundles.

Proposition 1.2.3 therefore implies that the stabilisation map $\mathcal{T}_{\lambda[n]_c} \to \mathcal{T}_{\lambda[n+1]_c}$ induces isomorphisms on homology in all degrees $\leqslant n/2 - 1$ with integral coefficients and in all degrees $\leqslant n/2$ with field coefficients, under our assumption that $\lambda = ()$.

To complete the proof of Theorem 1.3.1 for $\mathcal{T}_{\lambda[n]_c}$ we deduce the general case from the special case $\lambda = ()$ that we have just proven. To do this, we first observe that the constructions and results so far generalise directly to Gibbons-Manton torus bundles with *fixed points*. In this setting, we consider the subspace of the configuration space $C_{\lambda,n}(\mathbb{R}^3)$ where the λ-partitioned r-point configuration $\mathbf{x}$ is fixed and the unordered n-point configuration is free to move in the complement of $\mathbf{x}$; in other words, we consider the fibre of the projection $C_{\lambda,n}(\mathbb{R}^3) \to C_\lambda(\mathbb{R}^3)$ over $\mathbf{x} \in C_\lambda(\mathbb{R}^3)$. Let us denote this subspace by $C_{\lambda,n}(\mathbb{R}^3; \mathbf{x})$ and consider the restriction of $\mathcal{T}_{\lambda[n]_c} \to C_{\lambda,n}(\mathbb{R}^3)$ to $C_{\lambda,n}(\mathbb{R}^3; \mathbf{x})$, which we denote by $\mathcal{T}_{\lambda[n]_c}|_{\mathbf{x}}$. The difference between this setting and the $\lambda = ()$ setting considered above is that (1) the unordered n-point configuration now lies in $\mathbb{R}^3 \smallsetminus \mathbf{x}$, (2) there are r additional Gibbons-Manton circle factors encoding the pairwise interactions of the fixed points $\mathbf{x}$ with the free points and (3) the n Gibbons-Manton circle factors that encode the pairwise interactions of the n free points with each other are now modified to also take into account their interactions with the fixed points $\mathbf{x}$. The arguments above generalise directly to this setting and prove that restricted stabilisation maps

$$\mathcal{T}_{\lambda[n]_c}|_{\mathbf{x}} \longrightarrow \mathcal{T}_{\lambda[n+1]_c}|_{\mathbf{x}} \tag{1.29}$$

[4] Proposition 1.2.3 requires us to fix a basepoint on $Z = S^1$. This may initially appear problematic, since the circle fibres of the Gibbons-Manton circle factors (Definition 1.1.3) cannot be given consistent basepoints, since the Gibbons-Manton circle factors do not admit global sections. However, Proposition 1.2.3 only requires a choice of basepoint on a *single* fibre, namely the fibre over the base configuration, so this issue does not arise.

induce isomorphisms on homology in all degrees $\leqslant n/2 - 1$ with integral coefficients and in all degrees $\leqslant n/2$ with field coefficients. To deduce the same for the unrestricted stabilisation maps (1.25), we note that $\mathcal{T}_{\lambda[n]_c}|_{\mathbf{x}}$ is the fibre of the composite fibration

$$\mathcal{T}_{\lambda[n]_c} \longrightarrow C_{\lambda,n}(\mathbb{R}^3) \longrightarrow C_\lambda(\mathbb{R}^3),$$

where the second map forgets the unordered n-point configuration, consider the map of fibrations

$$\begin{array}{ccc} \mathcal{T}_{\lambda[n]_c}|_{\mathbf{x}} & \xrightarrow{(1.29)} & \mathcal{T}_{\lambda[n+1]_c}|_{\mathbf{x}} \\ \downarrow & & \downarrow \\ \mathcal{T}_{\lambda[n]_c} & \xrightarrow{(1.25)} & \mathcal{T}_{\lambda[n+1]_c} \\ & \searrow \quad \swarrow & \\ & C_\lambda(\mathbb{R}^3) & \end{array} \tag{1.30}$$

and apply a spectral sequence comparison argument to the corresponding map of Serre spectral sequences. □

Remark 1.3.7. For unordered configuration spaces, the stabilisation maps $C_n(\mathbb{R}^3) \to C_{n+1}(\mathbb{R}^3)$ have the additional property that they are split-injective on homology. This is essentially a consequence of the existence of forgetful maps $F_n(\mathbb{R}^3) \to F_r(\mathbb{R}^3)$ at the level of ordered configuration spaces that forget the last $n - r$ points of a configuration. Using these maps, standard techniques using transfer maps (see [McD75] or [MT14]) imply split-injectivity on homology for stabilisation maps of unordered configuration spaces. We record here the observation that the forgetful maps

$$\tau_{n,r} \colon F_n(\mathbb{R}^3) \longrightarrow F_r(\mathbb{R}^3) \tag{1.31}$$

do *not* naturally lift to Gibbons-Manton torus bundles (in contrast to the stabilisation maps, which do lift, by Corollary 1.3.5). In order to lift $\tau_{n,r}$ to Gibbons-Manton torus bundles $\widetilde{\mathcal{T}}_\lambda \to \widetilde{\mathcal{T}}_{\lambda|_r}$, where $\lambda = (k_1, \ldots, k_n)$ and $\lambda|_r = (k_1, \ldots, k_r)$, one would like it to be true that the pullback of the circle bundle $S_{\lambda|_r,j}$ along $\tau_{n,r}$ is $S_{\lambda,j}$ — given this, one would then be able to pre-compose the pullback of $\widetilde{\mathcal{T}}_{\lambda|_r}$ with the projection of $\widetilde{\mathcal{T}}_\lambda$ onto a sub-direct-sum. However, this is false. For every $i < j \leqslant r$, the pullback of the cohomology class α_{ij} along $\tau_{n,r}$ is α_{ij}, so we have

$$\tau_{n,r}^* \left(\sum_{\substack{i=1 \\ i \neq j}}^{r} k_i.\alpha_{ij} \right) = \sum_{\substack{i=1 \\ i \neq j}}^{r} k_i.\alpha_{ij}.$$

The left-hand side classifies the pullback of $S_{\lambda|_r,j}$ along $\tau_{n,r}$, but the right-hand side classifies $S_{\lambda,j}$ only if $k_{r+1} = \cdots = k_n = 0$, which is impossible since all k_i are assumed positive.

More informally, one could say that the reason why we cannot naturally lift forgetful maps to Gibbons-Manton torus bundles is because of the *non-local* nature of the additional circle parameters: each circle parameter is associated to *all* configurations points simultaneously, since it encodes the pairwise interactions of one of the points with all of the others. Thus there is no well-defined way of forgetting a subset of the configuration points in the presence of these non-local parameters.

1.3.2 Changing the fibre

For a sequence of spaces $\mathbf{Z} = \{Z_1, Z_2, \ldots\}$, we will consider the family of finite products of the form $Z_\lambda = Z_{k_1} \times \cdots \times Z_{k_r}$ for tuples $\lambda = (k_1, \ldots, k_r)$ of positive integers. If each Z_i is a G-space for some topological group G, we consider each Z_λ as a G-space via the diagonal action.

Definition 1.3.8. Let $\mathbf{Z}$ be a sequence of S^1-spaces and let $\lambda = (k_1, \ldots, k_r)$. Let $\widetilde{\mathcal{T}}_\lambda(\mathbf{Z})$ be the total space of the fibre bundle obtained from the principal T^r-bundle $\widetilde{\mathcal{T}}_\lambda$ by the Borel construction:

$$\widetilde{\mathcal{T}}_\lambda(\mathbf{Z}) = \widetilde{\mathcal{T}}_\lambda \times_{T^r} Z_\lambda \longrightarrow F_r(\mathbb{R}^3).$$

We then let $\mathcal{T}_\lambda(\mathbf{Z}) = \widetilde{\mathcal{T}}_\lambda(\mathbf{Z})/\Sigma_\lambda$, where Σ_λ acts diagonally on $\widetilde{\mathcal{T}}_\lambda$ and on the finite product Z_λ. The *Gibbons-Manton* $\mathbf{Z}$*-bundle* of weight λ is the space $\mathcal{T}_\lambda(\mathbf{Z})$. It is the total space of a fibre bundle

$$\mathcal{T}_\lambda(\mathbf{Z}) \longrightarrow F_r(\mathbb{R}^3)/\Sigma_\lambda \tag{1.32}$$

with fibre Z_λ.

In particular, we have $\mathcal{I}_\lambda = \mathcal{T}_\lambda(\mathbf{Z})$ for $\mathbf{Z} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots\}$. We now prove:

Theorem 1.3.9. *For any sequence* $\mathbf{Z} = \{Z_1, Z_2, \ldots\}$ *of path-connected* S^1*-spaces, there are natural stabilisation maps*

$$\mathcal{T}_{\lambda[n]_c}(\mathbf{Z}) \longrightarrow \mathcal{T}_{\lambda[n+1]_c}(\mathbf{Z}) \tag{1.33}$$

that induce isomorphisms on homology in all degrees $\leqslant n/2-1$ *with* $\mathbb{Z}$ *coefficients and in all degrees* $\leqslant n/2$ *with field coefficients.*

Theorem 1.3.1 corresponds to two special cases of Theorem 1.3.9, namely the sequences $\{S^1, S^1, \ldots\}$ and $\{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ of S^1-spaces. It therefore remains only to prove Theorem 1.3.9.

Proof of Theorem 1.3.9. The proof is a direct generalisation of the proof of Theorem 1.3.1 for $\mathcal{T}_{\lambda[n]_c}$, so we just explain the differences. First of all, the lifts of the stabilisation maps exist by the proof of Corollary 1.3.5, where we additionally apply the (functorial) Borel construction to the outer square of (1.28) before quotienting by the symmetric group actions.

We begin by assuming that $\lambda = ()$ and $r = 0$, so that $\lambda[n]_c = (c, c, \ldots, c)$ where there are n copies of $c \geqslant 1$. We are therefore in the setting of Proposition 1.2.3 with $Z = Z_c$. The two hypotheses of that proposition are satisfied by the same argument as in the proof of Theorem 1.3.1 for $\mathcal{T}_{\lambda[n]_c}$, together with the evident observation that applying the Borel construction that replaces each circle factor in the fibre with the S^1-space Z_c has the effect, on fibres, that permutation maps $T^n \to T^n$ and natural inclusions $T^n \to T^{n+1}$ are sent to the corresponding permutation maps $(Z_c)^n \to (Z_c)^n$ and natural inclusions $(Z_c)^n \to (Z_c)^{n+1}$. Thus Proposition 1.2.3 completes the proof in the case $\lambda = ()$.

This generalises to Gibbons-Manton $\mathbf{Z}$-bundles with *fixed points* exactly as for Gibbons-Manton torus bundles with fixed points, and one may then deduce the general case of the theorem from this by a spectral sequence comparison argument applied to the analogue of the diagram (1.30). ∎

Chapter 2

Stability for moduli spaces of manifolds with conical singularities

The results of this chapter are contained in the arXiv preprint [Pal18a], which is submitted for publication.

Introduction

Let M be a connected manifold with $\partial M \neq \varnothing$. As the number of points $n \to \infty$, the homology of the unordered configuration spaces $C_n(M)$ *stablises*: there are maps $C_n(M) \to C_{n+1}(M)$ inducing isomorphisms on integral homology in a diverging range of degrees. This was proved first for the plane $M = \mathbb{R}^2$ by Arnol'd [Arn70] and in general by McDuff and Segal [Seg73; McD75; Seg79]. This has been extended to *labelled* configuration spaces $C_n(M; X)$, where the configuration points are equipped with labels in a path-connected space X [Ran13], as well as homology with *polynomially twisted* coefficients [Pal18b].

Closely related, there are *moduli spaces of manifolds with marked points*. Formally, these are the classifying spaces $B\mathrm{Diff}(M, \{x_1, \ldots, x_n\})$ of the groups of diffeomorphisms of M fixing a given set of n points in its interior, and they may be realised more concretely as the spaces of submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to M and equipped with an n-point configuration in their interiors. Stability for the homology of these moduli spaces was proven in [Til16] using their connection to labelled configuration spaces on M.

We may think of marked points in M as *conical singularities* by writing a disc neighbourhood of each marked point as the cone on its boundary sphere – see the left-hand side of Figure 2.1 on the next page. However, the cone on a sphere is still a manifold, so these singularities are "inessential". More general conical singularities of an m-dimensional manifold are points where the manifold is locally homeomorphic to the cone on an $(m-1)$-dimensional manifold L, the *link* of the singularity. Hence it, in fact, fails to be a manifold at these points unless $L \cong S^{m-1}$. See the right-hand side of Figure 2.1 for an illustration of a conical singularity whose link is a 2-torus.

In this chapter we prove stability, as the number of singularities goes to infinity, for the homology of *moduli spaces of manifolds with conical* L*-singularities*, where $L = \partial T$ is the boundary of a tubular neighbourhood of a closed submanifold $P \subset M$ of sufficiently high codimension. These are the classifying spaces $B\mathrm{Diff}^L(\mathbf{M}_n)$ of the L-diffeomorphism groups of the singular manifolds $\mathbf{M}_n$ obtained by collapsing n tubular neighbourhoods of copies of P down to n conical singularities. Note that taking $P =$ point recovers the case of marked points (inessential singularities).

This will turn out to be a special case of our main result: stability for the homology of the

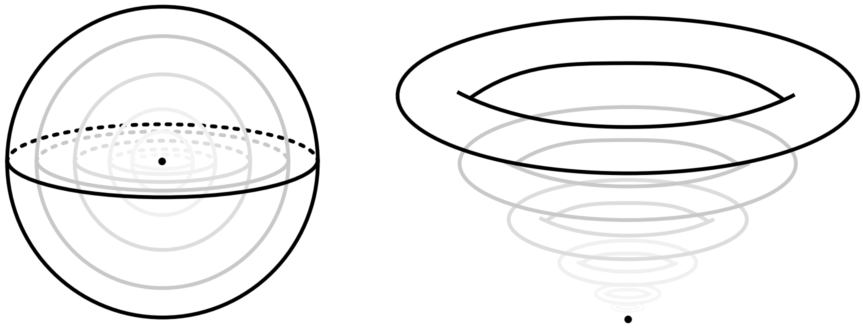


Figure 2.1 A marked point viewed as an inessential singularity given by the cone on a codimension-1 sphere; an essential singularity given by the cone on a torus.

classifying spaces of

$$\Sigma\mathrm{Diff}(M\sharp_P N\sharp_P \cdots \sharp_P N), \tag{2.1}$$

where $\sharp_P$ denotes *parametric connected sum* along a submanifold P and the *symmetric diffeomorphism group* $\Sigma\mathrm{Diff}$ is, roughly, the group of diffeomorphisms that preserve this iterated parametric connected sum decomposition. Parametric connected sum is a natural generalisation of connected sum, and one may encode surgery and, in dimension 3, Dehn surgery as the operation $-\sharp_P N$ by taking P and N to be appropriate spheres or lens spaces.

The essential input for our proofs, as in the case of marked points, is stability for the homology of labelled configuration spaces – except that we consider *configurations of disconnected submanifolds* of M rather than points, and the labels of the components take values in a bundle over a space of embeddings into M. Without labels (and with a bound on the dimension of the submanifolds), this was proven by the author in [Pal21]. In order to apply it to prove stability for (2.1) (and thus for moduli spaces of manifolds with conical singularities), we extend this stability result to configurations of disconnected submanifolds *with labels*. To do this, we first extend it to homology with *polynomially twisted coefficients* – a higher-dimensional analogue of [Pal18b].

A diagram illustrating this sequence of implications, starting from [Pal21] and [Pal18b], is given on page 35 below.

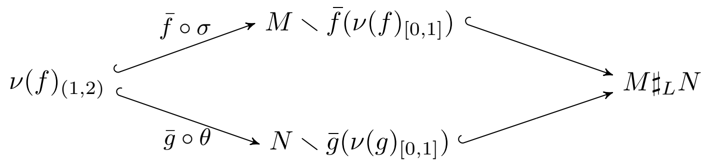
In the remainder of this introduction, we describe parametric connected sum in more detail, state precisely our main stability results (Theorems 2.A, 2.C, 2.D and Corollary 2.B) and then briefly discuss the stable homology and analogues of our results for mapping class groups.

Parametric connected sum

Definition 2.0.1. Given two smooth embeddings $f\colon L \hookrightarrow M$ and $g\colon L \hookrightarrow N$ with isomorphic normal bundles, we may form the *parametric connected sum* of M and N along L as follows. It depends also on the choice of a bundle isomorphism $\theta\colon \nu(f) \cong \nu(g)$ between the two normal bundles.

First choose a metric on the vector bundle $\nu(f) \to L$ and tubular neighbourhoods $\bar{f}\colon \nu(f) \hookrightarrow M$ and $\bar{g}\colon \nu(g) \hookrightarrow N$. (This is a contractible choice.) For a vector bundle E with a metric and an interval $I \subseteq [0,\infty)$, write E_I for the subbundle consisting of all vectors with norm in I.

The parametric connected sum $M\sharp_L N$ is formed by removing the neighbourhood $\bar{f}(\nu(f)_{[0,1]})$ of $f(L)$ from M and the neighbourhood $\bar{g}(\nu(g)_{[0,1]})$ of $g(L)$ from N and then gluing together the slightly larger neighbourhoods $\bar{f}(\nu(f)_{(1,2)})$ and $\bar{g}(\nu(g)_{(1,2)})$ by $\theta \circ \sigma$, where σ is the involution of $\nu(f)_{(1,2)}$ that acts radially on each fibre, sending vectors of norm r to vectors of norm $3-r$. This may be written as a pushout diagram:



When L is a point, this recovers the usual definition of connected sum of two manifolds.

Example 2.0.2. Take g to be the standard inclusion of S^p into S^m, for $p < m$ and let M have dimension m. Since $\nu(g)$ is trivial, an embedding $f\colon S^p \hookrightarrow M$ together with a bundle isomorphism θ as in Definition 2.0.1 is the same thing as a *framed* embedding. The parametric connected sum of M with S^m along S^p is the result of p-surgery along this framed embedding.

Example 2.0.3. Take N to be the lens space $L(p, q)$, thought of as the union of two solid tori along an appropriate identification of their boundaries, and take $g\colon S^1 \hookrightarrow L(p, q)$ to be the inclusion of the core of one of these solid tori. As in the previous example, $\nu(g)$ is trivial. Given a framed knot in a 3-manifold M, the parametric connected sum of M with $L(p, q)$ along this knot is the result of Dehn surgery of slope p/q along this knot.

Apart from these ubiquitous examples of surgery and Dehn surgery, instances of the more general parametric connected sum have also appeared numerous times before in the literature. For example, the notion of "connected sum along a k-skeleton" was used by Kreck [Kre85, pp. 25–26] (see also [Kre16, §5 and §6]) to give a geometric definition of the group structure on the set of $2n$-*manifolds with normal* $(n-1)$-*smoothing* up to stable diffeomorphism. This uses Wall's theory of *thickenings* [Wal66], which allows one, under certain conditions, to approximate a map from a CW-complex to a manifold M by a homotopy equivalence onto a compact, codimension-zero submanifold of M; these approximations play a role analogous to that of the tubular neighbourhoods in Definition 2.0.1 above. A version of the parametric connected sum in a context where all manifolds are equipped with embeddings into a fixed Euclidean space was defined in [Sko07, page 264]; this is also where the name *parametric connected sum* seems to have been first introduced. More details and further references may be found at [Man].

Results

Let M be a connected m-manifold with boundary and P a closed p-manifold. Choose embeddings $P \hookrightarrow \partial M$ and $P \hookrightarrow N$, where N is another compact m-manifold. Assume that the normal bundles of these embeddings (after pushing the first one into the interior of M) are isomorphic, and choose an isomorphism between them. We may then define

$$M \underset{nP}{\sharp} nN$$

to be the result of performing n parametric connected sum operations, each with a different copy of N, along pairwise disjoint embeddings of P into a collar neighbourhood of M, parallel to the chosen embedding into its boundary. (See Definition 2.1.2.)

Definition 2.0.4 (*Informal*)**.** A diffeomorphism of this manifold is called *symmetric* if it preserves, setwise, the decomposition into (a) M minus n tubular neighbourhoods of P and (b) n copies of N minus a tubular neighbourhood of P. Moreover, on the intersection of (a) and (b), it must act by fibrewise diffeomorphisms of the tubular neighbourhood (and permutations). We also require it to act by the identity on a neighbourhood of ∂M.

Write $\mathrm{Diff}_{\mathrm{fib}}(T)$ for the group of fibrewise diffeomorphisms of the tubular neighbourhood (normal bundle) $T \to P$, namely those diffeomorphisms of T that cover some diffeomorphism of P. We may more generally fix a subgroup $H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T)$ and say that a diffeomorphism is H-*symmetric* if it acts by H (and permutations) on the intersection of (a) and (b). We write

$$\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN)$$

for the group of H-symmetric diffeomorphisms, and we omit the subscript H if it is the whole group $\mathrm{Diff}_{\mathrm{fib}}(T)$. See Definition 2.2.1 for the precise definition.

Since M has boundary, and symmetric diffeomorphisms are required to fix a neighbourhood of it pointwise, there is an inclusion

$$M \underset{nP}{\sharp} nN \longleftrightarrow M \underset{(n+1)P}{\sharp} (n+1)N,$$

given by extending the collar neighbourhood of ∂M and performing another parametric connected sum with N along an embedding of P into this extension, and a homomorphism

$$\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow \Sigma_H \mathrm{Diff}(M \underset{(n+1)P}{\sharp} (n+1)N), \tag{2.2}$$

given by extending symmetric diffeomorphisms by the identity. Our main result will hold under the following hypotheses (see also §2.4):

(a) The dimensions p and m satisfy $p \leqslant \frac{1}{2}(m-3)$.
(b) Let $z\colon \mathrm{Diff}_{\mathrm{fib}}(T) \to \mathrm{Diff}(P)$ be the homomorphism that sends a fibrewise diffeomorphism to the diffeomorphism of P that it covers. Then $z(H)$ is open in $\mathrm{Diff}(P)$ and $H \cap \ker(z)$ is closed in $\ker(z)$. Moreover, we assume that the coset space $\ker(z)/(H \cap \ker(z))$ is path-connected.
(c) Every diffeomorphism in H (thought of as a diffeomorphism of a tubular neighbourhood of P in N) extends to the whole manifold N.

We will discuss these hypotheses in more detail shortly, but first we state the main results of the chapter.

Theorem 2.A. (Theorem 2.4.1) *Under these hypotheses, the homomorphism* (2.2) *induces isomorphisms on homology up to degree $\frac{n}{2}-1$ and split-injections in all degrees.*

Note. When we speak of the *homology* of a diffeomorphism group, we will always mean the homology of its classifying space (as a topological group, equipped with the Whitney C^∞ topology).

In §2.7 we reinterpret a special case of this as a result about diffeomorphism groups of manifolds with conical singularities. Write ∂T for the boundary of the disc bundle $T \to P$, in other words, the unit sphere bundle of the normal bundle of our chosen embedding of P into M. The group of fibrewise diffeomorphisms of T is naturally a subgroup of $\mathrm{Diff}(\partial T)$, and therefore so is H. In §2.7.4 we construct a sequence $\mathbf{M}_n$ of manifolds with conical singularities of "type" ∂T by collapsing tubular neighbourhoods of embedded copies of P in M. There is an inclusion $\mathbf{M}_n \hookrightarrow \mathbf{M}_{n+1}$, which may be thought of as adjoining a new singularity of type ∂T, and a homomorphism

$$\mathrm{Diff}_H^{\partial T}(\mathbf{M}_n) \longrightarrow \mathrm{Diff}_H^{\partial T}(\mathbf{M}_{n+1}) \tag{2.3}$$

given by extending diffeomorphisms by the identity. The notation $\mathrm{Diff}_H^{\partial T}(\ \ )$ means the group of diffeomorphisms of a manifold with conical ∂T-singularities that act by the cone on H near each singularity. See Definition 2.7.4 for more precise details.

Corollary 2.B. (Corollary 2.7.12) *Under the hypotheses* (a) *and* (b) *above, the homomorphism* (2.3) *induces isomorphisms on homology up to degree $\frac{n}{2}-1$ and split-injections in all degrees.*

These are proved as corollaries of an extension of the main theorem of [Pal21]. Let G be a subgroup of $\mathrm{Diff}(P)$ and let $C_{nP}(M;G)$ be the space whose points consist of an unordered collection of pairwise disjoint submanifolds in the interior of M that are each diffeomorphic to P and parametrised modulo G, and such that the whole collection is isotopic to a standard collection of parallel copies of P in a collar neighbourhood of ∂M. The main theorem of [Pal21] is that this sequence is homologically stable as $n \to \infty$, as long as:

(a) The dimensions p and m satisfy $p \leqslant \frac{1}{2}(m-3)$.
(b*) The subgroup $G \leqslant \mathrm{Diff}(P)$ is open.

In this chapter we extend this result to moduli spaces of *labelled* disconnected submanifolds. Let Z be a right G-space and $\pi\colon Z \to \mathrm{Emb}(P,M)$ a G-equivariant Serre fibration with path-connected fibres (plus some auxiliary data; see §2.5 for the precise definitions). Then $C_{nP}(M,Z;G)$ is the space of submanifolds of the interior of M (as above) that are equipped with labels in the appropriate fibres of Z/G.

Theorem 2.C. (Theorem 2.5.8) *There are natural stabilisation maps*

$$C_{nP}(M,Z;G) \longrightarrow C_{(n+1)P}(M,Z;G)$$

that induce split-injections on homology in all degrees, and – under the hypotheses (a) *and* (b*) *– isomorphisms up to degree* $\frac{n}{2}-1$*.*

This, in turn, follows from the fact that the sequence $C_{nP}(M;G)$ of (unlabelled) moduli spaces is homologically stable also with respect to (polynomial) twisted coefficient systems:

Theorem 2.D. (Theorem 2.8.3) *The stabilisation maps* $C_{nP}(M;G) \to C_{(n+1)P}(M;G)$ *induce split-injections on homology twisted by any polynomial functor* T*. Under the hypotheses* (a) *and* (b*)*, if* T *has finite degree* d*, they induce isomorphisms on homology twisted by* T *up to degree* $\frac{n-d}{2}$*.*

See §2.8 for the definition of "polynomial functor" ("finite-degree twisted coefficient system") in this context. We prove this *twisted* homological stability result as a consequence of the *untwisted* homological stability result of [Pal21], adapting the techniques of [Pal18b] to do so.

In summary, the sequence of implications that we prove is:

$$[\text{Pal21}] \xrightarrow[\text{[Pal18b]}]{\S 2.8 \text{ and}} \text{Theorem 2.D} \xrightarrow{\S 2.9} \text{Theorem 2.C} \xrightarrow{\S 2.6} \text{Theorem 2.A} \xrightarrow{\S 2.7} \text{Corollary 2.B}$$

Remark 2.0.5. In the special case where P is a point, Theorem 2.A is Theorem 1.3 of [Til16], and states that the symmetric diffeomorphism groups of a sequence of manifolds obtained by iterated connected sum (in the usual sense) is homologically stable. When P is a point, the manifolds $\mathbf{M}_n$ considered in Corollary 2.B are in fact *non*-singular, since in this case $\partial T = S^{m-1}$ and a singularity whose link (type) is a sphere is locally Euclidean. However, $\mathbf{M}_n$ is still a manifold with marked points, so in this case Corollary 2.B is homological stability for diffeomorphism groups of manifolds with marked points, with respect to the number of marked points: this is Theorem 1.1 of [Til16].

Remark 2.0.6. In the special case where $P \hookrightarrow N$ is the inclusion $S^p \hookrightarrow S^m$ (see Example 2.0.2), Theorem 2.A is homological stability for the symmetric diffeomorphism groups of a sequence of manifolds obtained from M by iterated p-surgery along a collection of pairwise disjoint, mutually isotopic, framed embeddings $S^p \times D^{m-p} \hookrightarrow M$, for $p \leqslant \frac{1}{2}(m-3)$.

The hypotheses

Remark 2.0.7. The dimension condition (a) is not used in any of the proofs of the four implications in the diagram above; the only place where it is used is in the proof of homological stability for $C_{nP}(M;G)$ in [Pal21].[1] Thus we may apply the same implications to the main result of [Kup20] instead of [Pal21], with the result that Theorems 2.A, 2.C, 2.D and Corollary 2.B are also true if the dimension assumption (a) is replaced with the assumption that $P = S^1$, M is a 3-manifold and the embedding $P = S^1 \hookrightarrow \partial M$ extends to a 2-disc.

Putting together Remarks 2.0.6 and 2.0.7, we have the following special cases of Theorem 2.A.

Corollary 2.0.8. *We have homological stability for the symmetric diffeomorphism groups of sequences of manifolds obtained from* M *by:*

[1] See Remark 1.8 of [Pal21] for a discussion of exactly where the dimension hypothesis is used in that paper.

- *iterated surgery along a framed p-sphere that may be isotoped into* ∂M*, if* $p \leqslant \frac{1}{2}(\dim(M)-3)$,
- *iterated Dehn surgery along an unknot, if* $\dim(M) = 3$.

Remark 2.0.9. The subgroup condition (b) is always satisfied if we take H to be the full group $\mathrm{Diff}_{\mathrm{fib}}(T)$ of fibrewise diffeomorphisms of T. This is because the map $\mathrm{Diff}_{\mathrm{fib}}(T) \to \mathrm{Diff}(P)$, given by restriction along the zero-section, is a fibre bundle, so its image is open.

Remark 2.0.10. When P is a point, the fibrewise diffeomorphism group is the orthogonal group $O(m)$, and condition (b) says that H must be a closed subgroup of $O(m)$, not contained in $SO(m)$. Condition (c) says that any element of H must extend to a diffeomorphism of N. If P is a point and $H \subseteq SO(m)$, then this is always possible. However, in view of condition (b), we know that H must not be contained in $SO(m)$, and in this case condition (c) is satisfied if and only if N is either non-orientable or admits an orientation-reversing diffeomorphism.

Stable homology

Knowing that a sequence of groups or spaces is homologically stable motivates the question of identifying the *stable homology* of the sequence, i.e. the colimit of the homology of the sequence. As far as the author is aware, this question is open, both for symmetric diffeomorphism groups (corresponding to Theorem 2.A) and for diffeomorphism groups of manifolds with conical singularities (corresponding to Corollary 2.B). However, the stable homology in the latter case may be related to the work of Perlmutter [Per15; Per13] on cobordism categories of manifolds with Baas-Sullivan singularities.

For a discussion of the (also mostly open) question of identifying the stable homology of the moduli spaces of disconnected submanifolds $C_{nP}(M;G)$, see §⟨vi.⟩ of the introduction of [Pal21].

Remark 2.0.11. The author has been informed of forthcoming work of James Griffin and Allen Hatcher on the homology (both stable and unstable) of a space closely related to $C_{nS^1}(\mathbb{D}^3)$. Here, we suppress mention of the subgroup $G \leqslant \mathrm{Diff}(S^1)$, which we take to be the full group of diffeomorphisms, so this is the moduli space of unoriented n-component unlinks in $\mathbb{R}^3$. This has a subspace $\mathcal{R}_n$ of all unlinks whose components are all *round* (meaning a rotation, translation and dilation of the standard inclusion of S^1 into $\mathbb{R}^3$), and it is shown in [BH13] that the inclusion $\mathcal{R}_n \hookrightarrow C_{nS^1}(\mathbb{D}^3)$ is a homotopy equivalence. If we write $\mathcal{PR}_n$ for the ordered version of this space, where the components of the unlink are numbered by $\{1,\ldots,n\}$, there is a projection $\mathcal{PR}_n \to (\mathbb{RP}^2)^n$ given by remembering the normal vectors of a configuration of round circles, which was shown in [BH13] to be a quasifibration. The fibre is the space of ordered configurations of pairwise disjoint round circles in $\mathbb{R}^3$ that are each contained in $\mathbb{R}^2 \times \{h\}$ for some $h \in \mathbb{R}$. The forthcoming result of Griffin and Hatcher is a computation of the integral homology of this fibre.

Mapping class groups

In §5 of [Til16] it is shown how to modify the methods of that paper — whose main result is homological stability for symmetric diffeomorphism groups of manifolds with respect to connected sum at a point — to prove homological stability instead for the symmetric *mapping class groups*, in other words, the (discrete) groups of path-components of the symmetric diffeomorphism groups.

This depends on knowing homological stability for the fundamental groups $\pi_1(C_n(M))$ of configuration spaces (instead of homological stability for the configuration spaces themselves) as an input for the argument. The reason why $\pi_1(C_n(M))$ is homologically stable is:

(i) If M is a surface (which we are assuming to be connected and with non-empty boundary), then $C_n(M)$ is an aspherical space, so its homology is the same as the homology of its fundamental group.

(ii) If $\dim(M) \geqslant 3$, then $C_n(M)$ is not aspherical, but its fundamental group nevertheless has a configuration space model for its classifying space. Namely, $\pi_1(C_n(M))$ decomposes as the wreath product $\pi_1(M) \wr \Sigma_n$, as shown in Lemma 4.1 of [Til16], and a model for the classifying space of this wreath product is the labelled configuration space $C_n(\mathbb{R}^\infty, B\pi_1(M))$, which is homologically stable.

When $P \neq \mathrm{pt}$, homological stability is in general *not* known for the *motion groups* $\pi_1(C_{nP}(M))$. The author is not aware of any $C_{nP}(M)$ that is aspherical, except when P is a point and M is a surface, so argument (i) does not help us. In particular, the moduli spaces $C_{nS^1}(\mathbb{D}^3)$ are not aspherical. However, their fundamental groups nevertheless *are* known to be homologically stable, by different means: they are isomorphic to certain quotients of mapping class groups of 3-manifolds, and a special case of the main result of [HW10] implies that they are homologically stable. We may therefore adapt the methods of the present chapter, as in §5 of [Til16], using as input the result of [HW10], to deduce homological stability for the *symmetric mapping class groups* of any sequence of 3-manifolds obtained from $\mathbb{D}^3$ by iterated parametric connected sum (with copies of a fixed manifold) along the components of an unlink.

It seems likely that argument (ii) above, i.e. the argument of §4 of [Til16], could be extended to obtain homological stability for the motion groups $\pi_1(C_{nP}(M))$ whenever the dimensions of P and M satisfy condition (a), i.e., $\dim(M) \geqslant 2.\dim(P)+3$, and then to deduce homological stability for the corresponding symmetric mapping class groups with respect to iterated parametric connected sum.

Outline. In sections 2.1 and 2.2 we give precise definitions of *iterated parametric connected sum* and *symmetric diffeomorphism groups.* Then in the short sections 2.3 and 2.4 we explain how to define stabilisation maps and state our first main result, Theorem 2.A. In section 2.5 we give a careful definition of the notion of *moduli spaces of disconnected* ***labelled*** *submanifolds* that we will need, and state Theorem 2.C. In sections 2.6–2.9 we prove our main results, as explained in the diagram of implications above, which we repeat here for convenience:

$$[\text{Pal21}] \xrightarrow[\text{[Pal18b]}]{\S 2.8 \text{ and}} \text{Theorem 2.D} \xrightarrow{\S 2.9} \text{Theorem 2.C} \xrightarrow{\S 2.6} \text{Theorem 2.A} \xrightarrow{\S 2.7} \text{Corollary 2.B}$$

where the longest and most delicate step is the deduction in §2.6 of Theorem 2.A from Theorem 2.C.

2.1 Iterated parametric connected sum

Let M be a smooth connected m-dimensional manifold with boundary and write

$$\hat{M} = M \underset{\partial M \times [0,\infty]}{\cup} (\partial M \times [-1,\infty]),$$

where $\partial M \times [0,\infty]$ is considered as a subspace of M via a choice of collar neighbourhood, namely a proper embedding $\mathrm{col}\colon \partial M \times [0,\infty] \hookrightarrow M$ sending $\partial M \times \{0\}$ to ∂M by the obvious map. Also let N be a smooth compact m-dimensional manifold and P be a smooth closed p-dimensional manifold. Fix embeddings

$$i\colon P \longhookrightarrow \partial M \subseteq \hat{M},$$
$$j\colon P \longhookrightarrow \mathring{N} = \mathrm{int}(N),$$

assume that the normal bundles $\nu_i \to P$ and $\nu_j \to P$ are isomorphic, and choose an isomorphism between them. Choose a metric on the bundle ν_i so that its structure group is $O(m-p)$, and give ν_j the corresponding metric via the chosen isomorphism. Let $D(\nu_i) \subseteq \nu_i$ and $D(\nu_j) \subseteq \nu_j$ denote the subbundles consisting of vectors of norm at most one. We implicitly identify $D(\nu_i)$ and $D(\nu_j)$ via the chosen isomorphism, and write

$$\xi\colon T = D(\nu_i) = D(\nu_j) \longrightarrow P,$$

which is a fibre bundle with fibres diffeomorphic to the closed disc D^{m-p} and structure group $O(m-p)$. Write $o\colon P \hookrightarrow T$ for the zero-section. We now also choose tubular neighbourhoods for

ν_i and ν_j, namely embeddings

$$\tau_i \colon T \hookrightarrow \hat{M},$$
$$\tau_j \colon T \hookrightarrow \mathring{N},$$

such that $i = \tau_i \circ o$ and $j = \tau_j \circ o$, and assume that $\tau_i(T) \subseteq \partial M \times (-\frac{1}{2}, \frac{1}{2}) \subset \hat{M}$. We may define an embedding

$$\Phi \colon nT = \{1, \ldots, n\} \times T \hookrightarrow \mathring{M}$$

of n disjoint, parallel copies of the tubular neighbourhood T in the interior of M by

$$\Phi(\alpha, x) = \tau_i(x) + \alpha - \tfrac{1}{2},$$

where for $r \in [0, \infty)$ the notation $+r$ denotes the self-map $\partial M \times [-1, \infty] \to \partial M \times [-1, \infty]$ given by the identity on ∂M and adding r in the second coordinate.

Notation 2.1.1. For a space X, we will henceforth use the notations nX and $\{1, \ldots, n\} \times X$ interchangeably. We will also write $\hat{n}X$ for $\{0, \ldots, n\} \times X$. Note that $\hat{n}X \supsetneq nX$!

Definition 2.1.2. With this data, we may form the *parametric connected sum*

$$M \underset{nP}{\sharp} nN \;=\; (M \setminus \Phi(nT')) \underset{n(\mathring{T} \setminus T')}{\cup} n(N \setminus \tau_j(T')),$$

where $\mathring{T} \to P$ is the interior of T, equivalently the subbundle of vectors of length less than 1, and $T' \to P$ is the subbundle of T of vectors of length at most one half. The union is formed along $n(\mathring{T} \setminus T')$, which is viewed as a subspace of $M \setminus \Phi(nT')$ via Φ, and as a subspace of $n(N \setminus \tau_j(T'))$ via $\mathrm{id} \times \tau_j$ precomposed with the involution of $\mathring{T} \setminus T' = \partial T \times (0.5, 1)$ given by $(x, t) \mapsto (x, 1.5 - t)$.

Remark 2.1.3. The boundary of the parametric connected sum is $\partial\big(M \underset{nP}{\sharp} nN\big) \cong \partial M \sqcup n(\partial N)$.

2.2 Symmetric diffeomorphism groups

Recall that we have a disc bundle $\xi \colon T \to P$ with structure group $O(m-p)$. Let

$$\mathrm{Diff}_{\mathrm{fib}}(T) \leqslant \mathrm{Diff}(T)$$

denote the subgroup of diffeomorphisms φ such that $\xi \circ \varphi = \bar{\varphi} \circ \xi$ for some diffeomorphism $\bar{\varphi}$ of P and φ restricts to a linear isometry on each fibre of ξ. Write

$$z \colon \mathrm{Diff}_{\mathrm{fib}}(T) \longrightarrow \mathrm{Diff}(P)$$

for the continuous homomorphism given by $\varphi \mapsto \bar{\varphi}$, equivalently, by restricting fibrewise diffeomorphisms of T to P via the zero-section of ξ. Its kernel is the group of bundle automorphisms of ξ. Choose a subgroup

$$H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T),$$

and write $G = z(H) \leqslant \mathrm{Diff}(P)$ and $K = H \cap \ker(z) \leqslant \ker(z)$.

Definition 2.2.1. The *symmetric diffeomorphism group* $\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) \leqslant \mathrm{Diff}(M \underset{nP}{\sharp} nN)$ consists of those diffeomorphisms that are the identity on a neighbourhood of ∂M, send the submanifold

$$n(\mathring{T} \setminus T') \;\subset\; M \underset{nP}{\sharp} nN$$

to itself setwise and act on this submanifold through the wreath product $H \wr \Sigma_n$.

The subgroup $\Sigma_H \mathrm{Diff}(M, nT)$ of $\mathrm{Diff}(M)$ consists of those diffeomorphisms that are the identity on a neighbourhood of the boundary, send the submanifold

$$\Phi(nT) \;\subset\; M$$

to itself setwise and act on this submanifold through the wreath product $H \wr \Sigma_n$.

Remark 2.2.2. Each diffeomorphism of $\Sigma_H \mathrm{Diff}(M, nT)$ is determined by its restriction to the submanifold $M_\Phi = M \smallsetminus \Phi(n\mathring{T})$, whose boundary splits as $\partial M_\Phi = \partial M \sqcup \Phi(n\partial T)$, so it may also be viewed as the subgroup of $\mathrm{Diff}(M_\Phi)$ of diffeomorphisms that act by the identity on a neighbourhood of $\partial M \subset \partial M_\Phi$ and by $H \wr \Sigma_n$ on $\Phi(n\partial T) \subset \partial M_\Phi$. We will call this the *boundary-permuting diffeomorphism group* of M_Φ. See §2.7 for another interpretation in terms of manifolds with conical singularities.

Remark 2.2.3. There is a continuous homomorphism

$$\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow \Sigma_H \mathrm{Diff}(M, nT) \tag{2.4}$$

given by restricting a diffeomorphism to $M \smallsetminus \Phi(nT')$ and then extending to M by extending linearly across each fibre of $\xi \colon T \to P$. Similarly, there is a continuous homomorphism

$$\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow \mathrm{Diff}_H(N) \wr \Sigma_n,$$

where $\mathrm{Diff}_H(N) \leqslant \mathrm{Diff}(N)$ denotes the subgroup of diffeomorphisms that send the submanifold $\tau_j(T) \subset N$ to itself setwise and act on it through the subgroup $H \leqslant \mathrm{Diff}(T)$. These homomorphisms fit into the following pullback square of topological groups:

$$\begin{array}{ccc} \Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) & \longrightarrow & \mathrm{Diff}_H(N) \wr \Sigma_n \\ \downarrow & & \downarrow \\ \Sigma_H \mathrm{Diff}(M, nT) & \longrightarrow & H \wr \Sigma_n \end{array} \tag{2.5}$$

Via this description, we could generalise the notion of *symmetric diffeomorphism group*, exactly as on page 133 of [Til16] (the diagram (2.5) corresponds exactly to the diagram at the top of that page), by replacing $\mathrm{Diff}_H(N)$ with an arbitrary topological group L equipped with a surjective[2] continuous homomorphism $L \to H$, and then taking the pullback of the diagram

$$\Sigma_H \mathrm{Diff}(M, nT) \longrightarrow H \wr \Sigma_n \longleftarrow L \wr \Sigma_n.$$

The results of this chapter hold also in this higher level of generality, but for concreteness we will stick to the symmetric diffeomorphism groups as defined above, with $L = \mathrm{Diff}_H(N)$ for some manifold N equipped with an embedding $T \hookrightarrow \mathring{N}$.

Remark 2.2.4. Up to a canonical homotopy equivalence, the boundary-permuting diffeomorphism group $\Sigma_H \mathrm{Diff}(M, nT)$ is a special case of the symmetric diffeomorphism group $\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN)$.

To see this, let us fix the initial data of an embedding $i \colon P \hookrightarrow \hat{M}$ whose image lies in $\partial M \subset \hat{M}$, a metric on the normal bundle $\nu_i \to P$ of this embedding and a tubular neighbourhood $T = D(\nu_i) \hookrightarrow \hat{M}$. Together with a choice of subgroup $H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T)$, this determines the group $\Sigma_H \mathrm{Diff}(M, nT)$.

We are now free to choose any compact m-dimensional manifold N, an embedding $j \colon P \hookrightarrow \mathring{N}$ and isomorphism $\nu_i \cong \nu_j$. In particular, we may choose $N = D_2(\nu_i)$, the total space of the subbundle of ν_i consisting of all vectors of norm at most two, and let $j \colon P \hookrightarrow \mathring{N} = \mathring{D}_2(\nu_i)$ be the zero-section. The normal bundles ν_i and ν_j are then canonically isomorphic. We also have to choose a tubular neighbourhood

$$T = D(\nu_j) \longleftrightarrow \mathring{D}_2(\nu_j) = \mathring{N},$$

which we simply take to be the inclusion. It is then easy to see that, in this case, the parametric connected sum $M \underset{nP}{\sharp} nD_2(\nu_i)$ deformation retracts onto its submanifold M_Φ (*cf.* Remark 2.2.2).

[2] We will shortly impose the assumption that $\mathrm{Diff}_H(N) \to H$ is surjective, in other words, that each element of $H \leqslant \mathrm{Diff}(T)$ may be extended over N. We do not, however, need this extension to preserve composition, i.e. we do not require this surjection to split.

Moreover, in this case, the continuous homomorphism (2.4) admits a continuous, homomorphic section given by extending fibrewise automorphisms of $D(\nu_i) \to P$ linearly to $D_2(\nu_i) \to P$, and this is a homotopy inverse (in the 2-category of topological groups) for (2.4).[3]

2.3 Stabilisation maps

We may extend Φ to an embedding

$$\hat{\Phi}\colon \hat{n}T = \{0,\ldots,n\} \times T \longrightarrow \hat{M}$$

defined by the same formula $\hat{\Phi}(\alpha,x) = \tau_i(x) + \alpha - \frac{1}{2}$ as before. In other words, we adjoin the embedding $\tau_i - \frac{1}{2}$ to Φ. Using this $\hat{\Phi}$ and $n+1$ copies of the tubular neighbourhood $\tau_j\colon T \hookrightarrow N$ we define

$$M \underset{\hat{n}P}{\sharp} \hat{n}N \;=\; (\hat{M} \smallsetminus \hat{\Phi}(\hat{n}T')) \underset{\hat{n}(\mathring{T}\smallsetminus T')}{\cup} \hat{n}(N \smallsetminus \tau_j(T'))$$

as above, as well as groups $\Sigma_H \mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$ and $\Sigma_H \mathrm{Diff}(M, \hat{n}T)$ and a continuous homomorphism

$$\Sigma_H \mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N) \longrightarrow \Sigma_H \mathrm{Diff}(M, \hat{n}T).$$

Extending diffeomorphisms by the identity along the inclusion $M \underset{nP}{\sharp} nN \hookrightarrow M \underset{\hat{n}P}{\sharp} \hat{n}N$, we obtain the horizontal maps in the commutative square

$$\begin{array}{ccc} \Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) & \longrightarrow & \Sigma_H \mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N) \\ \downarrow & & \downarrow \\ \Sigma_H \mathrm{Diff}(M, nT) & \longrightarrow & \Sigma_H \mathrm{Diff}(\hat{M}, \hat{n}T) \end{array} \tag{2.6}$$

of topological groups. These are the *stabilisation maps*.

2.4 Homological stability for symmetric diffeomorphism groups

We now make three assumptions (*cf.* the discussion of the hypotheses in the introduction).

(a) The dimensions p and m satisfy $p \leqslant \frac{1}{2}(m-3)$.
(b) The subgroup $G = z(H) \leqslant \mathrm{Diff}(P)$ is open and $K \leqslant \ker(z)$ is closed. Moreover, we assume that the coset space $\ker(z)/K$ is path-connected.
(c) Every diffeomorphism in H extends across N. More precisely, given any $h \in H \leqslant \mathrm{Diff}(T)$, there exists a diffeomorphism φ of N such that $\varphi(\tau_j(T)) = \tau_j(T)$ and $\varphi|_{\tau_j(T)} = \tau_j \circ h \circ \tau_j^{-1}$. In the notation of Remark 2.2.3, this says that the continuous homomorphism $\mathrm{Diff}_H(N) \to H$ is surjective (but not necessarily split).

The first main result of this chapter is the following theorem.

Theorem 2.4.1. (Theorem 2.A) *Under these assumptions, the two horizontal morphisms in* (2.6) *induce split-injections on homology in all degrees and isomorphisms in degrees* $* \leqslant \frac{n}{2} - 1$.

Remark 2.4.2. If we take coefficients in a field or $\mathbb{Q}/\mathbb{Z}$, the range of surjectivity improves to $* \leqslant \frac{n}{2}$.

[3] Here it is important that, in Definition 2.2.1, we require symmetric diffeomorphisms to act as the identity near ∂M, but there is no condition on how they act near the rest of the boundary of $M \underset{nP}{\sharp} nN$, i.e. the n copies of ∂N.

2.5 Moduli spaces of labelled disconnected submanifolds

We will deduce Theorem 2.A from a generalisation of the main theorem of [Pal21], so we will first state this generalisation precisely (see Theorem 2.5.8 on page 42 below).

Notation 2.5.1. By construction of $\hat{M}$, there is a smooth embedding

$$\mathrm{côl}\colon \partial M \times [-1,\infty] \hookrightarrow \hat{M}.$$

For any $r \in [-1,\infty)$ we will write $M(r) = \hat{M} \smallsetminus \mathrm{côl}([-1,r])$ and $M[r] = \hat{M} \smallsetminus \mathrm{côl}([-1,r))$.

Definition 2.5.2. Write $\mathrm{Diff}_{[-1,1]}(\mathbb{R})$ for the topological group of diffeomorphisms $\mathbb{R} \to \mathbb{R}$ whose support is contained in $[-1,1] \subset \mathbb{R}$. There is an evaluation map

$$\mathrm{ev}_0\colon \mathrm{Diff}_{[-1,1]}(\mathbb{R}) \longrightarrow (-1,1)$$

taking φ to $\varphi(0)$, which is a fibre bundle. We now make some choices that will be used in several constructions in this section and in subsequent sections. First, we choose an odd[4] diffeomorphism $\theta\colon \mathbb{R} \to (-1,1)$. Second, we choose a lift $\bar{\theta}\colon \mathbb{R} \to \mathrm{Diff}_{[-1,1]}(\mathbb{R})$ of θ (i.e. $\mathrm{ev}_0 \circ \bar{\theta} = \theta$) that is also a homomorphism with respect to addition on $\mathbb{R}$ and composition of diffeomorphisms.

Remark 2.5.3. Given θ, one may try to define $\bar{\theta}(r)\colon \mathbb{R} \to \mathbb{R}$ on $t \in (-1,1)$ by

$$\bar{\theta}(r)(t) = \theta(\theta^{-1}(t) + r)$$

and extend by the identity outside of $(-1,1)$. This will work as long as the function so defined is smooth at $t = 1$, which depends on how well θ has been chosen. For example, if we only care about C^1 diffeomorphisms, the function $\theta(t) = \frac{2}{\pi}\mathrm{arctan}(t)$ would work for this construction of $\bar{\theta}$.

Definition 2.5.4. Recall that we chose an embedding $i\colon P \hookrightarrow \partial M$. For any $r \in \mathbb{R}$ we denote the shifted embedding

$$\mathrm{côl} \circ (i(-), \theta(r))\colon P \longrightarrow \hat{M}$$

by i_r and we define a diffeomorphism

$$\mathrm{sh}_r\colon \hat{M} \longrightarrow \hat{M}$$

by $\mathrm{sh}_r(\mathrm{côl}(x,t)) = \mathrm{côl}(x, \bar{\theta}(r)(t))$ and $\mathrm{sh}_r(x) = x$ if $x \in \hat{M}$ is not in the image of côl.

Now write $E = \mathrm{Emb}(P, \hat{M})$. There is a continuous group homomorphism $\gamma\colon \mathbb{R} \to \mathrm{Homeo}(E)$ given by $\gamma(r)(\varphi) = \mathrm{sh}_r \circ \varphi$. Moreover, since $G \leqslant \mathrm{Diff}(P)$ acts on E by precomposition, and $\gamma(r)$ acts by postcomposition, we in fact have a continuous group homomorphism

$$\gamma\colon \mathbb{R} \longrightarrow \mathrm{Homeo}^G(E)$$

into the topological group of G-equivariant self-homeomorphisms of E. Note that $\gamma(r)(i_s) = i_{r+s}$ for all $r, s \in \mathbb{R}$.

Input 2.5.5. Now we fix the additional input data needed to define moduli spaces of disconnected submanifolds *with labels* (see Definition 2.5.6). Choose a G-equivariant Serre fibration $\pi\colon Z \to E$ and a continuous homomorphism

$$\bar{\gamma}\colon \mathbb{R} \longrightarrow \mathrm{Homeo}^G(Z)$$

such that $\pi \circ \bar{\gamma}(r) = \gamma(r) \circ \pi$ for all $r \in \mathbb{R}$. Also choose a basepoint $\bar{\imath}_0 \in Z$ such that $\pi(\bar{\imath}_0) = i_0$. For any $r \in \mathbb{R}$, define $\bar{\imath}_r = \bar{\gamma}(r)(\bar{\imath}_0)$ and note that $\pi(\bar{\imath}_r) = i_r$.

(The purpose of the data $(\bar{\gamma}, \bar{\imath}_0)$ is to allow us to lift the operation of "shifting" an embedding via $\mathrm{sh}_r \circ -$ from the space of embeddings to the total space Z of the fibration.)

[4] In the sense of odd functions, i.e. $\theta(-t) = -\theta(t)$.

Definition 2.5.6 (*Moduli spaces of labelled disconnected submanifolds*)**.** Fix a subgroup $G \leqslant \mathrm{Diff}(P)$ as above. Then $\pi^n \colon Z^n \to E^n$ is a $(G \wr \Sigma_n)$-equivariant fibration. Let $\pi_n \colon Z_n \to \mathrm{Emb}(nP, M(0))$ be its restriction to the $(G \wr \Sigma_n)$-invariant subspace $\mathrm{Emb}(nP, M(0)) \subset E^n$. We then define

$$C_{nP}(M, Z; G) \;=\; \big(Z_n/(G \wr \Sigma_n)\big)_{\{[\bar{\imath}_1],\ldots,[\bar{\imath}_n]\}},$$

the path-component of the quotient $Z_n/(G\wr\Sigma_n) \subseteq \mathrm{Sp}^n(Z/G)$ containing the element $\{[\bar{\imath}_1], \ldots, [\bar{\imath}_n]\}$. Similarly, we may restrict π^{n+1} to the subspace $\mathrm{Emb}(\hat{n}P, M(-1)) \subset E^{n+1}$ to obtain a $(G \wr \Sigma_{n+1})$-equivariant fibration $\pi_{\hat{n}} \colon Z_{\hat{n}} \to \mathrm{Emb}(\hat{n}P, M(-1))$, and define

$$C_{\hat{n}P}(M, Z; G) \;=\; \big(Z_{\hat{n}}/(G \wr \Sigma_{n+1})\big)_{\{[\bar{\imath}_0],\ldots,[\bar{\imath}_n]\}}.$$

Viewing these as subspaces of the symmetric powers $\mathrm{Sp}^n(Z/G)$ and $\mathrm{Sp}^{n+1}(Z/G)$ respectively, we may define a map

$$s_n \colon C_{nP}(M, Z; G) \longrightarrow C_{\hat{n}P}(M, Z; G)$$

by $s_n(\{[\varphi_1], \ldots, [\varphi_n]\}) = \{[\bar{\imath}_0], [\varphi_1], \ldots, [\varphi_n]\}$.

These constructions are functorial in P, M, G and Z in an appropriate sense. We will describe how they are functorial in Z when the other data P, M and G are fixed. For $i = 1, 2$ let $\pi_i \colon Z_i \to E$ be based, G-equivariant Serre fibrations and let $\bar{\gamma}_i \colon \mathbb{R} \to \mathrm{Homeo}^G(Z_i)$ be continuous homomorphisms such that $\pi_i \circ \bar{\gamma}_i(r) = \gamma(r) \circ \pi_i$ for all r. Given any based, G-equivariant map $F \colon Z_1 \to Z_2$ such that $\pi_2 \circ F = \pi_1$ and $F \circ \bar{\gamma}_1(r) = \bar{\gamma}_2(r) \circ F$ for all r, there are induced maps making the square

$$\begin{array}{ccc} C_{nP}(M, Z_1; G) & \xrightarrow{\;s_n\;} & C_{\hat{n}P}(M, Z_1; G) \\ \downarrow & & \downarrow \\ C_{nP}(M, Z_2; G) & \xrightarrow{\;s_n\;} & C_{\hat{n}P}(M, Z_2; G) \end{array}$$

commute. These form a category with terminal object given by $(\pi, \bar{\gamma}) = (\mathrm{id} \colon E \to E, \gamma)$. When the fibration $\pi \colon Z \to E$ (and the map $\bar{\gamma}$) are taken to be this terminal object, we drop the Z from the notation and write simply $s_n \colon C_{nP}(M; G) \to C_{\hat{n}P}(M; G)$. For any other choice of $(\pi \colon Z \to E, \bar{\gamma})$ there is a commutative square

$$\begin{array}{ccc} C_{nP}(M, Z; G) & \xrightarrow{\;s_n\;} & C_{\hat{n}P}(M, Z; G) \\ \downarrow & & \downarrow \\ C_{nP}(M; G) & \xrightarrow{\;s_n\;} & C_{\hat{n}P}(M; G). \end{array} \tag{2.7}$$

Theorem 2.5.7 ([Pal21, Theorem A])**.** *The map $s_n \colon C_{nP}(M; G) \to C_{\hat{n}P}(M; G)$ induces split-injections on homology in all degrees. It induces isomorphisms on homology up to degree $\frac{n}{2}$ if $p \leqslant \frac{1}{2}(m-3)$ and G is an open subgroup of* $\mathrm{Diff}(P)$*.*

Recall that p and m are the dimensions of P and M respectively. We will lift this to the top horizontal map in (2.7), under a condition on the fibration $\pi \colon Z \to E$.

Theorem 2.5.8. (Theorem 2.C) *The map $s_n \colon C_{nP}(M, Z; G) \to C_{\hat{n}P}(M, Z; G)$ induces split-injections on homology in all degrees. If $p \leqslant \frac{1}{2}(m-3)$, the fibres of π are path-connected and G is an open subgroup of* $\mathrm{Diff}(P)$*, then it induces isomorphisms on integral homology up to degree $\frac{n}{2} - 1$ and on homology with coefficients in a field up to degree $\frac{n}{2}$.*

Remark 2.5.9. If a map induces isomorphisms on homology (up to a certain degree) with coefficients in any field, then it also induces isomorphisms up to the same degree with coefficients in any abelian group that may be constructed from fields by iterated extensions and colimits. In particular, $\mathbb{Q}/\mathbb{Z}$ is such a group, so the conclusion of the above theorem also implies that s_n induces isomorphisms on homology with $\mathbb{Q}/\mathbb{Z}$ coefficients up to degree $\frac{n}{2}$.

2.6 Proof of stability for symmetric diffeomorphism groups

We will deduce Theorem 2.A from Theorem 2.C by a spectral sequence comparison argument. First we need some more constructions to set up the appropriate map of spectral sequences.

2.6.1 Some fibre bundles

Notation 2.6.1. For any real number r we will write $\mathbb{R}^{\infty,r} = \mathbb{R}^\infty \times [r,\infty)$. Most of the time r will be 0, -1 or -2.

Fix an embedding $b\colon \partial M \hookrightarrow \mathbb{R}^\infty$. Write $\mathrm{Emb}_b(M,\mathbb{R}^{\infty,0})$ for the space of embeddings $e\colon M \hookrightarrow \mathbb{R}^{\infty,0}$ such that

(i) $e \circ \hat{\mathrm{col}}|_{\partial M\times[0,\epsilon)} = b \times \mathrm{incl}$ for some $\epsilon > 0$.

We then define

$$X_n \subseteq \mathrm{Emb}_b(M,\mathbb{R}^{\infty,0}) \times \mathrm{Emb}(M \underset{nP}{\sharp} nN, \mathbb{R}^{\infty,0})$$

to be the subspace of pairs of embeddings (e,f) such that

(ii) $e|_{M\smallsetminus\Phi(nT')} = f|_{M\smallsetminus\Phi(nT')}$.

Note that there is a continuous action of the symmetric diffeomorphism group $\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ on X_n by precomposition in each factor (and the homomorphism (2.4) for the first factor). Similarly, we write $\mathrm{Emb}_b(\hat{M},\mathbb{R}^{\infty,-1})$ for the space of embeddings $e\colon \hat{M} \hookrightarrow \mathbb{R}^{\infty,-1}$ such that

(î) $e \circ \hat{\mathrm{col}}|_{\partial M\times[-1,-1+\epsilon)} = b \times \mathrm{incl}$ for some $\epsilon > 0$

and we define

$$X_{\hat{n}} \subseteq \mathrm{Emb}_b(\hat{M},\mathbb{R}^{\infty,-1}) \times \mathrm{Emb}(M \underset{\hat{n}P}{\sharp} \hat{n}N, \mathbb{R}^{\infty,-1})$$

to be the subspace of pairs of embeddings (e,f) such that

(îî) $e|_{\hat{M}\smallsetminus\hat{\Phi}(\hat{n}T')} = f|_{\hat{M}\smallsetminus\hat{\Phi}(\hat{n}T')}$.

There is a continuous action of the symmetric diffeomorphism group $\Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$ on $X_{\hat{n}}$, given by precomposition in each factor and the right-hand vertical map of (2.6) for the first factor.

Lemma 2.6.2. *The spaces X_n and $X_{\hat{n}}$ are contractible and the quotient maps*

$$X_n \longrightarrow X_n/\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN) \qquad\qquad X_{\hat{n}} \longrightarrow X_{\hat{n}}/\Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$$

are principal bundles with structure groups $\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ and $\Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$ respectively.

Proof. The contractibility of the spaces X_n and $X_{\hat{n}}$ may be seen by the usual argument for the contractibility of spaces of embeddings into $\mathbb{R}^\infty$: any family of such embeddings parametrised by S^i is contained in some finite-dimensional subspace of $\mathbb{R}^\infty$ (by compactness of S^i), and this may be used to extend it to D^{i+1}.

A mild extension of Proposition 4.15 of [Pal21] shows that the action of $\mathrm{Diff}_\partial(\mathbb{R}^{\infty,0})$ on the quotient $X_n/\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ is locally retractile. Then Proposition 4.8 of [Pal21] implies that the projection of X_n onto this quotient is a principal bundle. An identical argument implies that the other projection is also a principal bundle. □

This gives us explicit models for the classifying spaces of $\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ and $\Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$.

Write $\mathrm{Diff}_\partial(M)$ for the group of diffeomorphisms of M that act by the identity on a neighbourhood of its boundary. There is a forgetful map

$$\Psi\colon X_n/\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow \mathrm{Emb}_b(M,\mathbb{R}^{\infty,0})/\mathrm{Diff}_\partial(M), \tag{2.8}$$

and an analogous map $\hat{\Psi}$, replacing n with $\hat{n}$ and $\mathbb{R}^{\infty,0}$ with $\mathbb{R}^{\infty,-1}$.

Lemma 2.6.3. *The maps Ψ and $\hat{\Psi}$ are fibre bundles.*

Proof. As above, by a mild extension of Proposition 4.15 of [Pal21], the action of $\mathrm{Diff}_\partial(\mathbb{R}^{\infty,0})$ on the quotient $\mathrm{Emb}_b(M,\mathbb{R}^{\infty,0})/\mathrm{Diff}_\partial(M)$ is locally retractile, and then Theorem A of [Pal60] (see also Proposition 4.7 of [Pal21]) implies that Ψ is a fibre bundle (and an identical argument implies the same for $\hat{\Psi}$). □

2.6.2 Moduli spaces of submanifolds labelled by parametric connected sum data

Recall from §2.2 that $K = H \cap \ker(z) \leqslant H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T)$, and that $G = z(H) \leqslant \mathrm{Diff}(P)$, where

$$z\colon \mathrm{Diff}_{\mathrm{fib}}(T) \longrightarrow \mathrm{Diff}(P)$$

is the restriction along the zero-section $o\colon P \hookrightarrow T$ of $\xi\colon T \to P$.

Notation 2.6.4. Write $N' = N \smallsetminus \tau_j(T')$ and $U = \mathring{T} \smallsetminus T'$. There is an involution

$$\sigma\colon U \longrightarrow U$$

given by $(x,t) \mapsto (x, 1.5 - t)$, where we identify U with $\partial T \times (0.5, 1)$ (*cf.* Definition 2.1.2). Note also that τ_j restricts to an embedding $U \hookrightarrow N'$.

Definition 2.6.5. The subgroup $\mathrm{Diff}_K(N')$ of $\mathrm{Diff}(N')$ consists of those diffeomorphisms that send the subset $\tau_j(U) \subset N'$ to itself and act on it via $\tau_j K \tau_j^{-1}$.

Construction 2.6.6. Fix an embedding $e_0 \in \mathrm{Emb}_b(M,\mathbb{R}^{\infty,0})$. For convenience, we assume that

(a) $e_0(\mathrm{col}(x,t)) = (b(x),t)$ for $(t,x) \in \partial M \times [0,1]$,
(b) $e_0(M \smallsetminus \mathrm{col}(\partial M \times [0,1])) \subseteq \mathbb{R}^\infty \times (1,\infty)$.

The fact that we are making this assumption will not cause problems later, since $\mathrm{Emb}_b(M,\mathbb{R}^{\infty,0})$ is path-connected (in fact contractible). Also, it will be convenient to extend $\hat{M}$ slightly further to

$$\hat{\hat{M}} = M \underset{\partial M \times [0,\infty]}{\cup} (\partial M \times [-2,\infty]),$$

and write $\hat{\mathrm{col}}\colon \partial M \times [-2,\infty] \hookrightarrow \hat{\hat{M}}$ for the inclusion of the right-hand side of this pushout. We may extend e_0 to an embedding

$$\hat{\hat{e}}_0\colon \hat{\hat{M}} \longrightarrow \mathbb{R}^{\infty,-2},$$

defining $\hat{\hat{e}}_0(\hat{\mathrm{col}}(x,t)) = (b(x),t)$ for $(x,t) \in \partial M \times [-2,0]$. There is a diagram of topological spaces

$$\begin{array}{ccc} & & \mathrm{Emb}(N',\mathbb{R}^{\infty,-2})/\mathrm{Diff}_K(N') \\ & & \big\downarrow \\ \mathrm{Emb}^c(T,\hat{\hat{M}})/K & \longrightarrow & \mathrm{Emb}(U,\mathbb{R}^{\infty,-2})/K, \end{array} \tag{2.9}$$

where $\mathrm{Emb}^c(T,\hat{\hat{M}}) \subset \mathrm{Emb}(T,\hat{\hat{M}})$ is the subspace of embeddings $T \hookrightarrow \hat{\hat{M}}$ such that the image of the zero-section $o(P) \subset T$ is contained in $\hat{M} \subset \hat{\hat{M}}$. The vertical map is given by precomposition by $\tau_j \circ \sigma$ and the horizontal map is given by postcomposition by $\hat{\hat{e}}_0$ (and restriction of the domain).

Lemma 2.6.7. *The diagram (2.9) is a diagram of right G-spaces with respect to the following well-defined actions. For the bottom two spaces, the action of $g \in G$ is given by choosing any element $h \in H$ such that $z(h) = g$ and acting by precomposition by h. For the top-right space, the action is given by first choosing any element $h \in H$ such that $z(h) = g$ and then any diffeomorphism of N' whose restriction to U along the embedding τ_j is h.*

Proof. The well-defined-ness of the described G-actions on the bottom two spaces follows from the fact that we have a short exact sequence

$$1 \to K \hookrightarrow H \xrightarrow{z|_H} G \to 1,$$

and the horizontal map of (2.9) is clearly equivariant with respect to these actions. By one of our assumptions just before the statement of Theorem 2.4.1 on page 40, the map $\mathrm{Diff}_H(N') = \mathrm{Diff}_H(N) \to H$ given by restriction along the embedding τ_j is surjective. It is therefore also surjective after composing with $z|_H \colon H \to G$, and we have another short exact sequence

$$1 \to \mathrm{Diff}_K(N') \hookrightarrow \mathrm{Diff}_H(N') \longrightarrow G \to 1,$$

which shows that the described G-action on the top-right space of (2.9) is well-defined. To see that the vertical map of (2.9) is equivariant, we note that the action of $H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T)$ on $U \subset T$ commutes with the involution $\sigma \colon U \to U$. To see this, note that, under the identification $U \cong \partial T \times (0.5, 1)$, the H-action is trivial on the second component and the involution σ is trivial on the first component. ◻

Definition 2.6.8. Let Z be the pullback in the category of topological spaces of the diagram (2.9). Since this is a diagram of right G-spaces, Z is also a right G-space if we give it the diagonal action. In other words, we take Z to be the pullback in the category of right G-spaces.

Lemma 2.6.9. *The composite map*

$$\pi \colon Z \longrightarrow \mathrm{Emb}^c(T, \hat{\hat{M}})/K \longrightarrow \mathrm{Emb}(P, \hat{M}) = E$$

is a Serre fibration with path-connected fibres.

Proof. By a mild extension of Theorem B of [Pal60] (see also [Cer61]) allowing manifolds with boundary, the action of $\mathrm{Diff}_\partial(\hat{M})$ on $\mathrm{Emb}(P, \hat{M})$ is locally retractile. Similarly, a mild extension of Proposition 4.15 of [Pal21] implies that the action of $\mathrm{Diff}_\partial(\mathbb{R}^{\infty,-2})$ on $\mathrm{Emb}(U, \mathbb{R}^{\infty,-2})/K$ is locally retractile. Theorem A of [Pal60] then implies that the right-hand map $\mathrm{Emb}^c(T, \hat{\hat{M}})/K \to E$ above and the vertical map of (2.9) are fibre bundles. The left-hand map $Z \to \mathrm{Emb}^c(T, \hat{\hat{M}})/K$ above is a pullback of the vertical map of (2.9), so it is also a fibre bundle. A composition of two fibre bundles is not necessarily a fibre bundle, but it is at least a Serre fibration.

It is not hard to show that the fibres of the vertical map of (2.9) are path-connected, using the fact that we are considering embeddings into infinite-dimensional Euclidean space. The fibres of $Z \to \mathrm{Emb}^c(T, \hat{\hat{M}})/K$ are therefore also path-connected.

Fix an embedding $e \in \mathrm{Emb}(P, \hat{M})$ and denote the fibre of the map $\mathrm{Emb}^c(T, \hat{\hat{M}}) \to \mathrm{Emb}(P, \hat{M})$ over e by $\mathrm{Emb}(T, \hat{\hat{M}})_e$. Note that we do not yet take the quotient by K. This is almost the space $\mathrm{Tub}(e)$ of tubular neighbourhoods of e. More accurately, there is a fibration $\mathrm{Emb}(T, \hat{\hat{M}})_e \to \mathrm{Aut}(T)$ to the topological group of bundle automorphisms of T (as a disc bundle with structure group $O(m-p)$), and the fibre over the identity is $\mathrm{Tub}(e)$. This may be summarised as follows:

$$\begin{array}{ccccc} \mathrm{Tub}(e) & \longrightarrow & \mathrm{Emb}(T, \hat{\hat{M}})_e & \longrightarrow & \mathrm{Emb}^c(T, \hat{\hat{M}}) \\ & & \downarrow & & \downarrow \\ & & \mathrm{Aut}(T) & & \mathrm{Emb}(P, \hat{M}) \ni e \end{array} \tag{2.10}$$

Now, the space $\mathrm{Tub}(e)$ of tubular neighbourhoods of e is contractible [God07, Proposition 31], so the fibres of $\mathrm{Emb}^c(T, \hat{\hat{M}}) \to \mathrm{Emb}(P, \hat{M})$ are homotopy equivalent to $\mathrm{Aut}(T)$. Note that this group is exactly the kernel of the map

$$z \colon \mathrm{Diff}_{\mathrm{fib}}(T) \longrightarrow \mathrm{Diff}(P)$$

defined at the beginning of §2.2. To study the fibres of $\mathrm{Emb}^c(T,\hat{\hat{M}})/K \to \mathrm{Emb}(P,\hat{M})$, we quotient three of the spaces in (2.10) by the action of K:

$$\begin{array}{ccccc} \mathrm{Tub}(e) & \longrightarrow & \mathrm{Emb}(T,\hat{\hat{M}})_e/K & \longrightarrow & \mathrm{Emb}^c(T,\hat{\hat{M}})/K \\ & & \downarrow & & \downarrow \\ & & \mathrm{Aut}(T)/K & & \mathrm{Emb}(P,\hat{M}) \ni e \end{array} \tag{2.11}$$

As $\mathrm{Tub}(e)$ is contractible, the fibres of $\mathrm{Emb}^c(T,\hat{\hat{M}})/K \to \mathrm{Emb}(P,\hat{M})$ are homotopy equivalent to

$$\mathrm{Aut}(T)/K = \ker(z)/K.$$

But we assumed just before the statement of Theorem 2.4.1 on page 40 that this coset space is path-connected. Thus both of the fibrations $Z \to \mathrm{Emb}^c(T,\hat{\hat{M}})/K \to \mathrm{Emb}(P,\hat{M})$ have path-connected fibres, so the composite fibration π also has path-connected fibres. □

Note that, by construction, the map π is G-equivariant. Recall from §2.5 (see Input 2.5.5) that we also need to choose a basepoint $\bar{\imath}_0 \in Z$ and a continuous homomorphism $\bar{\gamma}\colon \mathbb{R} \to \mathrm{Homeo}^G(Z)$ in order to define the moduli space of labelled submanifolds $C_{nP}(M,Z;G)$.

Choose an embedding $\upsilon\colon N' \hookrightarrow \mathbb{R}^\infty \times (-0.5,0.5) \subset \mathbb{R}^{\infty,-2}$ so that the following diagram commutes:

$$\begin{array}{ccccc} U & \xrightarrow{\tau_j|_U} & N' & \overset{\upsilon}{\searrow} & \\ \sigma\updownarrow & & & & \mathbb{R}^{\infty,-2} \\ U & \xrightarrow[\tau_i|_U]{} & \hat{\hat{M}} & \underset{\hat{\hat{e}}_0}{\nearrow} & \end{array}$$

Remark 2.6.10. The choice of υ can be made independently of e_0, since we have prescribed how e_0 acts on $\mathrm{col}(\partial M \times [0,1])$, and therefore how $\hat{\hat{e}}_0$ acts on $\hat{\mathrm{col}}(\partial M \times [-2,1])$, and the image $\tau_i(U)$ is contained in $\hat{\mathrm{col}}(\partial M \times (-0.5,0.5))$.

Then $([\tau_i],[\upsilon]) \in Z$ and $\pi([\tau_i],[\upsilon]) = i = i_0$. So we may set $\bar{\imath}_0 = ([\tau_i],[\upsilon])$.

We may extend the "shift" map of Definition 2.5.4 by the identity to a diffeomorphism $\mathrm{sh}_r\colon \hat{\hat{M}} \to \hat{\hat{M}}$ for each $r \in \mathbb{R}$. We write $\mathrm{id} \times \bar{\theta}(r)$ for the self-diffeomorphism of $\mathbb{R}^{\infty,-2} = \mathbb{R}^\infty \times [-2,\infty)$ that is the identity on $\mathbb{R}^\infty$ and acts by (the restriction of) $\bar{\theta}(r)\colon \mathbb{R} \to \mathbb{R}$ on $[-2,\infty)$. With this notation, we define a map $\bar{\gamma}\colon \mathbb{R} \to \mathrm{Map}(Z,Z)$ by

$$\bar{\gamma}(r)\colon ([\alpha],[\beta]) \longmapsto ([\mathrm{sh}_r \circ \alpha], [(\mathrm{id} \times \bar{\theta}(r)) \circ \beta]).$$

One may easily check that $\bar{\gamma}$ is a well-defined, continuous map and that its image lies in $\mathrm{Homeo}^G(Z) \leqslant \mathrm{Map}(Z,Z)$. It is also a group homomorphism (since $\bar{\theta}$ is) and each $\bar{\gamma}(r)$ covers the self-homeomorphism $\gamma(r) = \mathrm{sh}_r \circ -$ of E. This completes the construction of the input data needed (see Input 2.5.5) in order to apply Definition 2.5.6.

Definition 2.6.11. We may now apply Definition 2.5.6 to the data $(\pi\colon Z \to E, \bar{\gamma}, \bar{\imath}_0)$ constructed above to obtain spaces $C_{nP}(M,Z;G)$ and $C_{\hat{n}P}(M,Z;G)$, as well as a stabilisation map

$$s_n\colon C_{nP}(M,Z;G) \longrightarrow C_{\hat{n}P}(M,Z;G).$$

These may be thought of as *moduli spaces of disconnected submanifolds labelled by parametric-connected-sum-data.*

In the rest of this subsection, we will show that the moduli space $C_{nP}(M,Z;G)$, for this particular fibration $\pi\colon Z \to E$, is homotopy equivalent to the fibres of the bundle (2.8): see Proposition 2.6.14. First we establish a lemma that we will need in the proof of this proposition.

Lemma 2.6.12. *The space* $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ *has a left-action of the group* $\mathrm{Diff}_\partial(M)$ *of diffeomorphisms of* M *that act by the identity on a neighbourhood of its boundary. The embedding* Φ *from* §2.1 *gives us a basepoint* $[\Phi]$ *for* $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$. *The orbit of this basepoint under the action of* $\mathrm{Diff}_\partial(M)$ *is path-connected.*

Remark 2.6.13. The map $[- \circ \Phi]\colon \mathrm{Diff}_\partial(M) \to \mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ is a fibre bundle, by Propositions 4.15 and 4.7 of [Pal21], so its image, the orbit of $[\Phi]$, must be a union of path-components. By Lemma 2.6.12 it is exactly the path-component of $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ containing $[\Phi]$.

Proof of Lemma 2.6.12. Let $\varphi \in \mathrm{Diff}_\partial(M)$; we will find a path of embeddings $nT \hookrightarrow M$ from $\varphi \circ \Phi$ to Φ. The image of Φ is contained in a collar neighbourhood of ∂M, so we may choose a path $t \mapsto \varphi_t$ in $\mathrm{Diff}_\partial(M)$ so that $\varphi_0 = \varphi$ and φ_1 restricts to the identity on this collar neighbourhood, in particular on the image of Φ. Then $t \mapsto \varphi_t \circ \Phi$ is the required path of embeddings. ◻

Proposition 2.6.14. *The fibre of* (2.8) *over* $[e_0] \in \mathrm{Emb}_b(M, \mathbb{R}^{\infty,0})/\mathrm{Diff}_\partial(M)$ *is homotopy-equivalent to* $C_{nP}(M, Z; G)$. *More precisely, there is a canonical inclusion* $\Psi^{-1}([e_0]) \hookrightarrow C_{nP}(M, Z; G)$, *which is a homotopy equivalence. The corresponding statement for* $\hat{\Psi}$ *also holds: Write* $\hat{e}_0 = \hat{\hat{e}}_0|_{\hat{M}}$. *There is a canonical inclusion* $\hat{\Psi}^{-1}([\hat{e}_0]) \hookrightarrow C_{\hat{n}P}(M, Z; G)$, *which is a homotopy equivalence.*

Proof. We will do this in three steps: (1) give an explicit description of $\Psi^{-1}([e_0])$ and note that it is path-connected, (2) give an explicit description of $C_{nP}(M, Z; G)$ and show that it contains a homeomorphic copy of $\Psi^{-1}([e_0])$, and (3) show that the inclusion is a homotopy equivalence.

Step 0. Before this, however, we recall a basic fact that we will use in the next step. Let X be a left G-space, and assume that the G-action on X is *locally retractile* (see for example Definition 4.5 of [Pal21]). Then for any $x \in X$ there is a G-equivariant homeomorphism $G/\mathrm{stab}_G(x) \cong \mathrm{orbit}(x)$. To see this, first note that the action map $- \cdot x\colon G \to X$ induces a continuous bijection $G/\mathrm{stab}_G(x) \to \mathrm{orbit}(x) \subseteq X$, which is G-equivariant. Then Theorem A of [Pal60] implies that this map is a fibre bundle, in particular an open map, and so it is a homeomorphism.

Step 1. Rewriting the definition of X_n a little, we may describe it as the subspace of

$$\mathrm{Pullback}\big(\, \mathrm{Emb}_b(M, \mathbb{R}^{\infty,0}) \longrightarrow \mathrm{Emb}(nU, \mathbb{R}^{\infty,0}) \longleftarrow \mathrm{Emb}(nN', \mathbb{R}^{\infty,0}) \,\big)$$

of pairs of embeddings (e, f) such that $f(nN')$ is disjoint from the closure[5] of $e(M \smallsetminus \Phi(n\mathring{T}))$. The first map in the pullback diagram is given by restriction along Φ and the second map is given by restriction along τ_j followed by the involution σ of U. The quotient $X_n/\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ may therefore be described as the subspace of

$$\mathrm{Pullback}\left(\frac{\mathrm{Emb}_b(M, \mathbb{R}^{\infty,0})}{\Sigma_H \mathrm{Diff}(M, nT)} \longrightarrow \frac{\mathrm{Emb}(nU, \mathbb{R}^{\infty,0})}{H \wr \Sigma_n} \longleftarrow \frac{\mathrm{Emb}(nN', \mathbb{R}^{\infty,0})}{\mathrm{Diff}_H(N) \wr \Sigma_n} \right)$$

of pairs $([e], [f])$ satisfying the same disjointness condition (*cf.* the pullback square (2.5) in §2.2). The fibre $\Psi^{-1}([e_0])$ is the subspace where the image of e agrees with the image of e_0, so it may be described as the subspace of

$$\mathrm{Pullback}\left(\frac{\mathrm{Diff}_\partial(M)}{\Sigma_H \mathrm{Diff}(M, nT)} \longrightarrow \frac{\mathrm{Emb}(nU, \mathbb{R}^{\infty,0})}{H \wr \Sigma_n} \longleftarrow \frac{\mathrm{Emb}(nN', \mathbb{R}^{\infty,0})}{\mathrm{Diff}_H(N) \wr \Sigma_n} \right)$$

of pairs $([\eta], [f])$ such that $f(nN')$ is disjoint from the closure of $e_0(M \smallsetminus \eta(\Phi(n\mathring{T})))$. A mild extension of Proposition 4.15 of [Pal21] shows that $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ is $\mathrm{Diff}_\partial(M)$-locally retractile. The stabiliser of $[\Phi] \in \mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ is the subgroup $\Sigma_H \mathrm{Diff}(M, nT) \leqslant \mathrm{Diff}_\partial(M)$, so via the "topological orbit-stabiliser theorem" (see Step 0 above) we deduce that $\mathrm{Diff}_\partial(M)/\Sigma_H \mathrm{Diff}(M, nT)$ is homeomorphic to the orbit of $[\Phi]$ in $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$. By Lemma 2.6.12 and Remark 2.6.13, the orbit of $[\Phi]$ in $\mathrm{Emb}(nT, M)/(H \wr \Sigma_n)$ is the path-component of $[\Phi]$ in

[5] We need to take the closure here since M was not assumed to be compact.

$\mathrm{Emb}(nT,M)/(H\wr\Sigma_n)$. Thus $\mathrm{Diff}_\partial(M)/\Sigma_H\mathrm{Diff}(M,nT)$ is homeomorphic to the path-component of $[\Phi]$ in $\mathrm{Emb}(nT,M)/(H\wr\Sigma_n)$. We may therefore describe $\Psi^{-1}([e_0])$ as the subspace of

$$\mathrm{Pullback}\left(\frac{\mathrm{Emb}(nT,M)}{H\wr\Sigma_n}\longrightarrow\frac{\mathrm{Emb}(nU,\mathbb{R}^{\infty,0})}{H\wr\Sigma_n}\longleftarrow\frac{\mathrm{Emb}(nN',\mathbb{R}^{\infty,0})}{\mathrm{Diff}_H(N)\wr\Sigma_n}\right)$$

of pairs $([\Phi'],[f])$ such that $f(nN')$ is disjoint from the closure of $e_0(M\smallsetminus\Phi'(n\mathring{T}))$ and there exists a path $[\Phi']\rightsquigarrow[\Phi]$ in $\mathrm{Emb}(nT,M)/(H\wr\Sigma_n)$.

It is now not hard to show that the fibre $\Psi^{-1}([e_0])$ is path-connected. Choose a basepoint $([\Phi_0'],[f_0])$ for it and consider any other point $([\Phi'],[f])$. There is a path $[\Phi']\rightsquigarrow[\Phi_0']$ in the left-hand space of the pullback diagram. The image of this path in the middle space may be lifted to a path $[f]\rightsquigarrow[f_1]$ in the right-hand space, since the right-hand map of the pullback diagram is a fibre bundle, and therefore a Serre fibration. Since we are considering embeddings into $\mathbb{R}^{\infty,0}$ we may easily ensure that this path of embeddings satisfies the disjointness condition, at each point in time during the path, with respect to the path $[\Phi']\rightsquigarrow[\Phi_0']$, so we have a path $([\Phi'],[f])\rightsquigarrow([\Phi_0'],[f_1])$ in $\Psi^{-1}([e_0])$. Choose a path $f_1\rightsquigarrow f_0$ of embeddings $nN'\hookrightarrow\mathbb{R}^{\infty,0}$ disjoint from the closure of $e_0(M\smallsetminus\Phi_0'(n\mathring{T}))$ and constant when restricted to nU. This gives us a path $([\Phi_0'],[f_1])\rightsquigarrow([\Phi_0'],[f_0])$ in $\Psi^{-1}([e_0])$. Thus we have shown that $\Psi^{-1}([e_0])$ is path-connected.

Step 2. Recall from Definition 2.5.6 that $C_{nP}(M,Z;G)$ is a certain path-component of $Z_n/(G\wr\Sigma_n)$. Unravelling this definition for the fibration $\pi\colon Z\to E$ of Lemma 2.6.9, we may describe Z_n as the subspace of

$$\mathrm{Pullback}\left(\frac{\mathrm{Emb}(T,\hat{\mathring{M}})^n}{K^n}\longrightarrow\frac{\mathrm{Emb}(U,\mathbb{R}^{\infty,-2})^n}{K^n}\longleftarrow\frac{\mathrm{Emb}(N',\mathbb{R}^{\infty,-2})^n}{\mathrm{Diff}_K(N')^n}\right)$$

of tuples of embeddings $(([\varphi_1],\dots,[\varphi_n]),([f_1],\dots,[f_n]))$ such that each $\varphi_\alpha(P)$ is contained in $\mathring{M}$ and the images $\varphi_1(P),\dots,\varphi_n(P)$ are pairwise disjoint. The quotient $Z_n/(G\wr\Sigma_n)$ is therefore the subspace of

$$\mathrm{Pullback}\left(\frac{\mathrm{Emb}(T,\hat{\mathring{M}})^n}{H\wr\Sigma_n}\longrightarrow\frac{\mathrm{Emb}(U,\mathbb{R}^{\infty,-2})^n}{H\wr\Sigma_n}\longleftarrow\frac{\mathrm{Emb}(N',\mathbb{R}^{\infty,-2})^n}{\mathrm{Diff}_H(N)\wr\Sigma_n}\right)$$

of collections of embeddings $(\{[\varphi_1],\dots,[\varphi_n]\},\{[f_1],\dots,[f_n]\})$ satisfying the same two conditions. Comparing this with the final description of $\Psi^{-1}([e_0])$ in the previous step, we see that there is a canonical inclusion $\Psi^{-1}([e_0])\hookrightarrow Z_n/(G\wr\Sigma_n)$. Moreover, $\Psi^{-1}([e_0])$ is path-connected and contains the basepoint configuration $\{[\bar{\iota}_1],\dots,[\bar{\iota}_n]\}$, so there is in fact a canonical inclusion

$$\Psi^{-1}([e_0])\longhookrightarrow C_{nP}(M,Z;G).$$

Step 3. A point in $C_{nP}(M,Z;G)$ lies in the subspace $\Psi^{-1}([e_0])$ if and only if

(a) $\varphi_1(T),\dots,\varphi_n(T)$ are pairwise disjoint and contained in M,
(b) $f_1(N'),\dots,f_n(N')$ are pairwise disjoint and contained in $\mathbb{R}^{\infty,0}$,
(c) $\bigcup_\alpha f_\alpha(N')$ is disjoint from the closure of $e_0(M\smallsetminus\bigcup_\beta\varphi_\beta(\mathring{T}))$,
(d) there is a path in $\mathrm{Emb}(nT,M)/(H\wr\Sigma_n)$ from $\{[\varphi_1],\dots,[\varphi_n]\}$ to $[\Phi]$.

In fact, property (d) is automatic once we have property (a). Since $C_{nP}(M,Z;G)$ is path-connected, there is a path in $\mathrm{Emb}(T,\hat{\mathring{M}})^n/(H\wr\Sigma_n)$ from $\{[\varphi_1],\dots,[\varphi_n]\}$ to $[\Phi]$, and the n embedded copies of $P\subset T$ in $\hat{\mathring{M}}$ are pairwise disjoint and contained in $\mathring{M}$ at each point in time during this path. We may therefore shrink the tubular neighbourhoods $T\supset P$ by an appropriate amount at each point during the path, to obtain a new path in $\mathrm{Emb}(nT,M)/(H\wr\Sigma_n)$ from $\{[\varphi_1],\dots,[\varphi_n]\}$ to $[\Phi]$.

Thus, we would like to define a deformation retraction that begins with a point in $C_{nP}(M,Z;G)$ and ends with a new point in $C_{nP}(M,Z;G)$ satisfying the disjointness properties (a), (b) and (c). In fact, we will not do this for $C_{nP}(M,Z;G)$, but instead for its ordered analogue $F_{nP}(M,Z;G)$, the covering space in which the n embedded copies of T in $\hat{\mathring{M}}$ are equipped with an ordering. If we write $\widetilde{\Psi}^{-1}([e_0])\subset F_{nP}(M,Z;G)$ for the restriction of this covering space to $\Psi^{-1}([e_0])\subset C_{nP}(M,Z;G)$, we have a map of fibre sequences:

$$\begin{array}{ccccc} F_{nP}(M,Z;G) & \longrightarrow & C_{nP}(M,Z;G) & \longrightarrow & B\Sigma_n \\ \uparrow & & \uparrow & & \| \\ \widetilde{\Psi}^{-1}([e_0]) & \longrightarrow & \Psi^{-1}([e_0]) & \longrightarrow & B\Sigma_n \end{array}$$

If we can define a deformation retraction for the inclusion $\widetilde{\Psi}^{-1}([e_0]) \subset F_{nP}(M,Z;G)$, then the map of long exact sequences of homotopy groups will imply that the inclusion $\Psi^{-1}([e_0]) \subset C_{nP}(M,Z;G)$ is also a (weak) homotopy equivalence.

We now sketch a deformation retraction that begins with a point in $F_{nP}(M,Z;G)$ and ends with a new point in $F_{nP}(M,Z;G)$ satisfying properties (a), (b) and (c). Since the "cores" $\varphi_\alpha(P)$ are pairwise disjoint and contained in $\mathring{M}$, we may shrink the tubular neighbourhoods $T \supset P$ as above to ensure property (a); this may moreover be done in a canonical way, so that the deformation retraction is continuous. To ensure property (c), we may choose k such that $e_0(M) \subseteq \mathbb{R}^k \subset \mathbb{R}^{\infty,0}$ and modify the embedded copies of N' in $\mathbb{R}^{\infty,-2}$ so that their $(k+1)$-st coordinate is non-zero on $N \smallsetminus \tau_j(T) \subset N'$.

Finally, we need to ensure property (b). We may push $\mathbb{R}^{\infty,-2}$ into $\mathbb{R}^{\infty,0}$ by increasing its last coordinate to ensure that the embedded copies of N' are contained in $\mathbb{R}^{\infty,0}$. To ensure that they are pairwise disjoint, we modify them by straight-line homotopies so that, for each $r \in \{1,\ldots,n\}$, the $(k+1+r)$-th coordinate of the r-th copy of N' is non-zero on $N \smallsetminus \tau_j(T) \subset N'$, and the $(k+1+r)$-th coordinate of every other copy of N' is zero on $N \smallsetminus V \subset N'$, where V is a very small open neighbourhood of $\tau_j(T)$ in N. (This is where we need to use the ordering.) This will force the different copies of N' to be pairwise disjoint, except possibly on the subsets $V \smallsetminus \tau_j(T) \subset N'$. However, we may control these neighbourhoods to be very small, i.e. very close to the corresponding $\varphi_r(T)$, so, by shrinking these further if necessary, we may ensure that the different copies of N' are pairwise disjoint everywhere. □

2.6.3 A map of fibre bundles

We now define a continuous map $X_n \to X_{\hat{n}}$, in order to obtain a map of bundles from Ψ to $\hat{\Psi}$.

Definition 2.6.15. Define

$$\mathrm{st}_n \colon X_n \longrightarrow X_{\hat{n}}$$

to send a pair of embeddings (e,f) to $(\hat{e},\hat{f})$, where:

(i) $\hat{e} = e$ on $M \subset \hat{M}$ and $\hat{e}(\hat{\mathrm{col}}(x,t)) = (b(x),t)$ for $(x,t) \in \partial M \times [-1,0]$.

(ii) Recall that

$$M \underset{\hat{n}P}{\sharp} \hat{n}N \;=\; (\hat{M} \smallsetminus \hat{\Phi}(\hat{n}T')) \underset{\hat{n}U}{\cup} \hat{n}N'$$

(*cf.* Definition 2.1.2 and Notation 2.6.4). We define $\hat{f} = \hat{e}$ on the subspace $\hat{M} \smallsetminus \hat{\Phi}(\hat{n}T')$, and also $\hat{f} = f$ on the subspace $nN' \subseteq M \underset{nP}{\sharp} nN$, so it remains to define $\hat{f}$ on $\hat{n}N' \smallsetminus nN' = \{0\} \times N'$. Here we define it by

$$\{0\} \times N' = N' \xrightarrow{\ \upsilon\ } \mathbb{R}^\infty \times (-0.5, 0.5) \xrightarrow{(\mathrm{id},-0.5)} \mathbb{R}^\infty \times (-1,0) \subset \mathbb{R}^{\infty,-1}.$$

It is then an easy exercise to check that $(\hat{e},\hat{f})$ is an element of $X_{\hat{n}}$ and that st_n is continuous.

Remark 2.6.16. Recall that X_n and $X_{\hat{n}}$ have actions of $\Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN)$ and $\Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$ respectively. Since we defined $\hat{e}$ and $\hat{f}$ above so that $\hat{e} = e$ on M and $\hat{f} = f$ on $M \underset{nP}{\sharp} nN$, it follows that st_n is equivariant with respect to the top horizontal map (continuous homomorphism) of the diagram (2.6), which we now denote by

$$\sigma_n \colon \Sigma_H\mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow \Sigma_H\mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N).$$

It therefore follows from Lemma 2.6.2 that the induced map

$$\overline{\mathrm{st}}_n \colon X_n/\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) \longrightarrow X_{\hat{n}}/\Sigma_H \mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N)$$

is a model for $B\sigma_n$, the map induced on classifying spaces by σ_n. It also fits into a map of bundles

$$\begin{array}{ccc} X_n/\Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN) & \xrightarrow{\overline{\mathrm{st}}_n} & X_{\hat{n}}/\Sigma_H \mathrm{Diff}(M \underset{\hat{n}P}{\sharp} \hat{n}N) \\ \Psi \downarrow & & \downarrow \hat{\Psi} \\ \mathrm{Emb}_b(M, \mathbb{R}^{\infty,0})/\mathrm{Diff}_\partial(M) & \longrightarrow & \mathrm{Emb}_b(M, \mathbb{R}^{\infty,-1})/\mathrm{Diff}_\partial(M), \end{array} \tag{2.12}$$

where the bottom horizontal map is defined by $[e] \mapsto [\hat{e}]$, where $\hat{e}$ is defined as in (i) in Definition 2.6.15 above.

Lemma 2.6.17. *The map $\overline{\mathrm{st}}_n$ induces split-injections on homology in all degrees.*

Proof. In order to apply Lemma 2 of [Dol62] to deduce split-injectivity, it suffices to define maps

$$X_n/D_n \longrightarrow \mathrm{Sp}^{\binom{n}{k}}(X_k/D_k)$$

satisfying a certain equation up to homotopy, where we are using the temporary abbreviation $D_n = \Sigma_H \mathrm{Diff}(M \underset{nP}{\sharp} nN)$. We briefly sketch how to construct such maps. They may be defined by

$$[e, f] \longmapsto \sum_{S \subseteq \{1,\ldots,n\},\, |S|=k} [e'_S, f'_S],$$

where f'_S is the composition

$$M \underset{kP}{\sharp} kN \;\cong\; M \underset{S\times P}{\sharp} (S \times N) \hookrightarrow M \underset{nP}{\sharp} nN \xrightarrow{f} \mathbb{R}^{\infty,0} \tag{2.13}$$

and $e'_S = e \circ \eta_S$, where $\eta_S \in \mathrm{Diff}_\partial(M)$ corresponds to the left-hand diffeomorphism of (2.13) in the sense that these two diffeomorphisms agree on $M \smallsetminus \Phi(kT')$. ☐

2.6.4 Stability for symmetric diffeomorphism groups

Lemma 2.6.18. *The map of base spaces in (2.12) is a homotopy equivalence.*

Proof. In order to define a homotopy inverse, we need a path of compactly-supported embeddings $\gamma_t \colon [-1, \infty) \hookrightarrow [-1, \infty)$ such that $\gamma_0(-1) = 0$, $\gamma_t(0) \geqslant 0$ for all t and $\gamma_1 = \mathrm{id}$. Then applying this isotopy to the last coordinate of $\mathbb{R}^{\infty,-1}$ determines a homotopy inverse for the bottom horizontal map in (2.12). The finer details of the construction are similar to those of Lemma 5.26 of [Pal21]. ☐

Now fix a point $[e_0]$ in the base space of Ψ as in Construction 2.6.6. There is a commutative square[6]

$$\begin{array}{ccc} C_{nP}(M, Z; G) & \xrightarrow{s_n} & C_{\hat{n}P}(M, Z; G) \\ \uparrow & & \uparrow \\ \Psi^{-1}([e_0]) & \xrightarrow[\overline{\mathrm{st}}_n|_{[e_0]}]{} & \hat{\Psi}^{-1}([\hat{e}_0]) \end{array} \tag{2.14}$$

[6] In fact this square does not commute on the nose, but if we replace the top horizontal map s_n with a homotopic map s'_n, then it does commute on the nose. Recall that s_n adjoins the element $[\bar{\iota}_0] = [[\tau_i], [\upsilon]] \in Z/G$ to an unordered tuple of elements of Z/G. The map s'_n instead adjoins the element $[[\tau_i - 0.5], [\upsilon - 0.5]]$, where the -0.5 denotes a shift along the collar neighbourhood of $\hat{\hat{M}}$ (for τ_i) and in the last coordinate of $\mathbb{R}^{\infty,-2}$ (for υ).

whose vertical maps are homotopy equivalences by Proposition 2.6.14. The map of bundles (2.12) induces a map of Serre spectral sequences, which is an isomorphism on E^2 pages up to vertical degree $\frac{n}{2}-1$ (or $\frac{n}{2}$ if we take field coefficients), by Theorem 2.C and Lemma 2.6.9. The Zeeman comparison theorem [Zee57] (*cf.* also [Iva93, Theorem 1.2] or [CDG13, Remarque 2.10]) then implies that $\overline{\mathrm{st}}_n = B\sigma_n$ induces isomorphisms on homology up to degree $\frac{n}{2}-1$ (or $\frac{n}{2}$ with field coefficients). Together with Lemma 2.6.17, this proves homological stability for the top horizontal map of (2.6). By Remark 2.2.4, a special case of this implies homological stability also for the bottom horizontal map of (2.6).[7] This completes the proof of Theorem 2.A, assuming Theorem 2.C.

2.7 Diffeomorphism groups of manifolds with conical singularities

Before proving Theorem 2.C, we first discuss manifolds with (discrete) conical singularities and their diffeomorphism groups. These are a special case of manifolds with *Baas-Sullivan singularities*, introduced in [Sul67; Baa73], for which the singular set is not necessarily discrete, but may itself be a smooth manifold of positive dimension (see §2.7.3 for a brief overview). We then prove homological stability for diffeomorphism groups of manifolds with conical singularities, with respect to adding new singularities of a fixed type — in fact, we will see that this is nothing more than a special case of homological stability for symmetric diffeomorphism groups, already contained in Theorem 2.A.

2.7.1 Manifolds with conical singularities

Fix a smooth, closed $(m-1)$-dimensional manifold L. The *open cone* on L is

$$\mathrm{cone}(L) = (L \times [0,\infty))/(L \times \{0\})$$

and we write $\star = [L \times \{0\}] \in \mathrm{cone}(L)$ for the point at the tip of the cone.

Definition 2.7.1. An *m-dimensional smooth manifold with conical L-singularities* consists of a space M equipped with a discrete subset $\Sigma \subseteq M$ and the structure of a smooth manifold on $M \smallsetminus \Sigma$. In addition, the points $\sigma \in \Sigma$ are equipped with pairwise disjoint open neighbourhoods $\sigma \in U_\sigma \subseteq M$ and homeomorphisms $U_\sigma \cong \mathrm{cone}(L)$ sending σ to $\star$ and restricting to diffeomorphisms $U_\sigma \smallsetminus \{\sigma\} \cong L \times (0,\infty)$.

For example, when $m = 1$, this amounts to a graph of fixed valency k (i.e. every vertex has the same valency k). In this case Σ is the set of vertices of the graph and L is the 0-manifold $\{1,\ldots,k\}$. Two other examples are as follows.

Example 2.7.2. If M is a smooth m-dimensional manifold, then we may take $\Sigma = \varnothing$. Alternatively, we may also take Σ to be any discrete subset and $L = S^{m-1}$, although in this case we forget the smooth structure at the marked points Σ.

Example 2.7.3. If $\ell \subset \mathbb{R}^3$ is a link with k components, then $M = \mathbb{R}^3/\ell$ is a manifold with a single conical singularity (Σ is the single point $[\ell] \in M$) of type $L = \bigsqcup_k (S^1 \times S^1)$.

2.7.2 Diffeomorphisms of singular manifolds

Now we also fix a subgroup $H \leqslant \mathrm{Diff}(L)$. Note that each self-diffeomorphism ψ of L determines a self-homeomorphism $\mathrm{cone}(\psi)$ of $\mathrm{cone}(L)$ via $\mathrm{cone}(\psi)([x,t]) = [\psi(x),t]$.

[7] We need to check that the third assumption stated just before Theorem 2.4.1 is satisfied for this special case (i.e. for this choice of N), but this is clear (*cf.* the last sentence of Remark 2.2.4).

Definition 2.7.4. Let M be a manifold with conical L-singularities. A *diffeomorphism* of M is then a homeomorphism $\varphi\colon M \to M$ such that $\varphi(\Sigma) = \Sigma$ and the restriction $\varphi|_{M\smallsetminus\Sigma}$ is a diffeomorphism. Moreover, for each $\sigma \in \Sigma$ we require that $\varphi(U_\sigma) = U_{\varphi(\sigma)}$ and that the induced homeomorphism $\mathrm{cone}(L) \cong U_\sigma \to U_{\varphi(\sigma)} \cong \mathrm{cone}(L)$ is of the form $\mathrm{cone}(h)$ for some $h \in H$. These form a subgroup

$$\mathrm{Diff}_H^L(M) \leqslant \mathrm{Homeo}(M).$$

2.7.3 Relation to manifolds with Baas-Sullivan singularities

Manifolds with discrete conical singularities are a special case ($s = 0$) of the more general notion of a *manifold with Baas-Sullivan singularities*, in which the singular set is a smooth s-dimensional manifold. Fix a smooth, closed manifold L of dimension $m - s - 1$, which will be called the *type*, or *link*, of the singular set.

Definition 2.7.5. A *manifold with Baas-Sullivan singularities* [Sul67; Baa73] consists of a topological space M equipped with the following data: a subset $\Sigma \subseteq M$, a structure of a smooth s-dimensional manifold (without boundary) on Σ and a structure of a smooth m-dimensional manifold (possibly with boundary) on $M \smallsetminus \Sigma$, an open neighbourhood $U \supseteq \Sigma$ in M and a homeomorphism

$$\theta\colon (U, \Sigma) \longrightarrow (\Sigma \times \mathrm{cone}(L), \Sigma \times \{\star\})$$

whose restriction to $U \smallsetminus \Sigma \longrightarrow \Sigma \times L \times (0, \infty)$ is a diffeomorphism.

Definition 2.7.6. Given a manifold $\mathbf{M} = (M, \Sigma, U, \theta)$ with Baas-Sullivan singularities of type L, a *diffeomorphism* of $\mathbf{M}$ is a homeomorphism $\varphi\colon M \to M$ fixing Σ and U setwise, such that the restrictions $\varphi|_\Sigma$ and $\varphi|_{M\smallsetminus\Sigma}$ are diffeomorphisms and $\varphi|_U = \theta^{-1} \circ (\varphi|_\Sigma \times \mathrm{cone}(\psi)) \circ \theta$ for some diffeomorphism $\psi\colon L \to L$. We may also fix a subgroup $H \leqslant \mathrm{Diff}(L)$ and require ψ to be an element of H, in which case this is an *H-diffeomorphism* of $\mathbf{M}$.

Remark 2.7.7. There is a another viewpoint on manifolds with Baas-Sullivan singularities, where, instead of a singular set $\Sigma \subseteq M$ equipped with a conical open neighbourhood, one instead has a smooth manifold M with boundary, equipped with a collar neighbourhood and an embedding $\Sigma \times L \hookrightarrow \partial M$ whose image is a union of components of ∂M. A morphism $\varphi\colon M \to M'$ between such objects is then defined to be a smooth map, compatible with the collar neighbourhoods, sending $\Sigma \times L$ to $\Sigma' \times L$, such that the restriction $\varphi|_{\Sigma\times L}$ is a product of smooth maps $\Sigma \to \Sigma'$ and $L \to L$. A diffeomorphism of M is an automorphism in this category. The definitions above are recovered by taking the quotient of M by the equivalence relation corresponding to the partition

$$\{\{\sigma\} \times L \mid \sigma \in \Sigma\} \cup \{\{x\} \mid x \in M \smallsetminus (\Sigma \times L)\}.$$

In particular, the conical neighbourhood of Σ is the image of the collar neighbourhood under this quotient. For more details, see [Bot92] or [Per15]. (Note: our definition of a diffeomorphism of a manifold with Baas-Sullivan singularities is a mild generalisation of that of [Per15, Definition 3.1], where the restriction $\varphi|_{\Sigma\times L}$ is required to be the product of a smooth map $\Sigma \to \Sigma'$ and the identity $L \to L$. In Definition 2.7.6 this corresponds to H-diffeomorphisms of $\mathbf{M}$ with $H = \{\mathrm{id}\} \leqslant \mathrm{Diff}(L)$.)

Remark 2.7.8. The definition of manifolds with Baas-Sullivan singularities may be generalised to allow a collection of smooth, closed manifolds L_k of dimension $m - s - 1$. In the case where $s = 0$ (corresponding to Definition 2.7.1), this amounts to saying that each open neighbourhood U_σ should be identified with $\mathrm{cone}(L_k)$ for some k. See [Bot92] or [Per13] for more details.

2.7.4 Relation to symmetric diffeomorphism groups

We now construct a specific sequence $\mathbf{M}_n$ of manifolds with n conical singularities of a fixed type.

Definition 2.7.9. As in §2.1, we fix an embedding $i\colon P \hookrightarrow \partial M \subseteq \hat{M}$, a metric on the normal bundle $\nu_i \to P$ of i and a tubular neighbourhood $T = D(\nu_i) \hookrightarrow \partial M \times (-\frac{1}{2}, \frac{1}{2}) \subseteq \hat{M}$. As described in §2.1, this induces an embedding

$$\Phi\colon nT = \{1, \dots, n\} \times T \longrightarrow \mathring{M}.$$

Recall that $\mathring{T}$ denotes the interior of T and $T' \subset T$ denotes the closed sub-disc-bundle of radius $\frac{1}{2}$.

Given these inputs, we construct $\mathbf{M}_n$, a manifold with n conical ∂T-singularities, as follows: starting with the manifold M, collapse the subset $\Phi(\{k\} \times T')$ to a point σ_k, for each $k \in \{1, 2, \ldots, n\}$. Then the singularity set is $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and each point σ_k is equipped with a conical neighbourhood $U_{\sigma_k} \cong \mathrm{cone}(\partial T)$ given by the image of $\Phi(\{k\} \times \mathring{T})$ under the collapse map $M \to \mathbf{M}_n$.

Remark 2.7.10. As a space, $\mathbf{M}_n$ is homeomorphic to the quotient of M_Φ obtained by collapsing each $\Phi(\{k\} \times \partial T) \subset \partial M_\Phi$ to a point — see Remark 2.2.2.

For a smooth fibre bundle $\xi \colon E \to B$ with structure group G, write $\mathrm{Diff}_{\mathrm{fib}}(E) \leqslant \mathrm{Diff}(E)$ for the subgroup of diffeomorphisms φ that respect the partition of E into fibres of ξ (it then follows that $\xi \circ \varphi = \bar{\varphi} \circ \xi$ for some diffeomorphism $\bar{\varphi}$ of B) and that act on fibres by elements of G. In particular we may apply this definition to the disc and sphere bundles (*cf.* §2.2)

$$\xi \colon T \longrightarrow P \qquad \text{and} \qquad \xi_\partial = \xi|_{\partial T} \colon \partial T \longrightarrow P,$$

whose structure groups are both $O(m-p)$. Since an element of $O(m-p)$ is determined by its action on $S^{m-p-1} = \partial D^{m-p}$, the restriction map $\mathrm{Diff}_{\mathrm{fib}}(T) \to \mathrm{Diff}_{\mathrm{fib}}(\partial T)$ is an isomorphism, and we will identify these groups via this isomorphism.

Lemma 2.7.11. *There is a natural isomorphism*

$$H_*(\mathrm{Diff}_H^{\partial T}(\mathbf{M}_n)) \;\cong\; H_*(\Sigma_H \mathrm{Diff}(M, nT))$$

for any subgroup $H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T) = \mathrm{Diff}_{\mathrm{fib}}(\partial T) \leqslant \mathrm{Diff}(\partial T)$.

Proof. This follows directly from the construction of $\mathbf{M}_n$, unravelling Definitions 2.2.1 and 2.7.4. ◻

Now Theorem 2.A and Lemma 2.7.11 immediately imply:

Corollary 2.7.12. (Corollary 2.B) *Suppose that* $p \leqslant \frac{1}{2}(m-3)$ *and the subgroup* $H \leqslant \mathrm{Diff}_{\mathrm{fib}}(T) \leqslant \mathrm{Diff}(\partial T)$ *has been chosen so that condition* (b) *of* §2.4 *holds. Then there are isomorphisms*

$$H_*(\mathrm{Diff}_H^{\partial T}(\mathbf{M}_n)) \;\cong\; H_*(\mathrm{Diff}_H^{\partial T}(\mathbf{M}_{n+1}))$$

for $* \leqslant \frac{n}{2} - 1$*, and for* $* \leqslant \frac{n}{2}$ *if we take field coefficients.*

2.8 Twisted homological stability

We will prove Theorem 2.C in §2.9 as a corollary of a twisted homological stability theorem for moduli spaces of disconnected submanifolds, which we prove in this section. This is a direct analogue of the main result of [Pal18b], which deals with configuration spaces of points, so we will not include all possible details in this section, since most of the constructions and proofs go through verbatim as in [Pal18b], with just minor changes of notation.

We return to the setup of §2.5, but without any labels for now. So we have a smooth, connected m-dimensional manifold M with non-empty boundary, a collar neighbourhood $\mathrm{col} \colon \partial M \times [0, \infty] \hookrightarrow M$ and an embedding $i \colon P \hookrightarrow \partial M$. Recall also that we have extended M by lengthening its collar:

$$\hat{M} = M \underset{\partial M \times [0,\infty]}{\cup} (\partial M \times [-1, \infty]),$$

and in Definition 2.5.4 we constructed “shifted” embeddings $i_r \colon P \hookrightarrow \hat{M}$ for any $r \in \mathbb{R}$, where $i_0 = i$.

Fix a subgroup $G \leqslant \mathrm{Diff}(P)$ and write $E = \mathrm{Emb}(P, \hat{M})$. Then in Definition 2.5.6 we constructed moduli spaces $C_{nP}(M; G) \subseteq \mathrm{Sp}^n(E/G)$ and $C_{\hat{n}P}(M; G) \subseteq \mathrm{Sp}^{n+1}(E/G)$ and a stabilisation map

$$s_n \colon C_{nP}(M; G) \longrightarrow C_{\hat{n}P}(M; G) \cong C_{(n+1)P}(M; G) \tag{2.15}$$

between them.

Remark 2.8.1. The identification on the right-hand side of (2.15) is given by identifying the interior $M(-1)$ of $\hat{M}$ (*cf.* Notation 2.5.1) with the interior $M(0)$ of M via a diffeomorphism supported on the collar neighbourhood $\mathrm{col}(\partial M \times [-1,\infty]) \subset \hat{M}$. It will be more convenient in this section to think of the target of the stabilisation map as $C_{(n+1)P}(M;G)$. There is a natural basepoint $\{[i_1],\ldots,[i_n]\}$ of $C_{nP}(M;G)$, and s_n is basepoint-preserving.

Definition 2.8.2. The category $\mathcal{B}_P(M)$ associated to these data has non-negative integers as objects, and a morphism $m \to n$ is given by a choice of $k \leqslant \min(m,n)$ and a path ℓ in $C_{kP}(M;G)$ with $\ell(0) \subseteq \{[i_1],\ldots,[i_m]\}$ and $\ell(1) \subseteq \{[i_1],\ldots,[i_n]\}$, up to endpoint-preserving homotopy. The identities are given by constant paths and composition is defined by concatenation of paths, as well as forgetting any "strand" of a path that does not match up with a "strand" of the other path, analogous to the composition of partially-defined functions. See (2) on page 151 of [Pal18b] for an illustration. The construction in §3.1 of [Pal18b] extends directly to this setting, and equips this category with an endofunctor

$$S\colon \mathcal{B}_P(M) \longrightarrow \mathcal{B}_P(M),$$

whose effect on objects is $n \mapsto n+1$, together with a natural transformation $\iota\colon \mathrm{id}_{\mathcal{B}_P(M)} \to S$. Any functor $T\colon \mathcal{B}_P(M) \to \mathsf{Ab}$ to the category of abelian groups (or any abelian category) may then be given a *degree* by defining $\deg(0) = -1$ and recursively

$$\deg(T) = \deg(\mathrm{coker}(T \to T \circ S)) + 1$$

for $T \neq 0$. The automorphism group of n in $\mathcal{B}_P(M)$ is $\pi_1(C_{nP}(M;G))$, so there are well-defined twisted homology groups $H_*(C_{nP}(M;G);T(n))$. The homomorphism $T(\iota_n)\colon T(n) \to T(n+1)$ is equivariant with respect to the map induced on π_1 by (2.15), so there are induced maps on twisted homology

$$H_*(C_{nP}(M;G);T(n)) \longrightarrow H_*(C_{(n+1)P}(M;G);T(n+1)). \tag{2.16}$$

Theorem 2.8.3. (Theorem 2.D) *If* $p \leqslant \frac{1}{2}(m-3)$, *$G$ is an open subgroup of* $\mathrm{Diff}(P)$ *and* $T\colon \mathcal{B}_P(M) \to \mathsf{Ab}$ *is a functor of degree* $d < \infty$, *then* (2.16) *is split-injective in all degrees, and an isomorphism for* $* \leqslant \frac{n-d}{2}$.

The proof of Theorem 2.D is a direct generalisation of §3 and §6 of [Pal18b], so we will just sketch the steps involved. Fix a functor $T\colon \mathcal{B}_P(M) \to \mathsf{Ab}$.

Definition 2.8.4. For $S \subseteq \{1,\ldots,n\}$ let $f_S\colon n \to n$ be the morphism of $\mathcal{B}_P(M)$ given by the constant path in $C_{(n-|S|)P}(M)$ at the point $\{[i_s] \mid s \in \{1,\ldots,n\} \smallsetminus S\}$. Then $T(f_S)$ is an endomorphism of $T(n)$ and we may define a subgroup

$$T_n^k = \mathrm{im}(T(f_{\{1,\ldots,n-k\}})) \cap \bigcap_{i=n-k+1}^{n} \ker(T(f_{\{i\}}))$$

of $T(n)$ for any $0 \leqslant k \leqslant n$. We write

$$C_{(n-k,k)P}(M;G) \longrightarrow C_{nP}(M;G) \tag{2.17}$$

for the covering space in which k copies of P are coloured red and the remaining $n-k$ copies are coloured green. Equivalently, this is the connected covering space of $C_{nP}(M;G)$ corresponding to the subgroup

$$\mathrm{end}_n^{-1}(\Sigma_{n-k} \times \Sigma_k) \subseteq \pi_1(C_{nP}(M;G)),$$

where $\mathrm{end}_n\colon \pi_1(C_{nP}(M;G)) \to \Sigma_n$ is the homomorphism that remembers just the permutation of $\{[i_1],\ldots,[i_n]\}$ induced by a path. We also write

$$\mathrm{red}_{(n-k,k)}\colon C_{(n-k,k)P}(M;G) \longrightarrow C_{kP}(M;G) \tag{2.18}$$

for the map that forgets all green parts of a configuration.

Proposition 3.5 and Lemma 6.4 of [Pal18b] generalise directly to the following.

Proposition 2.8.5. *Each T_n^k is invariant under the action of $\pi_1(C_{(n-k,k)P}(M;G))$ on $T(n)$, and therefore gives a twisted coefficient system for $C_{(n-k,k)P}(M;G)$. The pullback of the coefficient system T_k^k along the map (2.18) is naturally isomorphic to T_n^k. There is a natural isomorphism of $\mathbb{Z}[\pi_1(C_{nP}(M;G))]$-modules*

$$T(n) \;\cong\; \bigoplus_{k=0}^{n}\Bigl(\mathbb{Z}[\pi_1(C_{nP}(M;G))] \underset{\mathbb{Z}[\pi_1(C_{(n-k,k)P}(M;G))]}{\otimes} T_n^k\Bigr).$$

Lemma 3.16 of [Pal18b] also generalises directly:

Lemma 2.8.6. *We have $\deg(T) \leqslant d$ if and only if $T_n^k = 0$ for all $n \geqslant 0$ and all $k > d$.*

The final lemma that we will need before proving Theorem 2.D is the following.

Lemma 2.8.7. *The map (2.18) is a fibre bundle.*

Proof. By Proposition 4.15 of [Pal21], the action of $\mathrm{Diff}_c(\mathring{M})$ on $\mathrm{Emb}(kP,\mathring{M})/(G \wr \Sigma_k)$ is locally retractile. By Lemma 4.6(i) of [Pal21], the restriction of this action to the path-component of the identity $\mathrm{Diff}_c(\mathring{M})_0$ is also locally retractile. This restricted action obviously fixes setwise the path-component $C_{kP}(M;G) \subseteq \mathrm{Emb}(kP,\mathring{M})/(G \wr \Sigma_k)$, so by Lemma 4.6(iii) of [Pal21], the action of $\mathrm{Diff}_c(\mathring{M})_0$ on $C_{kP}(M;G)$ is locally retractile. Hence Theorem A of [Pal60] implies that (2.18) is a fibre bundle. ∎

Proof of Theorem 2.D. The idea is exactly the same as on pages 172–173 of [Pal18b], so we just give a sketch of how to adapt it. By Proposition 2.8.5, Lemma 2.8.6 and Shapiro's lemma for covering spaces (see Lemma 6.1 of [Pal18b]) there are natural isomorphisms

$$H_*(C_{nP}(M;G);T(n)) \;\cong\; \bigoplus_{k=0}^{d} H_*(C_{(n-k,k)P}(M;G);T_n^k). \tag{2.19}$$

It therefore suffices to show that the lift of the stabilisation map

$$C_{(n-k,k)P}(M;G) \longrightarrow C_{(n+1-k,k)P}(M;G) \tag{2.20}$$

that adds a new green copy of P to the configuration induces isomorphisms on twisted homology with respect to the coefficient systems T_n^k and T_{n+1}^k up to degree $\frac{n-k}{2}$. This stabilisation map is a map of fibre bundles (by Lemma 2.8.7) over the space $C_{kP}(M;G)$. Our default basepoint of this space is $\{[i_1],\ldots,[i_k]\}$, but for the next argument it will be more convenient to choose a different basepoint $\{[i'_1],\ldots,[i'_k]\}$, where each embedding $i'_\alpha \colon P \hookrightarrow M$ has image disjoint from the image of the collar neighbourhood. We may then define

$$M' = M \smallsetminus \bigcup_{\alpha=1}^{k} i'_\alpha(P)$$

and take the same collar neighbourhood for M' as for M. The restriction of the map (2.20) to the fibres over $\{[i'_1],\ldots,[i'_k]\}$ is the stabilisation map

$$C_{(n-k)P}(M';G) \longrightarrow C_{(n+1-k)P}(M';G).$$

There is a subtlety in this statement: it is not hard to see that there are topological embeddings

$$\begin{array}{ccc}
\mathrm{red}^{-1}_{(n-k,k)}(\{[i'_1],\ldots,[i'_k]\}) & \longrightarrow & \mathrm{red}^{-1}_{(n+1-k,k)}(\{[i'_1],\ldots,[i'_k]\}) \\
\uparrow & & \uparrow \\
C_{(n-k)P}(M';G) & \longrightarrow & C_{(n+1-k)P}(M';G)
\end{array}$$

making the square commute, defined by adjoining $\{[i'_1], \ldots, [i'_k]\}$ to a configuration. It remains to see that they are surjective: this follows from Proposition 5.10 of [Pal21]. Using this identification and the first part of Proposition 2.8.5, we therefore have a map of twisted Serre spectral sequences (*cf.* Proposition 5.7 of [Pal18b]), which is as follows on the E^2 pages:

$$H_s(C_{kP}(M;G); H_t(C_{(n-k)P}(M';G); T_k^k)) \longrightarrow H_s(C_{kP}(M;G); H_t(C_{(n+1-k)P}(M';G); T_k^k)),$$

where T_k^k is a *constant* coefficient system for the fibres, and which converges to the map on twisted homology induced by (2.20) with respect to the coefficient systems T_n^k and T_{n+1}^k. By Theorem A of [Pal21] and the universal coefficient theorem, the map of E^2 pages is an isomorphism for $t \leqslant \frac{n-k}{2}$. The Zeeman comparison theorem therefore implies that the map in the limit is also an isomorphism up to degree $\frac{n-k}{2}$. This completes the proof of Theorem 2.D, except for the split-injectivity statement.

The proof of split-injectivity in §7 of [Pal18b] generalises verbatim to establish the split-injectivity statement of Theorem 2.D. ◻

2.9 Stability for moduli spaces of labelled disconnected submanifolds

We now prove Theorem 2.C as a corollary of Theorem 2.D. This will be another spectral sequence comparison argument, using a map of Serre spectral sequences induced by the square (2.7), so as a first step we prove:

Lemma 2.9.1. *The vertical maps in the square* (2.7) *are Serre fibrations.*

Proof. We will show that the map $C_{nP}(M,Z;G) \to C_{nP}(M;G)$ is a Serre fibration; an identical argument will then show that the other vertical map of (2.7) is also a Serre fibration.

By assumption (see Input 2.5.5), the map $\pi\colon Z \to E$ is a Serre fibration and also G-equivariant. The n-fold product $\pi^n\colon Z^n \to E^n$ is also a Serre fibration, and so is its pullback $\pi_n\colon Z_n \to E_n$ along the inclusion $E_n = \mathrm{Emb}(nP, \mathring{M}) \subset E^n = \mathrm{Emb}(P, \hat{M})^n$. Since the map $\pi_n\colon Z_n \to E_n$ is also $(G \wr \Sigma_n)$-equivariant, there is an induced square

$$\begin{array}{ccc} Z_n & \longrightarrow & Z_n/(G \wr \Sigma_n) \\ {\scriptstyle \pi_n}\downarrow & \lrcorner & \downarrow{\scriptstyle \bar{\pi}_n} \\ E_n & \longrightarrow & E_n/(G \wr \Sigma_n), \end{array} \tag{2.21}$$

which is a pullback as indicated. By Propositions 4.15 and 4.8 of [Pal21], the bottom horizontal map is a principal $(G \wr \Sigma_n)$-bundle. So we know that the left-hand vertical map π_n and the bottom horizontal map in (2.21) are Serre fibrations, and the bottom horizontal map is also obviously surjective. Thus Lemma 4.19 of [Pal21] implies that the right-hand vertical map $\bar{\pi}_n$ is also a Serre fibration. Finally, the map $C_{nP}(M,Z;G) \to C_{nP}(M;G)$ is just the restriction of $\bar{\pi}_n$ to one path-component of its source and one path-component of its target, so it is also a Serre fibration. ◻

Let R be a ring. There is an induced map of Serre spectral sequences, converging to

$$H_*(C_{nP}(M,Z;G);R) \longrightarrow H_*(C_{\hat{n}P}(M,Z;G);R)$$

and whose map of E^2 pages is of the form

$$H_s(C_{nP}(M;G); H_t(f_n^{-1}(i_{\{1,\ldots,n\}});R)) \longrightarrow H_s(C_{\hat{n}P}(M;G); H_t(f_{\hat{n}}^{-1}(i_{\{0,\ldots,n\}});R)),$$

where f_n and $f_{\hat{n}}$ denote the vertical maps in the square (2.7) and where $i_{\{1,\ldots,n\}} = \{[i_1], \ldots, [i_n]\}$ and $i_{\{0,\ldots,n\}} = \{[i_0], [i_1], \ldots, [i_n]\}$ are the basepoints.

Remark 2.9.2. In this section (as in §2.8, see Remark 2.8.1) it will be more convenient to view the targets of the stabilisation maps in (2.7) as $C_{(n+1)P}(M,Z;G)$ and $C_{(n+1)P}(M;G)$ respectively, so the map of Serre spectral sequences is then

$$H_*(C_{nP}(M,Z;G);R) \longrightarrow H_*(C_{(n+1)P}(M,Z;G);R) \tag{2.22}$$

in the limit and

$$H_s(C_{nP}(M;G);H_t(f_n^{-1}(i_{\{1,\ldots,n\}});R)) \longrightarrow H_s(C_{(n+1)P}(M;G);H_t(f_{n+1}^{-1}(i_{\{1,\ldots,n+1\}});R)) \tag{2.23}$$

on the E^2 pages.

For these identifications, we use modifications of the maps $\gamma(1) = \mathrm{sh}_1 \circ - \colon E \to E$ and $\bar{\gamma}(1)\colon Z \to Z$ (*cf.* Definition 2.5.4). Namely, we choose a diffeomorphism $\kappa\colon \hat{M} \to M$ (note that there is no $\hat{}$ on the codomain) so that $\kappa = \mathrm{sh}_1$ on $M \subset \hat{M}$. This induces a G-equivariant endomorphism $\kappa \circ - \colon E \to E$, where we recall that $E = \mathrm{Emb}(P,\hat{M})$. We then choose a G-equivariant lift $\bar{\kappa}\colon Z \to Z$ of this so that $\bar{\kappa} = \bar{\gamma}(1)$ on $\pi^{-1}(\mathrm{Emb}(P,M)) \subset Z$. Then the identifications

$$C_{\hat{n}P}(M;G) \cong C_{(n+1)P}(M;G) \qquad\qquad C_{\hat{n}P}(M,Z;G) \cong C_{(n+1)P}(M,Z;G)$$

are defined by

$$\{[\varphi_0],\ldots,[\varphi_n]\} \longmapsto \{[\kappa\circ\varphi_0],\ldots,[\kappa\circ\varphi_n]\} \qquad\qquad \{[z_0],\ldots,[z_n]\} \longmapsto \{[\bar{\kappa}(z_0)],\ldots,[\bar{\kappa}(z_n)]\}$$

respectively.

Proposition 2.9.3. *Suppose that R is a principal ideal domain and that the fibration $\pi\colon Z \to E$ has path-connected fibres, whose homology is a flat R-module in each degree. Then for each $t \geqslant 0$ there is a functor $T_t\colon \mathcal{B}_P(M) \to \mathsf{Ab}$ of degree at most t such that, up to isomorphism, the map $T_t(\iota_n)\colon T_t(n) \to T_t(n+1)$ is the map*

$$H_t(f_n^{-1}(i_{\{1,\ldots,n\}});R) \longrightarrow H_t(f_{n+1}^{-1}(i_{\{1,\ldots,n+1\}});R) \tag{2.24}$$

induced by the restriction of the top horizontal map of (2.7).

Proof. As in [Pal18b], write Σ for the category with objects $\{1,\ldots,n\}$ for non-negative integers n (with $n=0$ corresponding to the empty set) and whose morphisms are partially-defined injections. This may be viewed as a special case of $\mathcal{B}_P(M)$: for example, $\mathcal{B}_{\mathrm{pt}}(\mathbb{R}^3) \cong \Sigma$.

The map $\pi\colon Z \to E$ is G-equivariant; write $\bar{\pi}\colon Z/G \to E/G$ for the induced map of orbit spaces. Let Y be the fibre $\bar{\pi}^{-1}([i_0])$ with basepoint $[\bar{\imath}_0]$. Example 4.1 of [Pal18b] gives us a functor

$$T_t\colon \Sigma \longrightarrow \mathsf{Ab}$$

such that $T_t(\iota_n)\colon T_t(n) \to T_t(n+1)$ is the map on H_t induced by the inclusion of pointed spaces $Y^n \hookrightarrow Y \times Y^n = Y^{n+1}$, in other words, the map $([\bar{\imath}_0],-,\ldots,-)$. By Lemma 4.2 and Remark 4.4 of [Pal18b] this functor has degree at most t. There is a functor $\mathcal{B}_P(M) \to \Sigma$ given by remembering just the partial injection $\{[i_1],\ldots,[i_m]\} \dashrightarrow \{[i_1],\ldots,[i_n]\}$ induced by a path ℓ of configurations as in Definition 2.8.2. Precomposition by this functor preserves the degree of functors into the category Ab,[8] so the composition

$$T_t\colon \mathcal{B}_P(M) \longrightarrow \Sigma \longrightarrow \mathsf{Ab}$$

also has degree at most t.

The map (2.24) is induced by the composition $f_n^{-1}(i_{\{1,\ldots,n\}}) \to f_{\hat{n}}^{-1}(i_{\{0,\ldots,n\}}) \to f_{n+1}^{-1}(i_{\{1,\ldots,n+1\}})$, which may equivalently be written

$$\prod_{\alpha=1}^{n} \bar{\pi}^{-1}([i_\alpha]) \longrightarrow \prod_{\alpha=0}^{n} \bar{\pi}^{-1}([i_\alpha]) \longrightarrow \prod_{\alpha=1}^{n+1} \bar{\pi}^{-1}([i_\alpha]),$$

[8] See §4.3 of [Pal17] for a more general discussion of when precomposition by a functor preserves degree.

where the first map is $([\bar{\iota}_0], -, \dots, -)$ and the second is a restriction of $\bar{\bar{\gamma}}(1)^{n+1}$ (*cf.* Remark 2.9.2), where $\bar{\bar{\gamma}}(r)$ denotes the map $Z/G \to Z/G$ induced by the G-equivariant map $\bar{\gamma}(r)\colon Z \to Z$. The domain may be identified with Y^n via the homeomorphism $\bar{\bar{\gamma}}(1) \times \cdots \times \bar{\bar{\gamma}}(n)$ and the codomain with Y^{n+1} via the homeomorphism $\bar{\bar{\gamma}}(1) \times \cdots \times \bar{\bar{\gamma}}(n+1)$. Under these identifications (using the fact that $\bar{\gamma}$ is a homomorphism), we see that (2.24) becomes the map induced on H_t by the inclusion $Y^n \hookrightarrow Y \times Y^n$, which is exactly $T_t(\iota_n)$, as required. ∎

Proof of Theorem 2.C. The argument in §5.2 of [Pal21] for the split-injectivity part of the statement generalises verbatim to the setting of moduli spaces of disconnected submanifolds with labels. All one needs, in order to apply Lemma 2 of [Dol62] to deduce split-injectivity, is to be able to define maps

$$C_{nP}(M, Z; G) \longrightarrow \mathrm{Sp}^{\binom{n}{k}}(C_{kP}(M, Z; G))$$

satisfying a certain equation up to homotopy. Viewing $C_{nP}(M, Z; G)$ as a subspace of the symmetric power $\mathrm{Sp}^n(Z/G)$ (*cf.* Definition 2.5.6), we construct such maps as restrictions of the maps

$$\mathrm{Sp}^n(Z/G) \longrightarrow \mathrm{Sp}^{\binom{n}{k}}(\mathrm{Sp}^k(Z/G))$$

that forget $n - k$ points in all possible ways. Thus (2.22) is always split-injective.

It remains to prove the second part of the statement, that when $\pi\colon Z \to E$ has path-connected fibres, $p \leqslant \frac{1}{2}(m-3)$ and G is an open subgroup of $\mathrm{Diff}(P)$, the map (2.22) is an isomorphism for $* \leqslant \frac{n}{2} - 1$, and also for $* \leqslant \frac{n}{2}$ if R is a field.

First let R be a field, so that every R-module is flat. Then these three assumptions, together with Theorem 2.D and Proposition 2.9.3, imply that the map (2.23) is an isomorphism for $s \leqslant \frac{n-t}{2}$, in particular for total degree $s + t \leqslant \frac{n}{2}$. The Zeeman comparison theorem then implies that (2.22) is also an isomorphism for $* \leqslant \frac{n}{2}$.

In general, if a continuous map $X \to Y$ induces isomorphisms on homology up to degree i with all field coefficients, then it induces isomorphisms on integral homology (and therefore with any untwisted coefficients, by the universal coefficient theorem) up to degree $i - 1$. This follows from the five-lemma applied to the natural long exact sequences induced by the short exact sequences

$$0 \to \mathbb{Z}/(p^r) \to \mathbb{Z}/(p^{r+1}) \to \mathbb{Z}/(p) \to 0 \qquad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \operatorname*{colim}_{r \to \infty} \mathbb{Z}/(p^r) \to 0$$

of coefficient groups. Thus the statement in the special case when R is a field implies the statement for general R. ∎

Chapter 3

Big mapping class groups with uncountable integral homology

The results of this chapter are accepted for publication as [PW22a] in joint work with Xiaolei Wu.

Introduction

There has been a recent wave of interest in *big mapping class groups* (mapping class groups of infinite-type surfaces); see [AV20] for a survey. In [PW22b], the authors recently computed the homology of a large family of big mapping class groups, namely the families of (1-*holed* or *punctured*) *binary tree surfaces* (see the introduction of [PW22b] for this terminology). Precisely, the mapping class group of every 1-holed binary tree surface is *acyclic* and the homology of the mapping class group of every punctured binary tree surface is periodic with $\mathbb{Z}$ in every even degree and zero in every odd degree. One instance of this result says that the mapping class group $\mathrm{Map}(\mathbb{D}^2 \smallsetminus \mathcal{C})$ is acyclic and that $H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C}))$ is $\mathbb{Z}$ for i even and zero for i odd, where $\mathcal{C}$ is a Cantor set embedded in the interior of the disc. In particular, in all of these examples, the homology groups $H_i(\mathrm{Map}(S))$ are *finitely generated* for each i. Some earlier results on the homology of big mapping class groups – in degrees 1 and 2 – include: $H_1(\mathrm{Map}(S \smallsetminus \mathcal{C})) \cong H_1(\mathrm{Map}(S))$ if $\mathcal{C}$ is a Cantor set embedded in the interior of a finite-type surface S [CC22] (see also [Vla21] for three special cases of this) and $H_2(\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}/2$ [CC21].

In this chapter we prove a contrasting result: for many infinite-type surfaces S, the group $H_i(\mathrm{Map}(S))$ is *uncountable* for all $i > 0$. In addition, we prove – for *all* infinite-type surfaces S – that $H_i(\overline{\mathrm{PMap}_c(S)})$ and $H_i(\mathcal{T}(S))$ are uncountable for all $i > 0$, where $\overline{\mathrm{PMap}_c(S)}$ and $\mathcal{T}(S)$ denote, respectively, the closure of the compactly-supported mapping class group and the Torelli group of S.

Our proofs are built on ideas from [APV20; Dom22; MT23]. In [APV20], Aramayona, Patel and Vlamis determined $H^1(\mathrm{PMap}(S))$ for any infinite-type surface S of genus at least 2; in particular, they showed that it is countable. (This was extended to genus 1 in [DP20], where it was also shown that $H^1(\mathrm{PMap}(S))$ is uncountable when S has genus 0.) Along the way they proved that, when S has infinitely many non-planar ends, its pure mapping class group $\mathrm{PMap}(S)$ admits a split-surjection onto the *Baer-Specker group* $\mathbb{Z}^{\mathbb{N}}$. Later, Domat proved that big pure mapping class groups $\mathrm{PMap}(S)$ are never perfect [Dom22]. Morover, he showed that $H_1(\mathrm{PMap}(S))$ is uncountable for many infinite-type surfaces S and that $H_1(\mathcal{T}(S))$ and $H_1(\overline{\mathrm{PMap}_c(S)})$ are uncountable for all infinite-type surfaces S. Malestein and Tao [MT23] were able to push the results of Domat further and prove that the first homology of the full mapping class group $H_1(\mathrm{Map}(S))$ is uncountable for a certain class of surfaces S, including $S = \mathbb{R}^2 \smallsetminus \mathbb{Z}$.

Uncountable homology

Given a surface S, recall that its pure mapping class group $\mathrm{PMap}(S)$ is the subgroup of its mapping class group $\mathrm{Map}(S) = \pi_0(\mathrm{Homeo}(S))$ consisting of all those mapping classes that fix the ends of S pointwise. Its Torelli group $\mathcal{T}(S)$ is the kernel of the natural homomorphism $\mathrm{Map}(S) \to \mathrm{Aut}(H_1(S))$. Recall also that $\mathrm{PMap}_c(S)$ denotes the subgroup of $\mathrm{Map}(S)$ of mapping classes that admit representative homeomorphisms with compact support, and $\overline{\mathrm{PMap}_c(S)}$ denotes its closure in $\mathrm{Map}(S)$ in the quotient topology induced by the compact-open topology on $\mathrm{Homeo}(S)$. We note that in general we have inclusions $\mathcal{T}(S) \subseteq \overline{\mathrm{PMap}_c(S)} \subseteq \mathrm{PMap}(S) \subseteq \mathrm{Map}(S)$. (The only non-obvious inclusion is the first one: it is explained during the proof of Theorem 3.4.10 below.) Our first result concerns the first two groups of this nested sequence and holds for *all* infinite-type surfaces S.

Theorem 3.A (Corollary 3.4.5 and Theorem 3.4.10)**.** *Let S be any infinite-type surface. Then the integral homology groups*

$$H_i(\overline{\mathrm{PMap}_c(S)}) \quad \textit{and} \quad H_i(\mathcal{T}(S))$$

are uncountable for every $i \geqslant 1$. Moreover, they each contain $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every degree, where $\mathfrak{c}$ denotes the cardinality of the continuum.

Remark 3.0.1. One might hope that our methods could be used to prove that the homology of the pure mapping class group $H_i(\mathrm{PMap}(S))$ is also uncountable for every $i \geqslant 1$ and for any infinite-type surface S. However, the methods of the present chapter can only prove this result in the case when S has at most one or infinitely many non-planar ends; see Remark 3.4.7 for more information. When S has n non-planar ends for $1 < n < \infty$, one can in fact prove that the (uncountably many) elements constructed in Domat's paper [Dom22, Theorem 6.1] all vanish in $H_1(\mathrm{PMap}(S))$; see Remark 3.4.9 for more information.

In order to state our result for the full mapping class groups $\mathrm{Map}(S)$, we first recall some background about ends of surfaces; more details are given in §3.1 and §3.2. Every surface S has a space of ends E, which is a compact, separable, totally disconnected topological space. The key hypothesis in our main theorem is a condition on the structure of the space E.

Definition 3.0.2. For points $x, y \in E$, we write $x \sim y$ and say that x is *similar* to y if and only if there are open neighbourhoods U, V of x, y respectively such that (U, x) and (V, y) are homeomorphic as based spaces. A point $x \in E$ is *topologically distinguished* if it is not equivalent to any other point of E under this equivalence relation.

Definition 3.0.3. For a topological space E, write $\Upsilon^+(E) = E\omega + 1$, where $E\omega$ means a countably infinite disjoint union of copies of E and $X + 1$ means the one-point compactification of X.

Theorem 3.B. *Let S be a connected, finite-genus surface with finitely many boundary components, whose space of ends E is of the form $E = E_1 \sqcup \Upsilon^+(E_2)$, where E_2 has a topologically distinguished point x and no point of E_1 is similar to x. Then the integral homology group*

$$H_i(\mathrm{Map}(S))$$

is uncountable for every $i \geqslant 1$. In fact, there is an injective homomorphism of graded abelian groups

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*(\mathrm{Map}(S)),$$

where Λ^* denotes the exterior algebra on an abelian group.

Remark 3.0.4. In the course of the proof of Theorem 3.B, we also prove the same statement with S replaced by the Loch Ness monster surface L, see Proposition 3.4.3.

Remark 3.0.5. All *countable* end spaces of surfaces (equivalently: countable compact Hausdorff spaces) are of the form $E = \omega^\alpha.n + 1$ for a countable ordinal α and a positive integer n [MS20]. Hence a surface S of finite genus with this end space satisfies the assumption of Theorem 3.B whenever $n = 1$ and α is a successor ordinal.

Thus for a large class of infinite-type surfaces S with countably many ends we know that $\mathrm{Map}(S)$ has uncountable integral homology in all positive degrees. This suggests the following question.

Question 3.0.6. Let S be an infinite-type surface with countably many ends. Is the homology of $\mathrm{Map}(S)$ uncountable in all positive degrees?

Remark 3.0.7. Without the hypothesis on the structure of the space of ends E of S, the conclusion of Theorem 3.B is false. For example, as mentioned above, we prove in [PW22b] that

$$H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C})) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Remark 3.0.8. The hypotheses of this chapter and the hypotheses of [PW22b] are in some sense opposite, with opposite conclusions. In [PW22b] we consider 1-holed binary tree surfaces, whose end spaces are *Cantor compactifications* $(E\omega)^{\mathcal{C}}$ (see [PW22b, §1.2] for the definition), which are highly self-similar (in particular $(E\omega)^{\mathcal{C}} \cong \mathcal{C}$ if $E = \varnothing$ or $E = \mathcal{C}$, which is homogeneous), and we prove that $H_i(\mathrm{Map}(S)) = 0$ for all $i > 0$. On the other hand, in this chapter we consider surfaces S whose end spaces E satisfy the "homogeneity breaking" hypothesis of Theorem 3.B (roughly: E has a limit point of topologically distinguished points), and conclude that $H_i(\mathrm{Map}(S))$ is uncountable for all $i > 0$.

Non-trivial torsion

So far, the elements that we have constructed in the homology of big mapping class groups all have infinite order. It would be interesting also to find some torsion elements. In fact, the following question was asked by Domat in [Dom22, Question 11.3].

Question 3.0.9. Let S be an infinite-type surface. Are there torsion elements in $H_1(\overline{\mathrm{PMap}_c(S)})$?

Recall that $\mathrm{PMap}_c(S)$ denotes the subgroup of $\mathrm{Map}(S)$ of mapping classes that admit representative homeomorphisms with compact support, and $\overline{\mathrm{PMap}_c(S)}$ denotes its closure in $\mathrm{Map}(S)$ in the quotient topology induced by the compact-open topology on $\mathrm{Homeo}(S)$. Also recall that $\overline{\mathrm{PMap}_c(S)} \subseteq \mathrm{PMap}(S)$ coincides with $\mathrm{PMap}(S)$ if and only if S has at most one non-planar end [PV18, Theorem 4]. Our third result answers Domat's question in the positive.

Theorem 3.C. *Let S be an infinite-type surface of genus 2 and with finitely many (possibly zero) boundary components. Then the homology groups $H_1(\mathrm{PMap}(S)) = H_1(\overline{\mathrm{PMap}_c(S)})$ and $H_1(\mathrm{Map}(S))$ both contain an order-10 element. Moreover, the cyclic group generated by this element is a direct summand.*

Remark 3.0.10. By comparing the stable homology of (orientable, finite-type) mapping class groups with rational coefficients [MW07] and with mod-p coefficients [Gal04], one sees that there are also many torsion elements in the integral homology of mapping class groups in the stable range. Using this and Lemma 3.6.2, one may find many higher-degree torsion elements in the homology of mapping class groups of infinite-type surfaces of finite genus.

In a sense, our answer to Domat's question is "cheating", since we simply show that a certain torsion element in the homology of the mapping class group of a finite-type subsurface of S injects into the homology of the mapping class group of S. Together with our uncountability results above (Theorems 3.A and 3.B), this suggests two refinements of Domat's question:

Question 3.0.11. Let S be an infinite-type surface. Do the homology groups $H_1(\overline{\mathrm{PMap}_c(S)})$ or $H_1(\mathrm{PMap}(S))$ contain torsion elements that are not supported on any finite-type subsurface of S?

Question 3.0.12. Let S be an infinite-type surface. Do the homology groups $H_1(\overline{\mathrm{PMap}_c(S)})$ or $H_1(\mathrm{PMap}(S))$ contain an uncountable torsion subgroup?

We note that a positive answer to Question 3.0.12 would imply a positive answer to Question 3.0.11, since torsion admitting finite-type support can only account for countably many torsion elements.

Outline

We begin with two sections of background: §3.1 on infinite-type surfaces and big mapping class groups and §3.2 on notions of *topologically distinguished points*. In §3.3 we prove a basic lemma that gives a sufficient criterion for the homology of a group to contain an embedded copy of the exterior algebra on a direct sum of continuum many copies of $\mathbb{Z}$. We also discuss techniques of [Dom22] that may be used to construct the inputs for this lemma.

Theorems 3.A and 3.B are then proven in §3.4–§3.5. In §3.4 we prove uncountability of the homology of the mapping class group of the Loch Ness monster surface, which is the first step in the proof of Theorem 3.B. We then adapt these techniques to prove Theorem 3.A on the homology of the closure of the compactly-supported mapping class group and the Torelli group of an arbitrary infinite-type surface S. In §3.5 we apply the results of §3.4, together with a covering space argument inspired by a technique of Malestein and Tao [MT23], to complete the proof of Theorem 3.B. The covering space argument in this section is the step in which we use in an essential way the hypothesis on the structure of the end space of the surface.

We prove Theorem 3.C on torsion elements in §3.6. Finally, in §3.7, we record some related open questions, in particular discussing the *co*homology of mapping class groups in §3.7.2. Appendix 3.8 gathers some basic facts about abelian groups that are needed in several of our proofs.

3.1 Surfaces, ends and mapping class groups

3.1.1 Infinite-type surfaces

All surfaces will be assumed to be second countable, connected, orientable and to have compact boundary. If the fundamental group of S is finitely generated, we say that S has *finite type*, otherwise it has *infinite type*. The classification of surfaces of possibly infinite type was proven by von Kerékjártó [Ker23] and Richards [Ric63]. Recall that an *end* of a surface S is an element of the set

$$\mathrm{Ends}(S) = \varprojlim \pi_0(S \setminus K), \tag{3.1}$$

where the inverse limit is taken over all compact subsets $K \subset S$. The *Freudenthal compactification* of S is the union

$$\overline{S} = S \sqcup \mathrm{Ends}(S)$$

equipped with the topology generated by $U \sqcup \{e \in \mathrm{Ends}(S) \mid e < U\}$ for all open subsets $U \subseteq S$. Here $e < U$ means that there is a compact subset $K \subset S$ such that U contains the component of $S \setminus K$ hit by e under the canonical map $\mathrm{Ends}(S) \to \pi_0(S \setminus K)$. The induced subspace topology on $\mathrm{Ends}(S)$ coincides with the limit topology induced from the discrete topology on each term in the inverse system. With this topology, $\mathrm{Ends}(S)$ is homeomorphic to a closed subset of the Cantor set. We call an end $e \in \mathrm{Ends}(S)$ *planar* if it has a neighbourhood (in the topology of $\overline{S}$) that embeds into the plane, otherwise we call it *non-planar*. The (closed) subspace of non-planar ends is denoted by $\mathrm{Ends}_{np}(S) \subseteq \mathrm{Ends}(S)$.

Theorem 3.1.1 ([Ric63, Theorems 1 and 2]). *Let S_1, S_2 be two surfaces of genus $g_1, g_2 \in \mathbb{N} \cup \{\infty\}$ and with $b_1, b_2 \in \mathbb{N}$ boundary components. Then $S_1 \cong S_2$ if and only if $g_1 = g_2$, $b_1 = b_2$ and there is a homeomorphism of pairs of spaces*

$$(\mathrm{Ends}(S_1), \mathrm{Ends}_{np}(S_1)) \;\cong\; (\mathrm{Ends}(S_2), \mathrm{Ends}_{np}(S_2)).$$

Conversely, given $g \in \mathbb{N} \cup \{\infty\}$, $b \in \mathbb{N}$ and a pair $X \subseteq Y$ of closed subsets of the Cantor set, where we require that $g = \infty$ if and only if $X \neq \varnothing$, there exists a surface S of genus g with b boundary components such that $(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S)) \cong (Y, X)$.

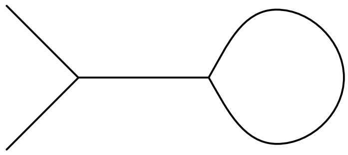


Figure 3.1 The 3-valent vertices of this graph are globally topologically distinguished but not topologically distinguished, since they are similar (but not globally similar) to each other.

3.1.2 Mapping class groups

For a surface S, the *mapping class group* of S is the group of isotopy classes of orientation-preserving diffeomorphisms of S fixing the boundary of S pointwise, i.e.

$$\mathrm{Map}(S) := \pi_0(\mathrm{Diff}^+(S, \partial S)).$$

The *pure mapping class group* $\mathrm{PMap}(S)$ of S is the subgroup of $\mathrm{Map}(S)$ consisting of all elements whose induced action on $\mathrm{Ends}(S)$ is the identity. It follows from the construction of [Ric63, Theorem 2] (or, more precisely, from the *naturality* of this construction) that every homeomorphism of $\mathrm{Ends}(S)$ sending the subspace $\mathrm{Ends}_{np}(S)$ onto itself is induced by some homeomorphism of S. This implies that we have the following short exact sequence.

Proposition 3.1.2. *Let S be any surface. Then there is a short exact sequence of groups*

$$1 \to \mathrm{PMap}(S) \longrightarrow \mathrm{Map}(S) \longrightarrow \mathrm{Homeo}(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S)) \to 1,$$

where $\mathrm{Homeo}(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S))$ *denotes the group of homeomorphisms of the pair of spaces* $(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S))$.

3.2 Topologically distinguished points

We now recall from the introduction the notion of *topologically distinguished points* (Definition 3.0.2) and compare it to a weaker notion of *globally topologically distinguished points*.

Definition 3.2.1. Let E be a topological space. Two points $x, y \in E$ are called *similar* if there are open neighbourhoods U and V of x and y respectively and a homeomorphism $U \cong V$ taking x to y. This is an equivalence relation on E. A point $x \in E$ is called *topologically distinguished* if its equivalence class under this relation is $\{x\}$, in other words it is similar only to itself.

Definition 3.2.2. Let E be a topological space. Two points $x, y \in E$ are called *globally similar* if there is a homeomorphism $\varphi \in \mathrm{Homeo}(E)$ with $\varphi(x) = y$. This is an equivalence relation on E. A point $x \in E$ is called *globally topologically distinguished* if its equivalence class under this relation is $\{x\}$, in other words it is globally similar only to itself. Equivalently, $x \in E$ is globally topologically distinguished if it is a fixed point of the action of $\mathrm{Homeo}(E)$ on E.

Remark 3.2.3. We record two immediate observations:

- If x and y are globally similar then they are similar.
- If x is topologically distinguished then it is globally topologically distinguished.

The converses of these two statements are false in general. For example, the two vertices of valence 3 in the graph pictured in Figure 3.1 are similar but not globally similar; also, both of them are globally topologically distinguished but not topologically distinguished. However, for zero-dimensional (Hausdorff) spaces the converse does hold:

Lemma 3.2.4. *Suppose that E is Hausdorff and zero-dimensional, i.e. it has a basis for its topology consisting of clopen subsets. Then two points $x, y \in E$ are similar if and only if they are globally similar. Thus $x \in E$ is topologically distinguished if and only if it is globally topologically distinguished.*

Proof. The second statement follows from the first one, so we only have to prove the first statement, that $x, y \in E$ are similar if and only if they are globally similar. One implication is obvious; we will prove the opposite implication. So let us assume that $x, y \in E$ are similar and choose open neighbourhoods U and V of x and y respectively and a homeomorphism $\varphi\colon U \to V$ taking x to y. Assume that $x \neq y$ (otherwise the result is obvious). Since E is zero-dimensional, we may assume, by shrinking them if necessary, that U and V are clopen. Since E is Hausdorff, we may assume, by shrinking them if necessary, that U and V are disjoint. We may therefore extend φ to a homeomorphism $\bar{\varphi} \in \mathrm{Homeo}(E)$ by:

- $\bar{\varphi}(e) = \varphi(e)$ for $e \in U$;
- $\bar{\varphi}(e) = \varphi^{-1}(e)$ for $e \in V$;
- $\bar{\varphi}(e) = e$ for $e \in E \smallsetminus (U \sqcup V)$.

This bijection is continuous since $\{U, V, E \smallsetminus (U \sqcup V)\}$ is an open cover of E and $\bar{\varphi}$ is continuous when restricted to each of these subsets. Its inverse is continuous for the same reason, so it is a homeomorphism of E taking x to y. Thus x and y are globally similar. □

Remark 3.2.5. End spaces of surfaces are always Hausdorff and zero-dimensional, so Lemma 3.2.4 implies that *topologically distinguished* and *globally topologically distinguished* are the same for end spaces.

Lemma 3.2.6. *If a space E has a topologically distinguished point, then $E\omega + 1$ has a globally topologically distinguished point. In fact, the point at infinity is globally topologically distinguished.*

Proof. Let ∞ denote the point at infinity of the one-point compactification $E\omega + 1$ of $E\omega = \bigsqcup_\omega E$. Let $\varphi \in \mathrm{Homeo}(E\omega + 1)$. We just need to show that $\varphi(\infty) = \infty$, since it will then follow that ∞ is a globally topologically distinguished point of $E\omega + 1$. Suppose for a contradiction that $\varphi(\infty) \neq \infty$. Write $E_i = E$ for each $i \in \mathbb{N}$, and identify $E\omega = \bigsqcup_{i \in \mathbb{N}} E_i$. By assumption, $\varphi(\infty) \in E_j$ for some $j \in \mathbb{N}$. Let $x \in E$ be a topologically distinguished point. Every open neighbourhood U of $\infty \in E\omega + 1$ contains infinitely many points that are similar to x, since, by definition of the one-point compactification, U must contain E_i for infinitely many i. Since φ is a homeomorphism, it must also be true that every open neighbourhood of $\varphi(\infty) \in E\omega + 1$ contains infinitely many points that are similar to x. But E_j is an open neighbourhood of $\varphi(\infty) \in E\omega + 1$ and it contains only one point that is similar to x, a contradiction. □

Corollary 3.2.7. *Suppose that E is Hausdorff and zero-dimensional. If E has a topologically distinguished point, then the point at infinity of $E\omega + 1$ is topologically distinguished.*

Proof. By Lemma 3.2.6, the point at infinity of $E\omega+1$ is globally topologically distinguished. Hausdorffness and zero-dimensionality of E automatically imply Hausdorffness and zero-dimensionality of $E\omega + 1$, so Lemma 3.2.4 then implies that the point at infinity of $E\omega + 1$ is topologically distinguished. □

Remark 3.2.8. There is another, a priori different, equivalence relation on topological spaces, defined by [MR22]. They define, for points $x, y \in E$:

$$x \leqslant y \iff \forall \text{ open neighbourhoods } U \ni y,\ \exists z \in U : z \sim x,$$

where $z \sim x$ means that z and x are similar in the sense of Definition 3.2.1. This is a pre-order on E, so it induces an equivalence relation

$$x \approx y \iff x \leqslant y \text{ and } y \leqslant x$$

on E and a poset structure on the quotient $E/{\approx}$. Clearly $x \sim y$ implies $x \approx y$. Also, if we now assume that E is the end space of a surface Σ, it is not hard to see (using Lemma 3.2.4 and Proposition 3.1.2) that $x \sim y$ if and only if there is a homeomorphism of Σ taking x to y. Theorem 1.2 of [MR22] says that if $x \approx y$ then there is a homeomorphism of Σ taking x to y. It follows that $\sim$ and $\approx$ are the same equivalence relation on E if it is the end space of a surface. In [MR23], the authors often consider the condition that "Σ has a unique maximal end", i.e. there is a unique maximal equivalence class $[x] \in E/{\approx}$ and the equivalence class $[x]$ has size

1. The condition that we require in this chapter is however much weaker, namely that "Σ has a topologically distinguished end", i.e. there is an equivalence class $[x] \in E/\approx$ of size 1 (but it need not be maximal in the poset structure of $E/\approx$).

3.3 Tools for proving uncountability

We start with a key lemma, which we use several times to conclude uncountability of the homology of a given group G in all positive degrees.

Notation 3.3.1. Let us fix some notation that will be used throughout the rest of the chapter.

- For an abelian group A, denote by $\Lambda^*(A)$ the exterior algebra on A.
- We denote by $\mathfrak{c}$ the cardinality of the continuum.

Lemma 3.3.2. *Let G be a group, denote by $\alpha\colon G \twoheadrightarrow G^{ab} = H_1(G)$ the quotient onto its abelianisation and let $\iota\colon \bigoplus_{\mathfrak{c}} \mathbb{Z} \to G$ be a homomorphism. Suppose that there is an embedding $f\colon \bigoplus_{\mathfrak{c}} \mathbb{Q} \hookrightarrow H_1(G)$ such that the diagram*

$$\begin{array}{ccc} \bigoplus_{\mathfrak{c}} \mathbb{Z} & \xrightarrow{\iota} & G \\ \downarrow & & \downarrow \alpha \\ \bigoplus_{\mathfrak{c}} \mathbb{Q} & \xhookrightarrow{f} & H_1(G), \end{array} \tag{3.2}$$

commutes, where $\bigoplus_{\mathfrak{c}} \mathbb{Z} \hookrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q}$ is the canonical inclusion. Then there is an injective homomorphism of graded abelian groups

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(G).$$

In particular, $H_i(G)$ is uncountable for all $i \geqslant 1$.

Proof. By Lemma 3.8.1, the embedding f admits a retraction. Hence the canonical inclusion

$$\bigoplus_{\mathfrak{c}} \mathbb{Z} \hookrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q} \tag{3.3}$$

factors through G. It follows that the induced homomorphism of graded abelian groups

$$H_*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}\Big) \tag{3.4}$$

factors through $H_*(G)$. The integral homology of a torsion-free abelian group A is naturally isomorphic to the exterior algebra $\Lambda^*(A)$ (see [Bro82, Theorem V.6.4(ii)]), so we have homomorphisms of graded abelian groups

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \longrightarrow H_*(G) \longrightarrow \Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}\Big), \tag{3.5}$$

whose composition is injective by Lemma 3.8.3. In particular the first map must be injective. ◻

In order to apply Lemma 3.3.2, we will need to be able to construct embeddings of direct sums of copies of $\mathbb{Q}$ into the first homology of big mapping class groups. The key topological input for this is a theorem of Domat, which we recall below and whose proof uses the machinery of Bestvina, Bromberg and Fujiwara [BBF15]. We first make some definitions that are implicit in the statement of [Dom22, Theorem 6.1].

Definition 3.3.3. Let S be a connected surface with at least two ends. Let us call a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ of isotopy classes of simple closed curves on S an *escaping sequence* if:

- each γ_i is end-separating, i.e., cutting along it disconnects S into two non-compact surfaces;

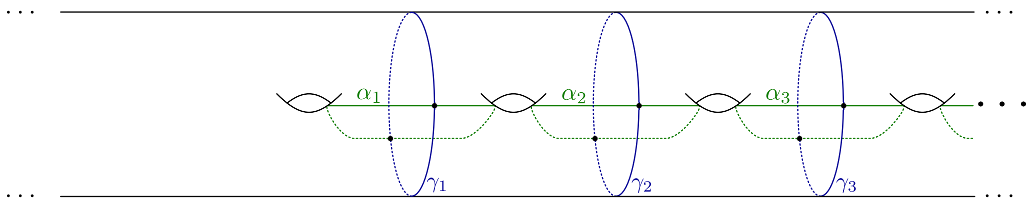


Figure 3.2 The once-punctured Loch Ness monster surface equipped with a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ of simple closed curves that is a *well-spaced*, *escaping* sequence in the sense of Definition 3.3.3. The fact that it is well-spaced is witnessed by the associated sequence of simple closed curves $\{\gamma_i'\}_{i\in\mathbb{N}}$ given by $\gamma_i' = T_{\alpha_i}(\gamma_i)$.

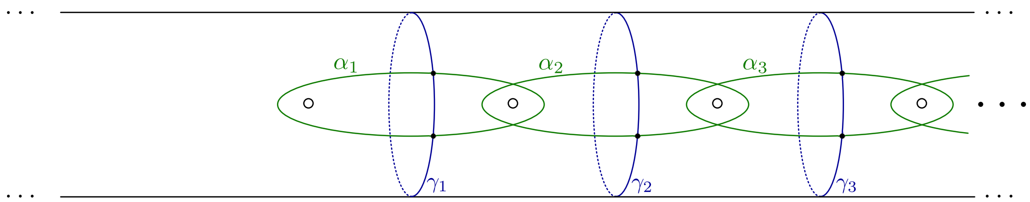


Figure 3.3 The flute surface equipped with a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ of simple closed curves that is an *escaping* sequence in the sense of Definition 3.3.3. After passing to the subsequence $\{\gamma_{2i}\}_{i\in\mathbb{N}}$, it becomes well-spaced, as explained in Example 3.3.6.

- γ_i and γ_j have pairwise-disjoint representatives for $i \neq j$;
- the sequence $\gamma_1, \gamma_2, \ldots$ eventually leaves every compact subset of S, i.e., if $K \subset S$ is a compact subset then only finitely many γ_i may be isotoped to lie in K.

An escaping sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ is *well-spaced* if there exists another escaping sequence $\{\gamma_i'\}_{i\in\mathbb{N}}$ such that:

- γ_i' is not isotopic to γ_i;
- γ_i' and γ_j have pairwise-disjoint representatives for $i \neq j$;
- there is a (necessarily non-trivial) element $g_i \in \mathrm{PMap}_c(S)$ taking γ_i to γ_i'.

Remark 3.3.4. It follows from the classification of surfaces that an escaping sequence exists on S if and only if S has infinite type. In addition, any escaping sequence becomes well-spaced after passing to an appropriate subsequence.

Example 3.3.5. In the key example of $S = L'$ the once-punctured Loch Ness monster surface, we may for example take $\{\gamma_i\}_{i\in\mathbb{N}}$ to be the sequence of curves pictured in Figure 3.2. Each γ_i is clearly end-separating, they are pairwise disjoint and no compact subset of L' contains more than finitely many of them, so this sequence is *escaping*. Moreover, taking $\gamma_i' = T_{\alpha_i}(\gamma_i)$ using the curves α_i also pictured in Figure 3.2, we obtain another escaping sequence $\{\gamma_i'\}_{i\in\mathbb{N}}$ witnessing that $\{\gamma_i\}_{i\in\mathbb{N}}$ is *well-spaced*.

Example 3.3.6. As another example, we may consider the *flute surface* depicted in Figure 3.3, together with the curves γ_i illustrated. These form an escaping sequence $\{\gamma_i\}_{i\in\mathbb{N}}$, but this is *not* a *well-spaced* escaping sequence: for example, one may attempt to construct another escaping sequence witnessing that it is well-spaced by setting $\gamma_i' = T_{\alpha_i}(\gamma_i)$ using the curves α_i illustrated, but then γ_i' intersects γ_{i+1}', so $\{\gamma_i'\}_{i\in\mathbb{N}}$ is not an escaping sequence as in Definition 3.3.3. However, the subsequence $\{\gamma_{2i}\}_{i\in\mathbb{N}}$ *is* well-spaced, as witnessed by the subsequence $\{\gamma_{2i}'\}_{i\in\mathbb{N}}$.

Theorem 3.3.7 ([Dom22, Theorem 6.1])**.** *Let S be an infinite-type surface with at least two ends and let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a well-spaced escaping sequence of simple closed curves on S. Let $a_1, a_2, \ldots$ be an unbounded sequence of positive integers. Then*

$$\prod_{i=1}^{\infty} (T_{\gamma_i})^{a_i} \in \overline{\mathrm{PMap}_c(S)}$$

projects to a non-zero element in $\left(\overline{\mathrm{PMap}_c(S)}\right)^{ab}$.

In fact, what is used in practice in [Dom22] is the following stronger fact, in the case when S has genus at least three. It is implicit in [Dom22, §8.1.1]; here we make the statement and the details of the proof explicit.

Corollary 3.3.8. *Let S be an infinite-type surface of genus at least three with at least two ends and let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a well-spaced escaping sequence of simple closed curves on S. Let $a_1, a_2, a_3, \ldots$ be a strictly increasing sequence of positive integers. Then there is an injective homomorphism $\varphi\colon \mathbb{Q} \hookrightarrow \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}$ sending $1/n \in \mathbb{Q}$ to the element*

$$\prod_{i=r_n}^{\infty} (T_{\gamma_i})^{a_i!/n} \in \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab},$$

where $r_n \geqslant 1$ is any integer sufficiently large so that $a_i \geqslant n$ for all $i \geqslant r_n$.

Proof. Using the presentation $\mathbb{Q} \cong \langle x_1, x_2, x_3, \ldots \mid (x_n)^n = x_{n-1}\rangle$, where x_n corresponds to $1/n! \in \mathbb{Q}$, we see that in order to define a homomorphism $\varphi\colon \mathbb{Q} \to G$, for any group G, it suffices to choose an element $\varphi(1)$ of G, a square root $\varphi(1/2!)$ of $\varphi(1)$, a cube root $\varphi(1/3!)$ of $\varphi(1/2!)$, etc. We begin by choosing

$$\varphi(1) = \prod_{i=1}^{\infty}(T_{\gamma_i})^{a_i!} \in \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}.$$

This is non-trivial by Theorem 3.3.7, since the sequence $(a_i!)$ is unbounded. In fact, Theorem 3.3.7 implies that $\varphi(1)$ has infinite order, since the sequence $(na_i!)$ is unbounded for all $n \geqslant 1$. We next need to choose a square root $\varphi(1/2!)$ of this element. First choose $r_2 \geqslant 1$ so that $a_i \geqslant 2$ for all $i \geqslant r_2$ (this is possible since (a_i) is strictly increasing). Then set

$$\varphi(1/2!) = \prod_{i=r_2}^{\infty} (T_{\gamma_i})^{a_i!/2!} \in \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}$$

and notice that

$$\frac{\varphi(1)}{2\varphi(1/2!)} = \prod_{i=1}^{r_2-1} (T_{\gamma_i})^{a_i!} \in \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}.$$

This is a finite product of Dehn twists, so it is the image of the corresponding element of $\mathrm{PMap}_c(S)^{ab}$. Restricting further, choose a compact subsurface $S' \subset S$ containing the curves $\gamma_1, \ldots, \gamma_{r_2-1}$ in its interior and having genus at least three. The element above is then the image of the corresponding element of $\mathrm{Map}(S')^{ab}$. But the mapping class group of any compact, orientable surface of genus at least three is perfect [Bir70; Pow78], so $\mathrm{Map}(S')^{ab} = 0$ and hence $\varphi(1) = 2\varphi(1/2!)$. Continuing in the same way, we construct a cube root of $\varphi(1/2!)$, etc. Thus we have constructed a homomorphism φ from $\mathbb{Q}$.

Recall that any homomorphism defined on $\mathbb{Q}$ is injective as long as its restriction to $\mathbb{Z} \subset \mathbb{Q}$ is injective. We observed above that $\varphi(1)$ has infinite order; hence φ is injective. Finally, the formula for $\varphi(1/n)$ in the statement follows immediately from the construction, noting again that we may remove finitely many terms from the infinite product without changing the element of the abelianisation. □

The following corollary is again implicit in [Dom22, §8.1.1], but we prefer to make the statement and the details of the proof explicit. Let the surface S and the sequences $\{\gamma_i\}_{i\in\mathbb{N}}$ and $\{a_i\}_{i\in\mathbb{N}}$ be as in Corollary 3.3.8. For any infinite subset $F \subseteq \mathbb{N}$, denote by

$$\varphi_F\colon \mathbb{Q} \longrightarrow \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}$$

the embedding obtained by applying Corollary 3.3.8 to the sequences $\{\gamma_i\}_{i\in\mathbb{N}}$ and $\{a_i\}_{i\in\mathbb{N}}$.

Corollary 3.3.9. *Let $\mathcal{F}$ be a family of infinite subsets of $\mathbb{N}$ such that any two of them have finite intersection. Then the homomorphism*

$$\Phi_{\mathcal{F}} = \bigoplus_{F\in\mathcal{F}} \varphi_F\colon \bigoplus_{F\in\mathcal{F}} \mathbb{Q} \longrightarrow \big(\overline{\mathrm{PMap}_c(S)}\big)^{ab}$$

is also injective.

Proof. Let $(r_F) \in \ker(\Phi_{\mathcal{F}})$. Since the domain of $\Phi_{\mathcal{F}}$ is a direct sum, there are only finitely many $F \in \mathcal{F}$ such that $r_F \neq 0$; let us enumerate these as $F_1, \ldots, F_s$. Also choose $n \geqslant 1$ so that $m_F := nr_F \in \mathbb{Z}$. We therefore have

$$0 = \Phi_{\mathcal{F}}(n(r_F)) = \Phi_{\mathcal{F}}((m_F)) = \prod_{i \in F_1} \left((T_{\gamma_i})^{a_i!}\right)^{m_{F_1}} \cdots \prod_{i \in F_s} \left((T_{\gamma_i})^{a_i!}\right)^{m_{F_s}}.$$

By Theorem 3.3.7, this product can only be zero if it is a *finite* product. But each $F_1, \ldots, F_s$ is infinite. Moreover, two terms of the product can only cancel if they are indexed by an element of one of the pairwise intersections $F_p \cap F_q$ for $p \neq q \in \{1, \ldots, s\}$, all of which are finite by assumption. Thus only finitely many cancellations can occur, so the only possible way for this product to be zero is if $s = 0$, which means that $(r_F) = 0$. Thus $\Phi_{\mathcal{F}}$ is injective. □

3.4 Proof of Theorem 3.A

We are now ready to prove Theorem 3.A. The tools of the previous section imply almost immediately the following result.

Proposition 3.4.1. *Let S be an infinite type surface of genus at least three with at least two ends. Then there is an injective homomorphism of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\overline{\mathrm{PMap}_c(S)}).$$

Proof. Choose a well-spaced escaping sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ of simple closed curves on S (such a sequence always exists by Remark 3.3.4) and set $a_i = i$. Choose a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that any two of them have finite intersection, and such that the family $\mathcal{F}$ has the cardinality of the continuum. (For example, we may identify $\mathbb{N}$ with $\mathbb{Q}$ and choose for each $a \in \mathbb{R}$ a sequence of distinct rationals converging to a.) There is then a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\mathcal{F}} \mathbb{Z} & \longrightarrow & \overline{\mathrm{PMap}_c(S)} \\ \big\downarrow & & \big\downarrow \\ \bigoplus_{\mathcal{F}} \mathbb{Q} & \xrightarrow{\Phi_{\mathcal{F}}} & \left(\overline{\mathrm{PMap}_c(S)}\right)^{ab}, \end{array} \tag{3.6}$$

where the bottom horizontal map $\Phi_{\mathcal{F}}$ is injective by Corollary 3.3.9 and its lift to $\overline{\mathrm{PMap}_c(S)}$ after restricting to $\mathbb{Z} \subset \mathbb{Q}$ in each summand is given by sending the generator $1 \in \mathbb{Z}$ of the summand corresponding to $F \in \mathcal{F}$ to the element

$$\prod_{i \in F} (T_{\gamma_i})^{i!} \in \overline{\mathrm{PMap}_c(S)}.$$

The result then follows by an application of Lemma 3.3.2. □

Remark 3.4.2. Proposition 3.4.1 also holds without the assumption that S has genus at least 3. This follows from an analogue of Corollary 3.3.8 that involves a sequence of pseudo-Anosov elements supported on pairwise-disjoint compact subsurfaces of S instead of Dehn twists; see [Dom22, §8.1.2] for more details of this construction. One then obtains a diagram of the form (3.6), where the horizontal maps are defined using infinite products of powers of these pseudo-Anosov elements instead of Dehn twists, and the result then follows from Lemma 3.3.2.

We next deduce the analogue of Theorem 3.B for the Loch Ness monster surface L and the surface L' obtained by removing one puncture from L.

Proposition 3.4.3. *The graded abelian groups $H_*(\mathrm{Map}(L'))$ and $H_*(\mathrm{Map}(L))$ each contain an embedded copy of the exterior algebra $\Lambda^*\big(\bigoplus_{\mathfrak{c}}\mathbb{Z}\big)$.*

Proof. Since L' has at most one non-planar end, [PV18, Theorem 4] implies that $\overline{\mathrm{PMap}_c(L')} = \mathrm{PMap}(L')$. We also have $\mathrm{PMap}(L') = \mathrm{Map}(L')$ since L' has only two punctures, which cannot be interchanged by a homeomorphism of L' since exactly one of them is non-planar. Thus the result for L' is a special case of Proposition 3.4.1. In this case, the sequence of simple closed curves γ_i may be taken to be those illustrated in Figure 3.2 (see Example 3.3.5).

In order to deduce the result for L, we use the Birman exact sequence, which takes the form

$$1 \to \pi_1(L) \longrightarrow \mathrm{Map}(L') \longrightarrow \mathrm{Map}(L) \to 1. \tag{3.7}$$

Since abelianisation is a right-exact functor, it follows that the kernel of $H_1(\mathrm{Map}(L')) \to H_1(\mathrm{Map}(L))$ is a quotient of $H_1(L)$; in particular it is *countable*. Consider the diagram

$$\begin{array}{ccccc} \bigoplus_{\mathcal{F}}\mathbb{Z} & \longrightarrow & \mathrm{Map}(L') & \longrightarrow & \mathrm{Map}(L) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \bigoplus_{\mathcal{F}}\mathbb{Q} & \xhookrightarrow{\Phi_{\mathcal{F}}} & (\mathrm{Map}(L'))^{ab} & \xrightarrow{(*)} & (\mathrm{Map}(L))^{ab}, \end{array} \tag{3.8}$$

where the left-hand square is (3.6) in the case $S = L'$ and the right-hand square is induced by (3.7). We know that $(*)$ has countable kernel by the discussion above, so Lemma 3.8.2 implies that, after removing countably many terms from the direct sum on the left-hand side, the composition across the bottom of (3.8) is also injective. We therefore obtain a diagram

$$\begin{array}{ccc} \bigoplus_{\mathfrak{c}}\mathbb{Z} & \longrightarrow & \mathrm{Map}(L) \\ \big\downarrow & & \big\downarrow \\ \bigoplus_{\mathfrak{c}}\mathbb{Q} & \xhookrightarrow{(*)'} & (\mathrm{Map}(L))^{ab}, \end{array} \tag{3.9}$$

where $(*)'$ is injective and the direct sums on the left-hand side are still indexed by a set with the cardinality of the continuum. The result for L thus follows from Lemma 3.3.2. □

Remark 3.4.4. We noted in Remark 3.4.2 that Proposition 3.4.1 holds without the assumption on the genus of S, i.e. it holds for any infinite type surface S with at least two ends. On the other hand, if S is an infinite type surface with at most one end, it must be the Loch Ness monster surface $S = L$, and the result then follows from Proposition 3.4.3 (see also [Dom22, Appendix]). Thus, in fact, Proposition 3.4.1 holds for any infinite type surface S. This is the first part of Theorem 3.A:

Corollary 3.4.5. *Let S be any infinite-type surface. Then the graded abelian group $H_*(\overline{\mathrm{PMap}_c(S)})$ contains an embedded copy of the exterior algebra $\Lambda^*\big(\bigoplus_{\mathfrak{c}}\mathbb{Z}\big)$, induced by an embedding $\bigoplus_{\mathfrak{c}}\mathbb{Z} \hookrightarrow \overline{\mathrm{PMap}_c(S)}$.*

Remark 3.4.6. There are two points where this proof is not entirely constructive. The first is the choice of the family $\mathcal{F} = \{\Lambda_a \mid a \in \mathbb{R}\}$ of infinite subsets of $\mathbb{N}$. However, this may easily be made explicit by choosing an explicit bijection between $\mathbb{N}$ and $\mathbb{Q}$ and then letting $\Lambda_a \subseteq \mathbb{Q}$, for $a \in \mathbb{R}$, be the sequence of rational numbers converging to $a \in \mathbb{R}$ given by truncating the binary expansion of a (to avoid ambiguity and to ensure that Λ_a is infinite, we specify that if a has a binary expansion ending in a sequence of 0's, we choose its *other* binary expansion ending in a sequence of 1's). The second point where it is non-constructive is in passing from diagram (3.8) to diagram (3.9) by throwing away countably many real numbers indexing the direct sum on the left-hand side. However, looking carefully at the proof of Lemma 3.8.2, one may make this step constructive too.

Remark 3.4.7. When S has at most one non-planar end, the pure mapping class group $\mathrm{PMap}(S)$ coincides with $\overline{\mathrm{PMap}_c(S)}$, by [PV18, Theorem 4]. Thus Corollary 3.4.5 says that $H_*(\mathrm{PMap}(S))$ is uncountable in every positive degree when S has at most one non-planar end. This statement also holds when S has infinitely many non-planar ends. Indeed, by [APV20, Corollary 6], we have in this case that

$$\mathrm{PMap}(S) \;\cong\; \overline{\mathrm{PMap}_c(S)} \rtimes \mathbb{Z}^{\mathbb{N}}.$$

In particular, $\mathbb{Z}^{\mathbb{N}}$ is a retract of $\mathrm{PMap}(S)$, so the natural induced map $H_i(\mathbb{Z}^{\mathbb{N}}) \to H_i(\mathrm{PMap}(S))$ is split-injective in every degree. The fact that that $H_*(\mathrm{PMap}(S))$ is uncountable in every positive degree in this case is therefore an immediate corollary of the following lemma.

Lemma 3.4.8. *The homology group $H_i(\mathbb{Z}^{\mathbb{N}})$ contains a direct summand isomorphic to $\mathbb{Z}^{\mathbb{N}}$ in every degree $i > 0$. Hence it contains a subgroup isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every degree $i > 0$.*

Proof. The first statement follows from the Künneth theorem applied to the decomposition $\mathbb{Z}^{\mathbb{N}} \cong \mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^i$. The second statement then follows from the fact that $\mathbb{Z}^{\mathbb{N}}$ contains free abelian groups of rank $\mathfrak{c}$. To see this, choose a family $\mathcal{F}$, of cardinality $|\mathcal{F}| = \mathfrak{c}$, of infinite subsets of $\mathbb{N}$ such that any pair have finite intersection. (For example, as in the proof of Proposition 3.4.1, we may identify $\mathbb{N}$ with $\mathbb{Q}$ and choose for each $a \in \mathbb{R}$ a sequence of distinct rationals converging to a.) It is then easy to check that the collection

$$\{\chi_F \in \mathbb{Z}^{\mathbb{N}} \mid F \in \mathcal{F}\},$$

where $\chi_F \colon \mathbb{N} \to \{0,1\} \subset \mathbb{Z}$ denotes the indicator function of $F \subseteq \mathbb{N}$, is $\mathbb{Z}$-linearly independent and hence generates a subgroup of $\mathbb{Z}^{\mathbb{N}}$ isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Z}$. ∎

Remark 3.4.9. When S has n non-planar ends with $1 < n < \infty$, by [APV20, Corollary 6] we have:

$$\mathrm{PMap}(S) \;\cong\; \overline{\mathrm{PMap}_c(S)} \rtimes \mathbb{Z}^{n-1}, \tag{3.10}$$

where $\mathbb{Z}^{n-1}$ is freely generated by $n-1$ handle shifts $h_1, \ldots, h_{n-1}$. As indicated in the proof of [APV20, Theorem 5], one may choose the handle shifts h_j to have pairwise disjoint support. Let Y_j be the support of h_j. Recall that each Y_j is a subsurface homeomorphic to the result of gluing handles onto $\mathbb{R} \times [0,1]$ periodically with respect to the transformation $(x,y) \mapsto (x+1,y)$. For convenience, we shall require that the i-th handle is attached to $[i,i+1] \times [0,1]$ and that h_j maps the i-th handle to the $(i+1)$-st handle. See Figure 3.4 for an illustration. The semi-direct product decomposition (3.10) implies that

$$H_1(\mathrm{PMap}(S)) \;\cong\; H_1(\overline{\mathrm{PMap}_c(S)})_{\mathbb{Z}^{n-1}} \oplus \mathbb{Z}^{n-1}, \tag{3.11}$$

where $(-)_{\mathbb{Z}^{n-1}}$ denotes the coinvariants under the action of the handle shifts. By Theorem 3.3.7, choosing the sequence of curves γ_i as illustrated in Figure 3.4 and any unbounded sequence of positive integers a_i, the infinite product of Dehn twists $f = \prod_{i=1}^{\infty} (T_{\gamma_i})^{a_i} \in \overline{\mathrm{PMap}_c(S)}$ represents a non-trivial element in the abelianisation $H_1(\overline{\mathrm{PMap}_c(S)})$. But it vanishes in $H_1(\mathrm{PMap}(S))$ – in other words, in the coinvariants under the action of the handle shifts – since $[f] = [g] - [h_1 g h_1^{-1}]$, where $g = \prod_{i=1}^{\infty} (T_{\gamma_i})^{b_i}$, $b_i = \Sigma_{j=1}^{i} a_j$.

Recall that the Torelli group $\mathcal{T}(S)$ is the kernel of the natural homomorphism $\mathrm{Map}(S) \to \mathrm{Aut}(H_1(S))$.

Theorem 3.4.10. *Let S be an infinite-type surface. The integral homology group $H_i(\mathcal{T}(S))$ is uncountable for every $i \geqslant 1$. In fact it contains an embedded copy of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ in every positive degree.*

Proof. By Corollary 3.4.5, there is an embedding

$$\textstyle\bigoplus_{\mathfrak{c}} \mathbb{Z} \longhookrightarrow \overline{\mathrm{PMap}_c(S)} \tag{3.12}$$

that induces on homology an embedding of $\Lambda^*(\bigoplus_{\mathfrak{c}} \mathbb{Z})$ into $H_*(\overline{\mathrm{PMap}_c(S)})$. It will therefore suffice to show that (3.12) factors through the Torelli group $\mathcal{T}(S)$.

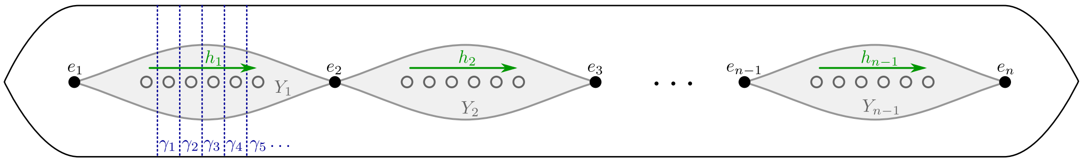


Figure 3.4 A surface with n non-planar ends $e_1, \ldots, e_n$ for $2 \leqslant n < \infty$. The top and bottom edges are identified to obtain a sphere, then the points $e_1, \ldots, e_n$ (together with a set of planar ends, which is not pictured) are removed, then we take a connected sum with a torus along each of the (infinitely many) small grey discs. The planar ends (not pictured) may have some or all of the non-planar ends $e_1, \ldots, e_n$ as limit points, but in any case lie *outside* of the subsurfaces $Y_1, \ldots, Y_{n-1}$, which support the handle shifts $h_1, \ldots, h_{n-1}$. The curves $\gamma_1, \gamma_2, \gamma_3, \ldots$ are chosen as illustrated such that the handle shift h_1 sends γ_i to γ_{i+1} (up to isotopy).

We first note that the Torelli group is contained in $\overline{\mathrm{PMap}_c(S)} \subset \mathrm{Map}(S)$: it clearly lies in $\mathrm{PMap}(S)$ since any non-trivial action on the space of ends of S implies a non-trivial action on $H_1(S)$; then the fact that it lies in $\overline{\mathrm{PMap}_c(S)}$ follows from [APV20, Corollary 6], which decomposes $\mathrm{PMap}(S)$ as a semi-direct product of $\overline{\mathrm{PMap}_c(S)}$ and a direct product of copies of $\mathbb{Z}$ generated by *handle shifts*, together with the fact that handle shifts act non-trivially on $H_1(S)$.

Finally, we just have to note that the elements of $\overline{\mathrm{PMap}_c(S)}$ used to define the homomorphism (3.12) actually lie in $\mathcal{T}(S)$. When the genus of S is at least 3, these elements are infinite products of Dehn twists around (pairwise disjoint) *separating* curves; hence they act trivially on $H_1(S)$. When the genus is at most 2, we instead use infinite products of (pairwise disjointly-supported) pseudo-Anosov elements, as explained in Remark 3.4.2. These elements are of the form $T_\alpha^2 T_\beta^2 T_\alpha^{-2} T_\beta^{-2}$ for a pair of separating curves α, β that fill a finite-type subsurface of S, as explained in [Dom22, p. 715], and they also act trivially on $H_1(S)$. □

Remark 3.4.11. In degree one, $H_1(\mathcal{T}(S))$ contains an embedded copy of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$, by [Dom22, Theorem 9.1].

3.5 Descending along double branched covers

In this section we generalise techniques of Malestein and Tao [MT23] – who proved uncountability of homology in degree 1 for the mapping class group of $\mathbb{R}^2 \smallsetminus \mathbb{N}$ – to higher degrees and to the more general class of surfaces from Theorem 3.B, completing the proof of that theorem. To do this, we will need the notion of a *ray surface* associated to a surface Σ.

Definition 3.5.1. Let Σ be any connected surface without boundary and write Σ_1 (respectively Σ_2) for the surface obtained by removing one (respectively two disjoint) open discs from Σ. The *ray surface* $\mathcal{R}(\Sigma)$ is the surface obtained by gluing together infinitely many copies of Σ_2 and "capping off" in one direction with a single copy of Σ_1. See the top half of Figure 3.5 for an example where $\Sigma = T^2$ is the torus; thus $\mathcal{R}(T^2)$ is the Loch Ness monster surface.

Remark 3.5.2. This is the same as the surface denoted by $\mathfrak{L}(\Sigma)$ in [PW22b] with its boundary capped off by a disc.

Before proving Theorem 3.B in general, we will prove it under certain stronger hypotheses on the surface S. Namely, we assume that the surface S has genus 0, empty boundary and that its space of ends is of the form $\Upsilon^+(E)$,[1] where E has a topologically distinguished point. This means that S may be written as $\mathcal{R}(\mathbb{S}^2 \smallsetminus E)$, using the construction $\mathcal{R}(-)$ of ray surfaces from Definition 3.5.1 above.

Denote by L the Loch Ness monster surface and consider its branched double covering $L \to \mathbb{R}^2$ depicted in Figure 3.5. This may also be written as

$$L \cong \mathbb{S}^2 \sharp \mathcal{R}(T^2) \longrightarrow \mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2) \cong \mathbb{R}^2. \tag{3.13}$$

[1] Recall that the notation $\Upsilon^+(-)$ was defined in Definition 3.0.3.

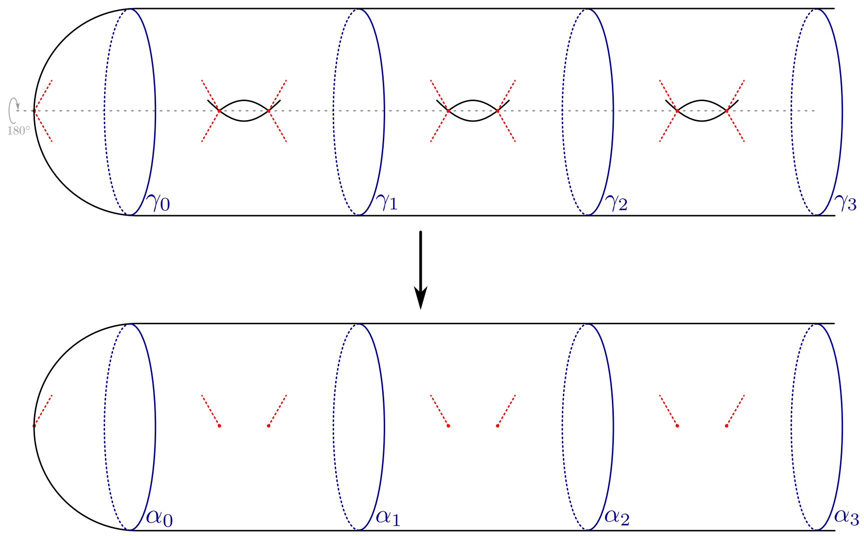


Figure 3.5 The branched double covering (3.13). After removing the subset marked in red (which includes the branch points), this restricts to the (genuine) double covering (3.14).

This decomposition corresponds to cutting along the curves depicted in the figure. Notice that there are exactly two branch points (of order 2) in each copy of $\mathbb{S}^2$ in $\mathcal{R}(\mathbb{S}^2)$ and one additional branch point in the copy of $\mathbb{S}^2$ in the extra connected summand. Let us now choose once and for all a topologically distinguished point $x \in E$ (this exists by hypothesis) and embed pairwise disjoint copies of E into $\mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2)$ so that:

- each copy of E lies entirely in one of the copies of $\mathbb{S}^2$,
- the point $x \in E$ is sent to a branch point of (3.13),
- each branch point of (3.13) is in the image of one of the embeddings of E.

We denote by X the complement of these embedded copies of E and we denote by $Y \subset \mathbb{S}^2 \sharp \mathcal{R}(T^2)$ the pre-image of $X \subset \mathbb{S}^2 \sharp \mathcal{R}(\mathbb{S}^2)$ under (3.13). Notice that:

$$Y \cong (\mathbb{S}^2 \smallsetminus V) \sharp \mathcal{R}(T^2 \smallsetminus (V \sqcup V))$$
$$X \cong (\mathbb{S}^2 \smallsetminus E) \sharp \mathcal{R}(\mathbb{S}^2 \smallsetminus (E \sqcup E)) \cong \mathcal{R}(\mathbb{S}^2 \smallsetminus E) \cong S,$$

where V denotes the wedge sum of two copies of E at the basepoint x. Since we have in particular removed all branch points of the branched double covering, we obtain by restriction a (genuine) double covering

$$Y \longrightarrow X \tag{3.14}$$

depicted in Figure 3.5.

We fix compatible basepoints on X and Y and denote by H the index-2 subgroup of $\pi_1(X)$ corresponding to this double covering. We also write $\mathrm{Map}_*(X)$ and $\mathrm{Map}_*(Y)$ for the *based* mapping class groups of X and Y, given by isotopy classes of self-homeomorphisms that fix the basepoint.

Lemma 3.5.3. *The action of* $\mathrm{Homeo}_*(X)$ *on* $\pi_1(X)$ *preserves the subgroup* H*.*

Proof. We first describe the subgroup $H \subset \pi_1(X)$ intrinsically. A based loop γ in X lies in H if and only if its lift to Y is a closed loop. This occurs if and only if the sum of its winding numbers around all branch points of the branched double covering (3.13) is even. We therefore have to show that *if* the sum of these winding numbers is even for γ, then the same is true for $\varphi \circ \gamma$, where φ is any based self-homeomorphism of X.

A subtle point here is the meaning of *winding number* (which we only need to define mod 2): a simple loop in the surface X has winding number ± 1 around an end $e \neq \infty$ if it separates X into two pieces, one containing e and the other containing the end ∞. Here ∞ denotes the end corresponding to going off to infinity to the right in Figure 3.5. More precisely, recall that the end space of X is the one-point compactification $\Upsilon^+(E) = E\omega + 1$ of a countably infinite disjoint union of copies of E and ∞ denotes the point at infinity of this one-point compactification. By Corollary 3.2.7 and our assumption that E has a topologically distinguished point, the point $\infty \in E\omega + 1$ is also topologically distinguished. Thus any self-homeomorphism φ of X fixes ∞, meaning that the notion of "winding number" is preserved by φ.

Let us now show that if the sum of the winding numbers of γ around all branch points of X is even, then the same is true for $\varphi \circ \gamma$. The end space $E\omega + 1$ of X has a topologically distinguished subset $\{x\}\omega$ given by the copy of the topologically distinguished point x in each copy of E. But this is precisely the set of branch point of the branched double covering (3.13). Thus the self-homeomorphism φ must send each end of X corresponding to a branch point to another end of X corresponding to a branch point. Its effect on winding numbers around branch points is therefore simply to permute them; so in particular their *sum* is preserved. Hence if the sum of winding numbers around branch points is even for γ, then the sum of winding numbers around branch points will also be even for $\varphi \circ \gamma$. ◻

Remark 3.5.4. The proof of Lemma 3.5.3 is where our assumption that the space E has a topologically distinguished point is used decisively. The lemma would be false without this assumption. See also Remark 3.5.5.

We may now complete the proof of Theorem 3.B under the stronger assumptions that we are currently making (we explain how to remove these assumptions at the end of this section).

Proof of Theorem 3.B under additional assumptions. It follows from Lemma 3.5.3 that each based homeomorphism of X lifts uniquely to a based homeomorphism of Y, giving us a continuous map $\mathrm{Homeo}_*(X) \to \mathrm{Homeo}_*(Y)$, which on π_0 induces

$$\mathrm{Map}_*(X) \longrightarrow \mathrm{Map}_*(Y). \tag{3.15}$$

Filling in all *planar* ends of a surface is a functorial operation on the category of surfaces, so by filling in all planar ends of Y we obtain a continuous map $\mathrm{Homeo}_*(Y) \to \mathrm{Homeo}_*(L)$ (see Proposition 3.9.2), which on π_0 induces

$$\mathrm{Map}_*(Y) \longrightarrow \mathrm{Map}_*(L). \tag{3.16}$$

Composing (3.15) and (3.16) with the forgetful map $\mathrm{Map}_*(L) \to \mathrm{Map}(L)$, we obtain

$$\mathrm{Map}_*(X) \longrightarrow \mathrm{Map}(L). \tag{3.17}$$

Let $\alpha_1, \alpha_2, \ldots$ be the collection of simple closed curves on X depicted in Figure 3.5. Since γ_i is a double covering of α_i, we see that

$$(T_{\alpha_i})^2 \longmapsto T_{\gamma_i}$$

under (3.17). Now recall that in §3.4 (see diagram (3.9)) we factored the inclusion $\bigoplus_{\mathfrak{c}} \mathbb{Z} \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$ through a map $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathrm{Map}(L)$ that sends the generator $1 \in \mathbb{Z}$ of each summand to a certain infinite product of Dehn twists around the curves γ_i from the top of Figure 3.5. Replacing each T_{γ_i} with $(T_{\alpha_i})^2$ in this infinite product, we obtain a map $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathrm{Map}_*(X)$ making the following triangle commute:

$$\begin{array}{ccc} & \bigoplus_{\mathfrak{c}} \mathbb{Z} & \\ \swarrow & & \searrow \\ \mathrm{Map}_*(X) & \xrightarrow{(3.17)} & \mathrm{Map}(L), \end{array} \tag{3.18}$$

where the right-hand diagonal map is part of a factorisation $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathrm{Map}(L) \to \bigoplus_{\mathfrak{c}} \mathbb{Q}$ of the standard inclusion. We have therefore shown that the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ also factors through $\mathrm{Map}_*(X)$. Now consider the diagram

$$\begin{array}{ccccc} \bigoplus_{\mathfrak{c}} \mathbb{Z} & \longrightarrow & \mathrm{Map}_*(X) & \xrightarrow{\varphi} & \bigoplus_{\mathfrak{c}} \mathbb{Q} \\ & \searrow & \downarrow & & \\ & & \mathrm{Map}(X), & & \end{array} \tag{3.19}$$

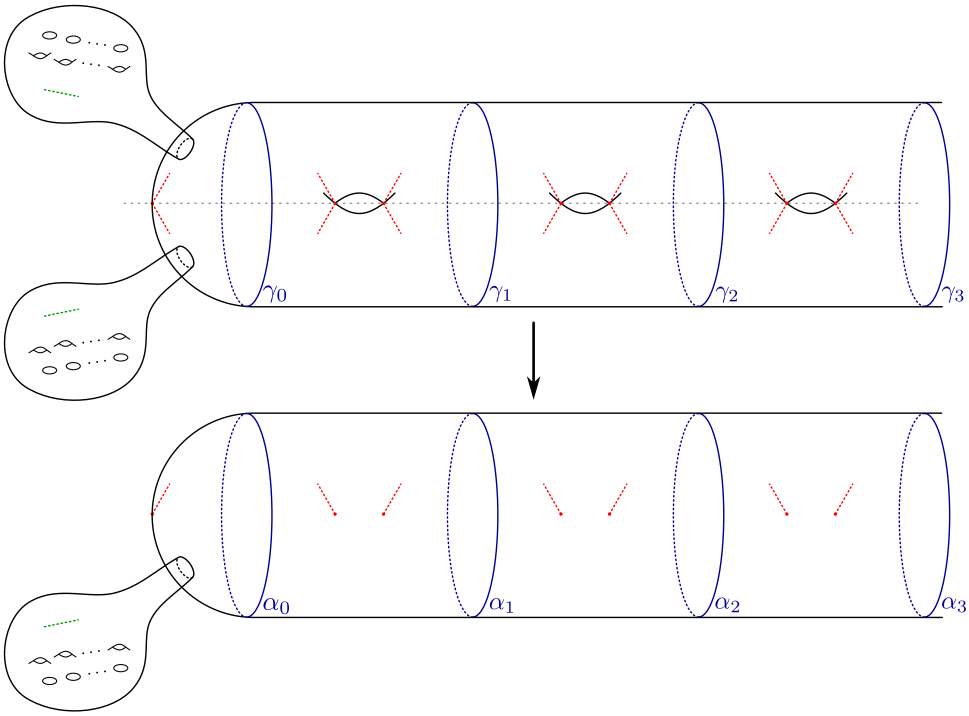


Figure 3.6 A modification of the branched double covering depicted in Figure 3.5.

where the middle vertical map forgets the basepoint. This is part of the Birman exact sequence for X, and its kernel is $\pi_1(X)$, which is in particular countable. Let us denote this kernel by K and consider its image $\varphi(K) \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$. Since $\varphi(K)$ is countable and each of its elements has only finitely many non-zero coordinates in $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ (because it is a direct *sum*), it is contained in the subgroup of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ given by the direct sum of only countably many of the copies of $\mathbb{Q}$. If we take the quotient by this subgroup, the resulting group is again isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ and the homomorphism φ now descends to $\mathrm{Map}(X)$. On the left-hand side of (3.19), we may compose with the inclusion of the corresponding sub-direct-summand of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ (which is again isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Z}$); this ensures that the composition across the top row of the following diagram is still the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$:

$$\begin{array}{ccccccccc} \bigoplus_{\mathfrak{c}} \mathbb{Z} & \longrightarrow & \bigoplus_{\mathfrak{c}} \mathbb{Z} & \longrightarrow & \mathrm{Map}_*(X) & \xrightarrow{\varphi} & \bigoplus_{\mathfrak{c}} \mathbb{Q} & \twoheadrightarrow & \bigoplus_{\mathfrak{c}} \mathbb{Q}. \\ & & & \searrow & \downarrow & & & \nearrow & \\ & & & & \mathrm{Map}(X) & & & & \end{array} \tag{3.20}$$

Thus we have shown that the standard inclusion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ into $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ factors through $\mathrm{Map}(X)$. This standard inclusion induces an injection on homology in all degrees, by Lemma 3.8.3 and the fact that $H_*(A) = \Lambda^*(A)$ for torsion-free abelian groups A, so it follows that we have an injection

$$\Lambda^*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) = H_*\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \hookrightarrow H_*(\mathrm{Map}(X)) = H_*(\mathrm{Map}(S)).$$

This completes the proof of Theorem 3.B under the additional assumptions on the surface S. ◻

We finish this section by showing how to modify the argument above to allow the more general surfaces S considered in the theorem.

Proof of Theorem 3.B in general. The proof follows exactly the same strategy as the proof in the special case above, so we just explain the steps that differ slightly.

In general, the surface S is of the form pictured at the bottom of Figure 3.6, where we have taken a connected sum of the surface considered previously with another surface of finite genus having finitely many boundary components, such that none of the points of its end space are similar to the topologically distinguished point $x \in E$. We may correspondingly modify the total space of the double covering by taking two connected sums with this surface (no new branch points are introduced).

Lemma 3.5.3 generalises directly to this setting, giving us a homomorphism that lifts (based) mapping classes up the double covering. Filling in all planar ends upstairs, as well as the finitely many boundary components, we obtain (as before) the Loch Ness monster surface L. With these modifications, the rest of the proof is identical to the proof in the special case given above, using the constructions of §3.4. □

Remark 3.5.5. It is essential to assume in Theorem 3.B that E_2 has a topologically distinguished point. Indeed, if we do not assume this, then the theorem is false. For example, without this assumption, the theorem would assert that the homology of $\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})$ is uncountable in all positive degrees, since $\Upsilon^+(\mathcal{C}) \cong \mathcal{C}$. However, the first and second homology groups of $\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})$ are known to be 0 and $\mathbb{Z}/2$ respectively [CC21].

3.6 Torsion elements

We prove in this section that, whenever S has genus 2, both $H_1(\mathrm{PMap}(S))$ and $H_1(\mathrm{Map}(S))$ contain an element of order 10 that generates a direct summand. We first recall that, for *compact* surfaces of genus 2, the first homology of their mapping class groups is precisely $\mathbb{Z}/10$. Denote by $S_{g,b}$ the connected, compact, orientable surface of genus g with $b \geqslant 1$ boundary components. When $g = 2$, we have the following.

Theorem 3.6.1 ([Kor02, §5]; see also [Mum67])**.** *For any $b \geqslant 0$, we have $H_1(\mathrm{Map}(S_{2,b})) \cong \mathbb{Z}/10$, generated by $[T_\alpha]$, where α is any non-separating simple closed curve in $S_{2,b}$.*

Proof of Theorem 3.C. If S has genus 2, there is an embedding $S_{2,1} \subseteq S$. Also, filling in the ends of S (all of which are planar since it has finite genus) to construct its Freudenthal compactification results in the compact surface $S_{2,b}$, where $b \geqslant 0$ is the number of boundary components of S. We therefore have homomorphisms

$$\mathrm{Map}(S_{2,1}) \longrightarrow \mathrm{PMap}(S) \subseteq \mathrm{Map}(S) \longrightarrow \mathrm{Map}(S_{2,b}), \tag{3.21}$$

where the first is given by extending homeomorphisms of $S_{2,1}$ by the identity on $S \smallsetminus S_{2,1}$ and the second is given by the unique extension of homeomorphisms to the Freudenthal compactification (see Proposition 3.9.2). Let α be a non-separating simple closed curve in $S_{2,1}$. By Theorem 3.6.1, the composition across (3.21) induces a map $\mathbb{Z}/10 \to \mathbb{Z}/10$ on first homology. Moreover, it clearly sends $[T_\alpha]$ to itself, so it sends a generator of the first $\mathbb{Z}/10$ to a generator of the second $\mathbb{Z}/10$; thus it is an isomorphism. Since we have factored an isomorphism of $\mathbb{Z}/10$ through $H_1(\mathrm{PMap}(S))$ and $H_1(\mathrm{Map}(S))$, it follows that these groups both contain $\mathbb{Z}/10$ as a direct summand. □

We record here a related general fact that (for example) allows one to embed torsion elements of the homology of mapping class groups of compact surfaces into the homology of mapping class groups of surfaces of infinite type.

Lemma 3.6.2. *Let S and S' be two surfaces with non-empty (compact) boundary and assume that S' is planar. Then there are embeddings of direct summands*

$$H_*(\mathrm{PMap}(S)) \hookrightarrow H_*(\mathrm{PMap}(S \natural S')) \quad \textit{and} \quad H_*(\mathrm{Map}(S)) \hookrightarrow H_*(\mathrm{Map}(S \natural S')), \tag{3.22}$$

where $-\natural-$ denotes the boundary connected sum along one chosen interval in the boundary of each of the two surfaces.

Proof. Since S' is planar, it must be of the form $S' = S_{0,b} \smallsetminus E$, where $b \geqslant 1$ is the number of its boundary components and E is its space of ends. We therefore have homomorphisms

$$\mathrm{PMap}(S) \longrightarrow \mathrm{PMap}(S \natural S') \longrightarrow \mathrm{PMap}(S \natural S_{0,b}) \longrightarrow \mathrm{PMap}(S \natural S_{0,1}) \cong \mathrm{PMap}(S), \tag{3.23}$$

where the first is given by extending homeomorphisms by the identity on S', the second is given by the unique extension of homeomorphisms to the Freudenthal compactification (see Proposition 3.9.2) and the third is given by filling in all boundary components of $S_{0,b}$ with discs, except the one along which we have taken the boundary connected sum, and extending homeomorphisms by the identity on these new discs. The isomorphism on the right-hand side is induced by a homeomorphism $S \natural S_{0,1} \cong S$ given by pushing the disc $S_{0,1}$ into a collar neighbourhood of the boundary of S. The composition across (3.23) is given by extending homeomorphisms of S by the identity on $S_{0,1}$ and then conjugating by the homeomorphism $S \natural S_{0,1} \cong S$. This is clearly isotopic to the identity, so, applying H_*, we have factored the identity map of $H_*(\mathrm{PMap}(S))$ through $H_*(\mathrm{PMap}(S \natural S'))$, which provides the first embedding of (3.22). The second embedding follows by an identical argument, replacing $\mathrm{PMap}(-)$ with $\mathrm{Map}(-)$ everywhere. ◻

3.7 Some open problems

In this section we propose some open questions, in addition to Questions 3.0.6, 3.0.11 and 3.0.12 discussed in the introduction. We divide them into §3.7.1 on homology and §3.7.2 on cohomology.

3.7.1 Homology

So far, our calculations suggest the answer to the following question could be positive.

Question 3.7.1. Let S be an infinite-type surface. Suppose that, for some $i \geqslant 1$, the group $H_i(\mathrm{Map}(S))$ is countable. Is $H_i(\mathrm{Map}(S))$ finitely generated for all i?

This would imply a dichotomy between those S for which $H_i(\mathrm{Map}(S))$ is finitely generated for all $i \geqslant 1$ and those S for which $H_i(\mathrm{Map}(S))$ is uncountable for all $i \geqslant 1$.

Question 3.7.2. Let $S_{g,1}$ be the connected, compact, orientable surface of genus g and with one boundary component. Does the forgetful map $\mathrm{Map}(S_{g,1} \smallsetminus \mathcal{C}) \to \mathrm{Map}(S_{g,1})$ induce isomorphisms on homology in all degrees?

Remark 3.7.3. When $g = 0$, a positive answer follows from [PW22b, Theorem B]. The answer in degree one (and for any g) has been proven to be positive in [CC22, Theorem 2.3]. On the other hand, the answer would be negative if we considered the sphere instead of $S_{g,1}$, since $H_2(\mathrm{Map}(\mathbb{S}^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}/2$ by [CC21, Theorem A.2]. It would also be negative if we took the plane instead of $S_{g,1}$, since $H_i(\mathrm{Map}(\mathbb{R}^2 \smallsetminus \mathcal{C})) \cong \mathbb{Z}$ for all even i by [PW22b, Theorem A].

By [PW22b, Theorem C], the mapping class groups of 1-holed binary tree surfaces are acyclic. One may wonder whether these are the *only* acyclic mapping class groups of infinite-type surfaces with connected boundary:

Question 3.7.4. Let S be an infinite-type surface with a single boundary component and suppose that its mapping class group $\mathrm{Map}(S)$ is acyclic. Is S necessarily a 1-holed binary tree surface?

3.7.2 Cohomology

Most of the results of this chapter may be summarised as follows. For any infinite-type surface S, the natural inclusion $\bigoplus_{\mathfrak{c}} \mathbb{Z} \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$ factors as

$$\bigoplus_{\mathfrak{c}} \mathbb{Z} \longrightarrow \mathcal{T}(S) \subseteq \overline{\mathrm{PMap}_c(S)} \longrightarrow \bigoplus_{\mathfrak{c}} \mathbb{Q}, \tag{3.24}$$

and similarly for the *full* mapping class group $\mathrm{Map}(S)$ if S satisfies the conditions of Theorem 3.B or if it is the Loch Ness monster surface (Proposition 3.4.3). Our results about integral homology then follow from the fact that the natural inclusion $\bigoplus_{\mathfrak{c}} \mathbb{Z} \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$ induces an injective homomorphism of exterior algebras $\Lambda^*(\bigoplus_{\mathfrak{c}} \mathbb{Z}) \subset \Lambda^*(\bigoplus_{\mathfrak{c}} \mathbb{Q})$ on homology (Lemma 3.8.3). It is therefore natural to consider also the effect of the factorisation (3.24) on integral *cohomology*. However, this factorisation does not tell us anything about cohomology, since the composition across (3.24) induces the zero map on cohomology:

Lemma 3.7.5. *For each $i \geqslant 1$, we have:*

$$H^i\Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}\Big) \cong \prod_{\mathfrak{c}} \mathbb{Z},$$

$$H^i\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}\Big) \cong \begin{cases} 0 & \text{if } i = 1, \\ \bigoplus_{2^{\mathfrak{c}}} \mathbb{Q} & \text{if } i \geqslant 2. \end{cases}$$

In particular, the inclusion $\bigoplus_{\mathfrak{c}} \mathbb{Z} \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$ induces the zero map on H^i.

Proof. The last statement follows from the two calculations, since the induced map on H^i has a rational vector space as its domain, which is a divisible group. Its image must therefore also be divisible, but the only divisible subgroup of $\prod_{\mathfrak{c}} \mathbb{Z}$ is the trivial group.

It therefore remains to check the two calculations. The first one follows from the fact that $H_i(\bigoplus_{\mathfrak{c}} \mathbb{Z}) \cong \bigoplus_{\mathfrak{c}} \mathbb{Z}$ for all $i \geqslant 1$, the universal coefficient theorem, the fact that $\mathrm{Hom}_{\mathbb{Z}}(-,-)$ and $\mathrm{Ext}_{\mathbb{Z}}(-,-)$ take direct sums to products in the first variable and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$.

For the second calculation, we again use the universal coefficient theorem, where this time we use the facts that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$ and that $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$ is a rational vector space of dimension $\mathfrak{c}$ (see for example [Wie69]). Thus for $i \geqslant 2$ we have $H^i(\bigoplus_{\mathfrak{c}} \mathbb{Q}) \cong \prod_{\mathfrak{c}}(\bigoplus_{\mathfrak{c}} \mathbb{Q})$, which is a divisible and torsion-free abelian group, hence a rational vector space, of cardinality (and hence also dimension) $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$. ◻

Since the composition across (3.24) induces the zero map on cohomology, we cannot deduce anything about $H^*(\overline{\mathrm{PMap}_c(S)})$ from this. However, we wonder whether the right-hand map of (3.24) is nevertheless injective on cohomology. If it is, it would positively answer the first part of the following question.

Question 3.7.6. Let S be an infinite-type surface and $i \geqslant 2$. Do the groups $H^i(\overline{\mathrm{PMap}_c(S)})$ or $H^i(\mathrm{PMap}(S))$ contain a rational vector space of dimension $2^{\mathfrak{c}}$?

The second part of this question is motivated by the observation that, in the case when S has infinitely many non-planar ends, the answer is yes. In fact, we have:

Proposition 3.7.7. *Let S be a surface with infinitely many non-planar ends and let $i \geqslant 2$. Then there is an embedding*

$$\bigoplus_{2^{\mathfrak{c}}} \mathbb{Q} \oplus \bigoplus_{2^{\mathfrak{c}}} \mathbb{Q}/\mathbb{Z} \longrightarrow H^i(\mathrm{PMap}(S)).$$

Proof. By [APV20, Corollary 6], $\mathrm{PMap}(S)$ admits a split-surjection onto the Baer-Specker group $\mathbb{Z}^{\mathbb{N}}$, so $H^i(\mathbb{Z}^{\mathbb{N}})$ is a summand of $H^i(\mathrm{PMap}(S))$. By the universal coefficient theorem, $H^i(\mathbb{Z}^{\mathbb{N}})$ has a direct summand of the form $\mathrm{Ext}_{\mathbb{Z}}(H_{i-1}(\mathbb{Z}^{\mathbb{N}}),\mathbb{Z})$ and we know by Lemma 3.4.8 that $H_{i-1}(\mathbb{Z}^{\mathbb{N}})$ contains a direct summand isomorphic to $\mathbb{Z}^{\mathbb{N}}$. Putting this together, it follows that $H^i(\mathrm{PMap}(S))$ has a direct summand isomorphic to $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}},\mathbb{Z})$. This group is isomorphic to $\bigoplus_{2^{\mathfrak{c}}} \mathbb{Q} \oplus \bigoplus_{2^{\mathfrak{c}}} \mathbb{Q}/\mathbb{Z}$, by [Nun61, Theorem 5] (see also [Fuc73, Exercise 2 of §99]). ◻

3.8 Appendix A: Abelian groups

We collect here a few facts about abelian groups that are needed in our proofs. For a comprehensive treatment of the theory of abelian groups, we refer to [Fuc70; Fuc73].

Recall that an abelian group A is called *divisible* if for each element $a \in A$ and positive integer n, there is another element $b \in A$ such that $a = nb$. An abelian group A is called *injective* if for every injective homomorphism of abelian groups $\iota \colon B \to C$ and homomorphism $f \colon B \to A$, there is a homomorphism $g \colon C \to A$ such that $g \circ \iota = f$. By [Fuc70, Theorems 21.1 and 24.5], an abelian group is divisible if and only if it is injective. In particular:

Lemma 3.8.1. *Every injective homomorphism from a divisible abelian group to another abelian group admits a retraction.*

Proof. Let A be a divisible abelian group and let $\iota \colon A \to C$ be an injective homomorphism. Since A is injective, taking $B = A$ and $f = \mathrm{id}$ above, we obtain a retraction of ι. ◻

Lemma 3.8.2. *Suppose that we have homomorphisms of abelian groups*

$$\bigoplus_{\mathfrak{c}} \mathbb{Q} \xrightarrow{\ f\ } A \xrightarrow{\ g\ } B,$$

where f is injective and g has countable kernel. Then, after restricting the direct sum on the left to a subcollection of the same cardinality, the composition $g \circ f$ is also injective.

Proof. Consider the subgroup $K := \ker(g \circ f) = f^{-1}(\ker(g)) \subset \bigoplus_{\mathfrak{c}} \mathbb{Q}$. Since $\ker(g)$ is countable and f is injective, K is a countable subgroup of $\bigoplus_{\mathfrak{c}} \mathbb{Q}$. Each element of K has only finitely many non-zero coordinates in the direct sum and K has countably many elements; thus K is contained in the sub-direct-sum given by countably many $\mathbb{Q}$ summands. After removing these summands from the direct sum, the composition $g \circ f$ is injective. ◻

Lemma 3.8.3. *For any set I, the canonical inclusion $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$ induces an injective map of graded abelian groups*

$$\Lambda^*\Big(\bigoplus_I \mathbb{Z}\Big) \hookrightarrow \Lambda^*\Big(\bigoplus_I \mathbb{Q}\Big). \tag{3.25}$$

To prove this, we first recall the following basic calculation:

Lemma 3.8.4. $\Lambda^*(\mathbb{Z}) \cong \mathbb{Z}[0] \oplus \mathbb{Z}[1]$ *and* $\Lambda^*(\mathbb{Q}) \cong \mathbb{Z}[0] \oplus \mathbb{Q}[1]$.

Proof. The only non-obvious statement is that $\Lambda^i(\mathbb{Q}) = 0$ for $i \geqslant 2$. To see this, first recall that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \tag{3.26}$$

via an isomorphism that sends $a_1 \otimes a_2 \otimes \cdots \otimes a_i \mapsto a_1 a_2 \cdots a_i$. The $\mathbb{Z}$-module $\Lambda^i(\mathbb{Q})$ is the quotient of this tensor power by the sub-$\mathbb{Z}$-module generated by all elements $a_1 \otimes a_2 \otimes \cdots \otimes a_i$ with $a_j = a_k$ for some $j \neq k$. Thus to prove that $\Lambda^i(\mathbb{Q}) = 0$ we have to show that every rational number is a $\mathbb{Z}$-linear combination of rational numbers of the form $b^2 a_3 \cdots a_i$. For $i \geqslant 3$ this is obvious, as we may take $b = 1$. For $i = 2$, consider a rational number $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ with $q \neq 0$. Lagrange's four-square theorem implies that we have $pq = a^2 + b^2 + c^2 + d^2$ for integers a, b, c, d. Dividing by q^2, we deduce that $\frac{p}{q}$ is a sum of four rational squares. ◻

Proof of Lemma 3.8.3. By [Bro82, §V.6.2, V.6.3], for any abelian group A we have

$$\Lambda^*\Big(\bigoplus_I A\Big) \cong \Lambda^*\Big(\operatorname*{colim}_{J \subseteq I} \bigoplus_J A\Big) \cong \operatorname*{colim}_{J \subseteq I} \Lambda^*\Big(\bigoplus_J A\Big) \cong \operatorname*{colim}_{J \subseteq I} \bigotimes_J \Lambda^*(A), \tag{3.27}$$

where the colimit is taken over finite subsets J of I. For any finite set J, the canonical map

$$\bigotimes_J \Lambda^*(\mathbb{Z}) \longrightarrow \bigotimes_J \Lambda^*(\mathbb{Q})$$

is injective by Lemma 3.8.4 and the natural isomorphisms (3.26). Thus (3.25) is also injective since the colimit on the right-hand side of (3.27), for $A = \mathbb{Z}$ or $A = \mathbb{Q}$, is taken over a direct system in which all maps are inclusions of direct summands. ◻

3.9 Appendix B: Extending homeomorphisms to Freudenthal compactifications

Notation 3.9.1. For a surface S, recall that we denote by $\overline{S}$ its *Freudenthal compactification* (see §3.1.1). We will write $\mathcal{P}(S) = \mathrm{Ends}(S) \smallsetminus \mathrm{Ends}_{np}(S)$ for its space of *planar* ends. We will also write $\widehat{S} \subseteq \overline{S}$ for the subspace $\overline{S} \smallsetminus \mathrm{Ends}_{np}(S)$ where we have removed all non-planar ends from $\overline{S}$. Equivalently, it is the subspace of $\overline{S}$ consisting of all of its locally Euclidean points: in other words it is the maximal subspace that is a surface. Intuitively, $\widehat{S}$ is the result of "filling in" all planar ends $\mathcal{P}(S)$ of S.

Since every homeomorphism of S extends (necessarily uniquely) to $\overline{S}$ and every homeomorphism of $\overline{S}$ sends $\widehat{S}$ onto itself, we have well-defined injective functions

$$\mathrm{Homeo}(S) \longrightarrow \mathrm{Homeo}(\overline{S}) \longrightarrow \mathrm{Homeo}(\widehat{S}). \tag{3.28}$$

Proposition 3.9.2. *With respect to the compact-open topology, the left-hand function in (3.28) is a topological embedding and the right-hand function is a homeomorphism.*

Proof. The fact that the left-hand map is a topological embedding follows from Proposition 3.9.5 below applied to $X = S$. To deal with the right-hand map, first note that $\overline{S}$ is the Freudenthal compactification of $\widehat{S}$ (as well as of S), so we have a well-defined function

$$\mathrm{Homeo}(\widehat{S}) \longrightarrow \mathrm{Homeo}(\overline{S}), \tag{3.29}$$

given by extending homeomorphisms uniquely. It is evidently a set-theoretic inverse for the restriction map $\mathrm{Homeo}(\overline{S}) \to \mathrm{Homeo}(\widehat{S})$; hence both (3.29) and this restriction map are bijections. Now applying Proposition 3.9.5 to $X = \widehat{S}$, we deduce that (3.29) is a topological embedding. Since it is also a bijection, this means that it is a homeomorphism, and hence so is its inverse, which is the restriction map on the right-hand side of (3.28). ◻

Corollary 3.9.3. *There is an isomorphism of topological groups* $\mathrm{Homeo}(S) \cong \mathrm{Homeo}(\widehat{S}, \mathcal{P}(S))$.

Proof. This follows directly from Proposition 3.9.2, together with the observation that the image of the composite topological embedding (3.28) is precisely $\mathrm{Homeo}(\widehat{S}, \mathcal{P}(S))$, the subspace of $\mathrm{Homeo}(\widehat{S})$ of homeomorphisms sending $\mathcal{P}(S)$ onto itself. ◻

Remark 3.9.4. Corollary 3.9.3 says that filling in the planar ends of a surface and then fixing them setwise does not change anything at the level of homeomorphism groups. This generalises the usual dichotomy between thinking of punctures (isolated planar ends) either as punctures or as marked points.

Proposition 3.9.5. *Let X be a connected, locally connected, locally compact, Hausdorff and second countable space, write $\overline{X}$ for its Freudenthal compactification and give all homeomorphism groups the compact-open topology. Then the injective function* $\mathrm{Homeo}(X) \to \mathrm{Homeo}(\overline{X})$ *given by unique extensions of homeomorphsims is a topological embedding, in particular it is continuous.*

Proof. We begin by rephrasing the statement. The topology on $\mathrm{Homeo}(X)$ induced from the compact-open topology on $\mathrm{Homeo}(\overline{X})$ via the injection $\mathrm{Homeo}(X) \to \mathrm{Homeo}(\overline{X})$ is called the *F-topology*. What we must show is that the F-topology coincides with the compact-open topology. (For the weaker statement that $\mathrm{Homeo}(X) \to \mathrm{Homeo}(\overline{X})$ is continuous, rather than a topological embedding, we would just have to show that the compact-open topology is finer than the F-topology.)

The collection of topologies on $\mathrm{Homeo}(X)$ making both the group operation and the evaluation map $\mathrm{Homeo}(X) \times X \to X$ continuous was studied in [Are46], where it was proven that there exists a *minimum* such topology if X is locally compact and Hausdorff. Moreover, if X is also locally connected, this minimal topology is the compact-open topology. On the other hand, it is proven in [Di 06] that, if X is rim-compact, Hausdorff and $\overline{X}$ is locally connected at any ideal point, then the F-topology is minimal. Thus, if both sets of hypotheses are satisfied, we may conclude that the F-topology coincides with the compact-open topology, as desired. Indeed, the assumptions of the proposition do imply both sets of hypotheses: in particular rim-compactness is weaker than local compactness (which we have assumed) and our assumptions also imply that the Freudenthal compactification $\overline{X}$ is locally connected at any ideal point; see the paragraph before Theorem 9 in [Di 13]. ◻

Part II

Motion groups and mapping class groups: representations and lower central series

Chapter 4

Lower central series of partitioned surface braid groups

The results of this chapter form part of the monograph [DPS22b] (joint work with Jacques Darné and Arthur Soulié), which is accepted for publication in the Memoirs of the American Mathematical Society.

Note. The results of [DPS22b] concern the lower central series of (partitioned) *surface braid groups*, *virtual braid groups* and *welded braid groups* (and further generalisations of the last of these); in the interests of space, only the part of [DPS22b] concerning surface braid groups is reproduced here.

Introduction

One of the most basic objects one needs to understand when studying the structure of a group G is its *lower central series* (shortened to "LCS") $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$. Its behaviour varies greatly from one group to another. For instance, if G is perfect (i.e. all its elements can be written as products of commutators), its LCS is completely trivial; this holds for instance for mapping class groups of closed surfaces of genus $g \geqslant 3$ [Kor02, Thm 5.1]. On the contrary, if G is nilpotent, or residually nilpotent, its LCS contains deep information about the structure of G; examples of residually nilpotent groups include free groups [MKS04, Chap. 5], pure braid groups [FR85; FR88], pure braid groups on surfaces [BB09b; BG16], pure welded braid groups [BP09, §5.5], and conjecturally pure virtual braid groups [Bar+16]. The LCS is also deeply connected to the structure of the group ring of G. In particular, Quillen [Qui68] proved that if we consider the filtration of the group ring $\mathbb{Q}G$ by the powers of its augmentation ideal, then the associated graded algebra is isomorphic to the universal enveloping algebra of the Lie algebra $\mathcal{L}(G) \otimes \mathbb{Q}$, where $\mathcal{L}(G)$ is the graded Lie ring obtained from $\Gamma_*(G)$.

The amount of information one can hope to extract from the study of a LCS depends in the first place on whether or not it *stops* in the following sense:

Definition 4.0.1. The LCS of a group G is said to *stop* if there exists an integer $i \geqslant 1$ such that $\Gamma_i(G) = \Gamma_{i+1}(G)$. We say that it *stops at* Γ_i if i is the smallest integer for which this holds. Otherwise, we say the LCS *does not stop* or else that it *stops at* ∞.

It follows from the definition of the LCS (recalled in §4.1.1 below) that if $\Gamma_i(G) = \Gamma_{i+1}(G)$ for some $i \geqslant 1$, then $\Gamma_k(G) = \Gamma_{k+1}(G)$ for all $k \geqslant i$, whence our choice of terminology.

Partitioned braid groups

In this chapter, we study the LCS of the surface braid group $\mathbf{B}_n(S)$ for any surface S, as well as its *partitioned* versions, in the sense we describe now.

There is a notion of the *underlying permutation* of an element of $\mathbf{B}_n(S)$, corresponding to a canonical surjection $\pi\colon \mathbf{B}_n(S) \twoheadrightarrow \mathfrak{S}_n$ to the symmetric group, from which we can define *partitioned* versions of $\mathbf{B}_n(S)$. Let us first fix our conventions concerning partitions of integers:

Definition 4.0.2. Let $n \geqslant 1$ be an integer. A *partition of* n is an l-tuple $\lambda = (n_1, \dots, n_l)$ of integers $n_i \geqslant 1$, for some $l \geqslant 1$ called the *length* of λ, such that n is the sum of the n_i. Given such a λ, for $j \leqslant l$, let us define $t_j := \sum_{i \leqslant j} n_i$, including $t_0 = 0$. Then the set $b_j(\lambda) := \{t_{j-1}+1, \dots, t_j\}$ is referred to as the *j-th block* of λ, and n_i is called the *size* of the i-th block.

For $\lambda = (n_1, \dots, n_l)$ a partition of n, we consider the preimage

$$\mathbf{B}_\lambda(S) := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right),$$

which is called the *λ-partitioned* version of $\mathbf{B}_n(S)$. There are two extremal situations: the trivial partition $\lambda = (n)$ simply gives the group $\mathbf{B}_n(S)$, whereas the discrete partition $\lambda = (1, 1, \dots, 1)$ corresponds to the subgroup of *pure* braids in $\mathbf{B}_n(S)$.

As we will see later on, the LCS of $\mathbf{B}_n(S)$ stops at $\varGamma_2$ or $\varGamma_3$ (when n is at least 3), whereas the LCS of the subgroup of pure braids is a very complex object (in particular, it does not stop, when n is at least 2 or 3, depending on the surface S). We can thus expect the partitioned braid groups $\mathbf{B}_\lambda(S)$ to display a range of intermediate behaviours when λ varies, and this is indeed what we observe.

Methods

A fundamental tool in the study of LCS is the graded Lie ring structure on the associated graded $\mathcal{L}(G) := \bigoplus_{i \geqslant 1} \varGamma_i(G)/\varGamma_{i+1}(G)$. Namely, this is a graded abelian group endowed with a Lie bracket induced by commutators in G. It is always generated, as a Lie algebra over $\mathbb{Z}$, by its degree one piece, which is the abelianisation $G^{\mathrm{ab}} = G/\varGamma_2(G)$. This often allows one to use *disjoint support arguments* to show that the LCS stops, when it does. Precisely, if one can show that pairs of generators of G^{ab} have commuting representatives in G (which is the case if they have representatives whose *supports* are disjoint, for a certain notion of support), then, by definition of the Lie bracket, they commute in $\mathcal{L}(G)$. In this case, $\mathcal{L}(G)$ is abelian, and it is generated by G^{ab}, which means that it is reduced to G^{ab}. In turn, that means that $\varGamma_i(G) = \varGamma_{i+1}(G)$ whenever $i \geqslant 2$. This type of argument is used throughout the chapter.

One other main line of argumentation for studying LCS is given by *looking for quotients whose LCS is well-understood*. Namely, if we can find a quotient of G whose LCS does not stop, then neither does the LCS of G; see Lemma 4.1.1. Typically, we look for a quotient that is a semi-direct product of an abelian group with $\mathbb{Z}$ or $\mathbb{Z}/2$, a free product of abelian groups or a wreath product of an abelian group with some $\mathfrak{S}_\lambda$, whose LCS can be computed completely; see Appendix 4.5.

Finally, a very important tool in our analysis is the study of the *quotient by the residue*. Precisely, if we denote by $\varGamma_\infty(G)$ (abbreviated $\varGamma_\infty$ when the context is clear) the intersection of the $\varGamma_i(G)$, the LCS of G is "the same" as the LCS of $G/\varGamma_\infty$: each $\varGamma_i(G)$ is the preimage of $\varGamma_i(G/\varGamma_\infty)$ by the canonical projection, and this projection induces an isomorphism between $\mathcal{L}(G)$ and $\mathcal{L}(G/\varGamma_\infty)$. In particular, one of $\varGamma_*(G)$ and $\varGamma_*(G/\varGamma_\infty)$ stops if and only if the other does, which happens exactly when $G/\varGamma_\infty$ is nilpotent. Considering $G/\varGamma_\infty$ instead of G can lead to very important simplifications. Let us illustrate this by an example, variations of which are used throughout the chapter. We know that $\varGamma_\infty = \varGamma_2$ for $\mathbf{B}_n$, and that $\varGamma_2$ contains the elements $\sigma_i\sigma_j^{-1}$ of $\mathbf{B}_n$. Thus whenever we have a morphism $\mathbf{B}_n \to G$, the subgroup $\varGamma_\infty(G)$ must contain the image of $\varGamma_\infty(\mathbf{B}_n)$, which contains the images of $\sigma_i\sigma_j^{-1}$, so all the σ_i have the same image in $G/\varGamma_\infty$.

Results

Does the LCS stop? We give a complete answer to this question for all of the families of groups listed above, with the single exception of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ with $m \geqslant 3$ (see Conjecture 4.4.95). We also obtain some information about the associated Lie rings: in particular, we completely compute the Lie ring of $\mathbf{B}_\lambda(S)$ in the stable case.

Besides their intrinsic value, these results have several applications, notably to the representation theory of braid groups and their relatives. See for instance [BGG17] and the work of the second and third authors [PS19], where the knowledge of the structure of such LCS is key in the construction and the study of representations of these groups using homological approaches. Furthermore, let us mention that one can see surface braid groups and their LCS as invariants of the surfaces themselves; as an application of this point of view, we recover the Riemann-Hurwitz formula for coverings of closed surfaces in Remark 4.4.47.

We summarise our results in three tables on pages 86–88, organised as follows:

- Table 4.1 gathers the *stable* cases, which are those where the blocks of the partitions are large enough for the *disjoint support argument* described above to be applied readily.
- Table 4.2 gathers the cases where there are blocks in the partitions which are too small for the disjoint support argument to be applied readily, but not too many of them, so that the LCS still stops.
- Table 4.3 gathers the cases where the LCS does not stop.

Some of our results have already been obtained in the literature, by different methods. Namely, the question of whether the lower central series stops has already been studied:

- by Gorin and Lin [GL69] for $\mathbf{B}_n$ and Kohno [Koh85] for the pure braid group $\mathbf{P}_n = \mathbf{B}_{1,\ldots,1}$, which is moreover known to be residually nilpotent by Falk and Randell [FR85; FR88].
- by Bellingeri, Gervais and Guaschi [BGG08] for $\mathbf{B}_n(S)$ where S is a compact, connected, orientable surface with or without boundary.
- by Bellingeri and Gervais [BG16] for the pure surface braid group $\mathbf{P}_n(S)$ where S is a compact, connected, non-orientable surface with or without boundary and different from the projective plane $\mathbb{P}^2$.
- by Gonçalves and Guaschi [GG09a; GG09b] for $\mathbf{B}_n(\mathbb{S}^2)$ and $\mathbf{B}_n(\mathbb{S}^2 - \mathcal{P})$ where $\mathcal{P}$ is a finite set of points in $\mathbb{S}^2$.
- by Guaschi and de Miranda e Pereiro [GM20] for $\mathbf{B}_n(S)$ where S is a compact, connected, non-orientable surface without boundary.
- by van Buskirk [Bus66] and by Gonçalves and Guaschi [GG04b; GG11; GG07], both for the braid group on the projective plane $\mathbf{B}_n(\mathbb{P}^2)$.

Notation in the tables

The letter $\lambda = (n_1, \ldots, n_l)$ denotes a partition of n of length $l \geqslant 1$. The letter μ denotes a partition that is either empty or whose blocks have size at least 3. On the other hand, ν denotes any partition (possibly empty, unless stated otherwise).

In Table 4.2, the function f is defined by $f(m) = \max\{v_2(m), 1\}$, where v_2 is the 2-adic valuation. The number ϵ is either 0 or 1 (the precise value may depend on the case, in particular on m, and is unknown, although we conjecture that it is always 1 for m even and 0 for m odd, so that $f(m) + \epsilon = v_2(m) + 1$ in all cases).

The letter S denotes any connected surface (not necessarily compact nor orientable, and possibly with boundary). Six exceptional surfaces are mentioned in the tables, denoted by $\mathbb{D}$ (the disc), $\mathbb{D} - pt$ (the disc minus an interior point), $\mathbb{T}^2$ (the torus), $\mathbb{M}^2$ (the Möbius strip), $\mathbb{S}^2$ (the 2-sphere) and $\mathbb{P}^2$ (the projective plane). A surface S is called *generic* if it is not one of these six exceptional surfaces.

The symbol (†) in front of a family of groups indicates that the result concerned is already partly known in the literature quoted above.

<table>
<tr><th colspan="6">The stable cases</th></tr>
<tr><th colspan="2">Family of groups</th><th>Partition</th><th>Stops at Γ_k</th><th>Ref.</th><th>Lie Alg.</th></tr>
<tr><td colspan="2">Classical braids $\mathbf{B}_\lambda$</td><td>$n_i \geqslant 3$ (†)</td><td>$k = 2$</td><td>4.3.6 (4.3.7)</td><td>4.3.5</td></tr>
<tr><td rowspan="4">Surface braids $\mathbf{B}_\lambda(S)$</td><td>$S \subseteq \mathbb{S}^2$</td><td>$n_i \geqslant 3$ (†)</td><td>$k = 2$</td><td rowspan="4">4.4.53</td><td>4.4.48</td></tr>
<tr><td>$S \nsubseteq \mathbb{S}^2$, orientable</td><td>$n_i \geqslant 3$ (†)</td><td>$k = 3$</td><td>§4.4.5</td></tr>
<tr><td rowspan="2">S non-orientable</td><td>$l = 1,\ n_1 \geqslant 3$</td><td>$k = 2$</td><td>4.4.33</td></tr>
<tr><td>$l \geqslant 2,\ n_i \geqslant 3$</td><td>$k = 3$</td><td>§4.4.5</td></tr>
</table>

Table 4.1 The stable cases.

The unstable cases for which the LCS stops

Family of groups		Partition	Stops at Γ_k	Ref.	Lie Alg.
Classical braids $\mathbf{B}_\lambda$		(2)	$k=2$	$\mathbf{B}_2 \cong \mathbb{Z}$	
		$(1,\mu)$, $(1,1,\mu)$		4.3.6 (4.3.10, 4.3.11)	4.3.5
Surface braids $\mathbf{B}_\lambda(S)$	$S=\mathbb{D}-pt$	$(1,\mu)$	$k=2$	4.4.64	4.3.5
	$S=\mathbb{T}^2$	(1)	$k=2$	$\mathbf{B}_1(\mathbb{T}^2)\cong\mathbb{Z}^2$	
		$(1,\mu)$, $\mu\neq\varnothing$	$k=3$	4.4.66	4.4.48 and 4.4.57
	$S=\mathbb{M}^2$	(1)	$k=2$	$\mathbf{B}_1(\mathbb{M}^2)\cong\mathbb{Z}$	
	$S=\mathbb{S}^2$	(2) or $(2,1)$	$k=2$	$\mathbf{B}_2(\mathbb{S}^2)\cong\mathbb{Z}/2$, $\mathbf{B}_{2,1}(\mathbb{S}^2)\cong\mathbb{Z}/4$	
		$(1,\mu)$, $(1,1,\mu)$, $(1,1,1,\mu)$		4.4.71	4.4.48
		$(2,m)$, $m\geqslant 3$	$k=f(m)+1+\epsilon$	4.4.74	–
	$S=\mathbb{P}^2$	$(1,m)$, $m\geqslant 3$	$k=f(m)+2+\epsilon$	4.4.81	–
		(1)	$k=2$	$\mathbf{B}_1(\mathbb{P}^2)\cong\mathbb{Z}/2$	
		$(1,1)$	$k=3$	$\mathbf{B}_{1,1}(\mathbb{P}^2)\cong Q_8$ (4.4.83)	
		(2)	$k=4$	$\mathbf{B}_2(\mathbb{P}^2)\cong Dic_{16}$ (4.4.83)	

Table 4.2 The unstable cases for which the LCS stops.

The unstable cases for which the LCS does not stop			
Family of groups		Partition	Ref.
Classical braids $\mathbf{B}_\lambda$		$(1,1,1,\nu)$ (†)	4.3.6 (4.3.8)
		$(2,\nu)$, $l \geqslant 2$	4.3.6 (4.3.12, 4.3.15, 4.3.18)
Surface braids $\mathbf{B}_\lambda(S)$	S generic	$(1,\nu)$, $(2,\nu)$ (†)	4.4.63
	$S = \mathbb{D} - pt$	$(1,1,\nu)$	4.4.64
		$(2,\nu)$	4.4.63
	$S = \mathbb{T}^2$	$(1,1,\nu)$	4.4.66
		$(2,\nu)$ (†)	4.4.66 (4.4.63)
	$S = \mathbb{M}^2$	$(1,\nu)$, $l \geqslant 2$	4.4.63, 4.4.68, 4.4.69
		$(2,\nu)$	4.4.63
	$S = \mathbb{S}^2$	$(1,1,1,1,\nu)$	4.4.71 (4.4.72)
		$(2,\nu)$, $l \geqslant 3$ or $(2,2)$	4.4.71 (4.4.73, 4.4.76)
	$S = \mathbb{P}^2$	$(1,\nu)$, $l \geqslant 3$	4.4.80
		$(2,\nu)$, $l \geqslant 3$ or $(2,2)$ or $(2,1)$	4.4.84, 4.4.92, 4.4.89

Table 4.3 The unstable cases for which the LCS does not stop.

4.1 General recollections

In this chapter, we recall some classical notions and tools to study the lower central series of groups. These will be used throughout the chapter.

4.1.1 Commutator calculus and lower central series

Let G be a group. Recall that the *lower central series* (LCS) of G is the descending sequence of normal subgroups $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$ of G, also denoted by $\Gamma_*(G)$, defined by

$$\Gamma_i(G) := \begin{cases} G & \text{if } i = 1, \\ [G, \Gamma_{i-1}(G)] & \text{if } i \geqslant 2, \end{cases}$$

where $[G, \Gamma_i(G)]$ is the subgroup of G generated by all commutators $[x,y] := xyx^{-1}y^{-1}$ with x in G and y in $\Gamma_{i-1}(G)$. The subgroups $\Gamma_i(G)$ are fully invariant, and in particular normal in G. As a consequence, one can also think of the LCS as an ascending chain of quotients $G/\Gamma_i(G)$ of G. Recall that the abelianisation G^{ab} of G is the first of these quotients, namely $G/\Gamma_2(G)$. In general, $G/\Gamma_{c+1}(G)$ is the universal c-nilpotent quotient of G (recall that G is called c-nilpotent if $\Gamma_{c+1}(G) = \{1\}$). The group G is called *residually nilpotent* if its *residue* $\Gamma_\infty(G) := \bigcap \Gamma_i(G)$ is equal to $\{1\}$. The quotient $G/\Gamma_\infty(G)$ is the universal (and, in particular, the largest) residually nilpotent quotient of G.

The following lemma is folklore and will be used very often in the sequel. In particular, we will often use its contrapositive: if a group G has a quotient whose LCS does not stop, then the LCS of G does not stop either.

Lemma 4.1.1. *Let H be a quotient of G. If $\Gamma_i(G) = \Gamma_{i+1}(G)$ for some i, then $\Gamma_i(H) = \Gamma_{i+1}(H)$.*

Proof. For all $k \geqslant 1$, it follows from the definition of the LCS that $\Gamma_k H = \pi(\Gamma_k G)$. As a consequence, $\Gamma_{i+1} H = \Gamma_i H$ whenever $\Gamma_{i+1} G = \Gamma_i G$. □

We now give two partial converses to Lemma 4.1.1. The first one (also the most obvious one) is the case of a quotient by a subgroup having some finiteness properties. Since most of the extensions that we consider in the sequel will *not* satisfy the required hypothesis, we will use it only once, in the very simple case when the kernel is cyclic of order two. Nevertheless, we state it in a general framework:

Lemma 4.1.2. *Let $K \hookrightarrow G \twoheadrightarrow H$ be a short exact sequence of groups. Suppose that there exists $l \geqslant 0$ such that every strictly decreasing central filtration of K stops after at most l steps, that is, if $K = K_1 \supsetneq K_2 \supsetneq \cdots \supsetneq K_m$ is a nested sequence of subgroups satisfying $[K, K_i] \subset K_{i+1}$, then $m \leqslant l$. Suppose moreover that for some $i \geqslant 1$, we have $\varGamma_{i+1}H = \varGamma_i H$. Then $\varGamma_{i+l+1}G = \varGamma_{i+l}G$.*

Proof. The filtration $K \cap \varGamma_*(G)$ is a central filtration of K, so it can strictly decrease only l times. If $\varGamma_{i+1}H = \varGamma_i H$, then $\varGamma_i(H) = \varGamma_{i+k}(H)$ is the image of $\varGamma_{i+k}(G)$ in H, for all $k \geqslant 0$. Recall that if L and M are subgroups of G such that $L \subseteq M$, then L and M are equal if and only if their image in H and their intersection with K are equal. As a consequence, for $\varGamma_{i+k}(G)$ to decrease when k grows, its intersection with the kernel must decrease, which can happen at most l times. So the LCS of G must stop at most at $\varGamma_{i+l}$. ◻

Example 4.1.3. If K is finite, such an l clearly exists. In fact, for any central filtration K_* on K, since the cardinal of K/K_m is the product of the cardinals of the K_i/K_{i+1}, one can take l to be the number of prime factors in the cardinal of K (a bound that is optimal if K is abelian). In particular, we will apply this with $K \cong \mathbb{Z}/2$ and $l = 1$ in the proof of Proposition 4.4.81.

Remark 4.1.4. Finite groups are not the only ones for which the hypothesis holds. We could for instance apply the Lemma with K simple, or more generally with K perfect (with $l = 0$). Also, the class of groups K satisfying this hypothesis is stable by extensions. In fact, an equivalent way of stating it is to ask that the maximal residually nilpotent quotient of K (that is, $K/\varGamma_\infty(K)$) is finite.

The other partial converse to Lemma 4.1.1 that we will use concerns quotients by central subgroups. This case is a bit more subtle, and requires the following result, which can be useful when calculating quotients by residues.

Proposition 4.1.5. *Let G be a group, and let N be a normal subgroup of G. Suppose that for some $i \geqslant 2$, $N \cap \varGamma_i(G) = \{1\}$. Then the canonical morphism $\varGamma_\infty(G) \to \varGamma_\infty(G/N)$ is an isomorphism. In particular, G is residually nilpotent if and only if G/N is. Moreover, we have a short exact sequence:*

$$N \hookrightarrow G/\varGamma_\infty \twoheadrightarrow (G/N)/\varGamma_\infty.$$

Proof. Let $\pi\colon G \twoheadrightarrow G/N$ be the canonical projection. Since $N \cap \varGamma_\infty(G) = \{1\}$, the induced morphism $\varGamma_\infty(G) \to \varGamma_\infty(G/N)$ is injective. Let us show that it is surjective. Let $y \in \varGamma_\infty(G/N)$. Since $\varGamma_k(G/N) = \pi(\varGamma_k G)$ by definition of the LCS, there is, for each $k \geqslant 1$, some $x_k \in \varGamma_k(G)$ such that $\pi(x_k) = y$. Then $x_k x_{k+1}^{-1} \in N \cap \varGamma_k(G)$, which implies that $x_k x_{k+1}^{-1} = 1$ whenever $k \geqslant i$. Thus the sequence (x_k) is stationary at $x := x_i$, which must be in $\varGamma_\infty(G)$ by definition of the x_k, and is sent to y by π. This proves the first part of the Proposition. Now, let us consider the commutative diagram of groups:

$$
\begin{array}{ccccc}
1 & \hookrightarrow & \varGamma_\infty(G) & \xrightarrow{\cong} & \varGamma_\infty(G/N) \\
\downarrow & & \downarrow & & \downarrow \\
N & \hookrightarrow & G & \twoheadrightarrow & G/N \\
\downarrow{=} & & \downarrow & & \downarrow \\
N & \dashrightarrow & G/\varGamma_\infty & \dashrightarrow & (G/N)/\varGamma_\infty.
\end{array}
$$

By the Nine Lemma, the bottom row must be a short exact sequence. ◻

The following corollary provides the promised partial converse to Lemma 4.1.1:

Corollary 4.1.6. *Let G be a group and A be a central subgroup of G. Suppose that for some $i \geqslant 2$, the canonical map $A \to G/\Gamma_i(G)$ is injective. If the LCS of G/A stops at Γ_k, then the LCS of G stops at Γ_k or at Γ_{k+1}.*

Proof. The extension $A \hookrightarrow G/\Gamma_\infty \twoheadrightarrow (G/A)/\Gamma_\infty$ from Proposition 4.1.5 is a central one. As a consequence, if $(G/A)/\Gamma_\infty$ is k-nilpotent, then G/Γ_∞ is nilpotent of class k or $k+1$. □

Remark 4.1.7. In explicit examples, the hypothesis of Corollary 4.1.6 is most easily checked when $i = 2$, since we typically have a complete understanding of G^{ab}; see for example the proof of Proposition 4.4.74.

4.1.2 Lie rings of lower central series

We now recall the definition and basic properties of a key tool for studying the LCS of a group, namely its associated Lie ring. We refer the reader to [Laz54, Chap. 1] for further details.

Note that, for all $i \geqslant 1$, $[\Gamma_i(G), \Gamma_i(G)] \subseteq [G, \Gamma_i(G)] \subseteq \Gamma_{i+1}(G) \subseteq \Gamma_i(G)$. Thus, $\Gamma_i(G)$ is a normal subgroup of G, and the quotient $\mathcal{L}_i(G) := \Gamma_i(G)/\Gamma_{i+1}(G)$ is an abelian group. Moreover, one can show that $[\Gamma_i(G), \Gamma_j(G)] \subseteq \Gamma_{i+j}(G)$ for all $i, j \geqslant 1$, which is the crucial property allowing us to define the Lie ring associated with $\Gamma_*(G)$:

Proposition 4.1.8 ([Laz54, Th. 2.1])**.** *The graded abelian group defined by $\mathcal{L}(G) := \bigoplus_{i \geqslant 1} \mathcal{L}_i(G)$ is a Lie ring, with the Lie bracket induced by the commutator map of G.*

Convention 4.1.9. Let g be an element of G. If there is an integer d such that $g \in \Gamma_d(G) - \Gamma_{d+1}(G)$, it is obviously unique. We then call d the *degree* of g with respect to $\Gamma_*(G)$. The notation $\overline{g}$ denotes the class of g in some quotient $\mathcal{L}_i(G)$. If the integer i is not specified, it is assumed that $i = d$, which means that $\overline{g}$ denotes the only non-trivial class induced by g in $\mathcal{L}(G)$. If such a d does not exist (that is, if $g \in \bigcap \Gamma_i(G)$), we say that g has degree ∞ and we put $\overline{g} = 0$.

With this convention, the Lie bracket $[-,-]$ of $\mathcal{L}(G)$ is given by the collection of bilinear maps $\mathcal{L}_i(G) \times \mathcal{L}_j(G) \to \mathcal{L}_{i+j}(G)$ defined by:

$$\forall x \in \mathcal{L}_i(G),\ \forall y \in \mathcal{L}_j(G),\ [\overline{x}, \overline{y}] := \overline{[x, y]} \in \mathcal{L}_{i+j}(G).$$

The following lemma, which will be used several times in the sequel to identify G/Γ_∞ for some group G, is one illustration of the use of Lie rings in studying the LCS:

Lemma 4.1.10. *Let $p\colon G \twoheadrightarrow Q$ be a surjective group morphism. If Q is a residually nilpotent group, then the following conditions are equivalent:*

- *$\mathcal{L}(p)\colon \mathcal{L}(G) \twoheadrightarrow \mathcal{L}(Q)$ is an isomorphism.*
- *p induces an isomorphism $G/\Gamma_\infty \cong Q$.*

Proof. If Q is residually nilpotent, p induces a map $G/\Gamma_\infty \twoheadrightarrow Q$ between two residually nilpotent groups. Since $G \twoheadrightarrow G/\Gamma_\infty$ induces an isomorphism between the associated Lie rings, the statement for G can be deduced from the statement for G/Γ_∞. Thus, we can assume that G is residually nilpotent and, under this hypothesis, we need to show that p is an isomorphism if and only if $\mathcal{L}(p)$ is. Clearly, if p is an isomorphism, then $\mathcal{L}(p)$ is too. Conversely, if p, which is surjective, is not an isomorphism, then there is some non-trivial element x in its kernel. Since G is residually nilpotent, x induces a non-trivial class $\overline{x}$ in $\mathcal{L}(G)$, which is sent to 0 by $\mathcal{L}(p)$. This implies that $\mathcal{L}(p)$ is not injective, which concludes our proof. □

4.1.3 Computing abelianisations from decompositions

Let us recall some classical tools for computing the abelianisation from some decomposition of a given group. The abelianisation functor $G \mapsto G^{\mathrm{ab}}$ is a left adjoint, hence right exact. In order to compute the abelianisation of an extension, one can say more. Given a short exact sequence of groups $H \hookrightarrow G \twoheadrightarrow K$, let us denote by $(H^{\mathrm{ab}})_K$ the coinvariants of H^{ab} with respect to the action of K on H^{ab} induced by conjugation in G.

Lemma 4.1.11. *The short exact sequence $H \hookrightarrow G \twoheadrightarrow K$ induces the following exact sequence of abelian groups:*

$$(H^{\mathrm{ab}})_K \to G^{\mathrm{ab}} \to K^{\mathrm{ab}} \to 0.$$

Proof. The conjugation action of G on H^{ab} factors through $G/H = K$, hence $(H^{\mathrm{ab}})_G = (H^{\mathrm{ab}})_K$. Since we have an exact sequence $H^{\mathrm{ab}} \to G^{\mathrm{ab}} \to K^{\mathrm{ab}} \to 0$, it suffices to show that the morphism $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ factors through $(H^{\mathrm{ab}})_K$. It is equivariant with respect to the action of G induced by conjugation (which is obviously trivial on G^{ab}), whence the result. □

For split exact sequences, we can say even more:

Lemma 4.1.12. *The abelianisation of a semidirect product $H \rtimes K$ is isomorphic to the product $(H^{\mathrm{ab}})_K \times K^{\mathrm{ab}}$.*

Proof. As a consequence of the usual formula $[x, yz] = [x, y](x[x, z]x^{-1})$, one sees that the commutator subgroup $[H \rtimes K, H \rtimes K]$ is normally generated by $[H, H]$, $[H, K]$ and $[K, K]$. We can take the quotient by these three sets of relations successively: $(H \rtimes K)/[H, H]$ is isomorphic to $H^{\mathrm{ab}} \rtimes K$, then killing $[H, K]$ gives $(H^{\mathrm{ab}})_K \times K$ and finally, $((H^{\mathrm{ab}})_K \times K)/[K, K] \cong (H^{\mathrm{ab}})_K \times K^{\mathrm{ab}}$. □

4.2 Strategy and first examples

In this chapter, we present some general ideas used to decide whether the LCS stops or not. As a first example, we then apply these ideas to Artin groups.

4.2.1 Generation in degree one – first consequences

The Lie ring associated to the LCS has the following fundamental property:

Proposition 4.2.1. *The Lie ring $\mathcal{L}(G)$ is generated in degree one. That is, it is generated by the abelianisation $\mathcal{L}_1(G) = G^{\mathrm{ab}}$ as a Lie algebra over $\mathbb{Z}$.*

Proof. It is a direct consequence of the definitions: the equality $\mathcal{L}_k(G) = [\mathcal{L}_1(G), \mathcal{L}_{k-1}(G)]$ is obtained directly from $\Gamma_k(G) = [\Gamma_1(G), \Gamma_{k-1}(G)]$, by passing to the appropriate quotients. □

A first consequence of this is the following:

Corollary 4.2.2. *Let G be a group. If G^{ab} is cyclic, then $\Gamma_2 G = \Gamma_3 G$.*

Proof. Proposition 4.2.1 implies that the Lie ring $\mathcal{L}(G)$ is a quotient of the free Lie ring on G^{ab}. Since G^{ab} is cyclic, the latter is the abelian Lie ring consisting only of G^{ab}. As a consequence, $\Gamma_2 G/\Gamma_3 G = \mathcal{L}_2(G) = \{0\}$. □

Example 4.2.3 (Braids)**.** Directly from their usual presentations, one computes the abelianisation of the braid groups: $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$ for $n \geqslant 2$. Thus $\Gamma_2(\mathbf{B}_n) = \Gamma_3(\mathbf{B}_n)$. This fact is originally due to Gorin and Lin [GL69], who proved it by different methods. This property of $\mathbf{B}_n$, which is also true for any quotient of $\mathbf{B}_n$ (such as the symmetric group $\mathfrak{S}_n$), may also be seen as a particular case of the computations for Artin groups below (Proposition 4.2.11).

Example 4.2.4 (Knot groups)**.** For any knot, the knot group has (infinite) cyclic abelianisation, thus its LCS stops at Γ_2. This generalises readily to the enveloping groups of any connected quandle; see for instance [BNS19, Prop. 3.3].

Example 4.2.5 (Automorphisms of free groups)**.** Consider the automorphism group $\mathrm{Aut}(\mathbf{F}_n)$ of the free group on n letters. The kernel IA_n of the projection from $\mathrm{Aut}(\mathbf{F}_n)$ onto $\mathrm{Aut}(\mathbf{F}_n^{\mathrm{ab}}) \cong \mathrm{GL}_n(\mathbb{Z})$ is generated by the usual K_{ij} and K_{ijk} from [MKS04, §3.5], which are easily seen to be commutators of automorphisms. Thus $\mathrm{Aut}(\mathbf{F}_n)^{\mathrm{ab}} \cong \mathrm{GL}_n(\mathbb{Z})^{\mathrm{ab}}$. Whenever $n \geqslant 3$, this group is cyclic of order two, so the LCS of $\mathrm{Aut}(\mathbf{F}_n)$ stops at Γ_2, and so does the one of $\mathrm{GL}_n(\mathbb{Z})$.

An easy generalisation of Corollary 4.2.2 is:

Corollary 4.2.6. *Let G be a group. Let S be a generating set of G^{ab}. Suppose that, for each pair $(s,t) \in S^2$, we can find representatives $\tilde{s}, \tilde{t} \in G$ of s and t such that $\tilde{s}$ and $\tilde{t}$ commute. Then $\Gamma_2 G = \Gamma_3 G$.*

Proof. The Lie ring $\mathcal{L}(G)$ is generated by S. Moreover, the fact that $\left[\tilde{s}, \tilde{t}\right] = 1$ in G readily implies that $[s,t] = 0$ in $\mathcal{L}(G)$. Since the brackets $[s,t]$ for $(s,t) \in S^2$ generate $\mathcal{L}_2(G) = [\mathcal{L}_1(G), \mathcal{L}_1(G)]$, we see that $\Gamma_2 G/\Gamma_3 G = \mathcal{L}_2(G) = \{0\}$. In fact, $\mathcal{L}(G)$ is an abelian Lie ring, reduced only to $\mathcal{L}_1(G) = G^{\mathrm{ab}}$. ◻

We have not made any effort to make the above corollary as general as possible. In particular, $\tilde{s}$ and $\tilde{t}$ may commute only up to an element of $\Gamma_3 G$, and the conclusion still holds. Also, one may think of similar statements showing that $\Gamma_3 G = \Gamma_4 G$, and so on. Weak as it may seem, our statement is already very useful. In particular, when applied to groups whose elements have a geometric interpretation, it will often happen that $\tilde{s}$ and $\tilde{t}$ can be chosen "*with disjoint support*", which readily implies that they commute. We will sometimes need a more refined version of the above, but we will discuss it in each particular situation.

Example 4.2.7 (Automorphisms of $\mathbf{F}_2$)**.** As an example of a case where Corollary 4.2.6 does not work, but the same kind of technique does apply, let us consider $\mathrm{Aut}(\mathbf{F}_2)$. We have mentioned that $\mathrm{Aut}(\mathbf{F}_n)^{\mathrm{ab}} \cong \mathrm{GL}_n(\mathbb{Z})^{\mathrm{ab}}$; see Example 4.2.5. For $n = 2$, this is no longer cyclic, but isomorphic to $(\mathbb{Z}/2)^2$, generated by the (equivalences classes of the) automorphisms σ and τ acting as follows (fixing free generators x and y of $\mathbf{F}_2$):

$$\sigma(x) = y, \qquad \sigma(y) = x, \qquad\qquad \tau(x) = x^{-1}, \qquad \tau(y) = y.$$

It follows that $\mathcal{L}_2(\mathrm{Aut}(\mathbf{F}_2))$ is generated by the (equivalence class of the) automorphism $\iota = [\sigma, \tau]$ acting by $\iota(x) = x^{-1}$ and $\iota(y) = y^{-1}$. It is easy to check that ι commutes with both σ and τ, so $\mathcal{L}_3(\mathrm{Aut}(\mathbf{F}_2)) = 0$. Thus, the LCS of $\mathrm{Aut}(\mathbf{F}_2)$ stops at Γ_3, as does that of $\mathrm{GL}_2(\mathbb{Z})$.

Let us spell out another useful consequence of Proposition 4.2.1:

Corollary 4.2.8. *Let G be a group and $d \geqslant 1$ be an integer. If $\mathcal{L}_k(G) = \Gamma_k G/\Gamma_{k+1} G$ is a d-torsion abelian group for some $k \geqslant 1$, then $\mathcal{L}_l(G) = \Gamma_l G/\Gamma_{l+1} G$ is too for all integers $l \geqslant k$.*

Proof. If an element $x \in \mathcal{L}(G)$ is of d-torsion, then for all $y \in \mathcal{L}(G)$, the bracket $[x,y]$ is too, because $d \cdot [x,y] = [d \cdot x, y] = 0$. Since $\mathcal{L}_{l+1}(G) = [\mathcal{L}_1(G), \mathcal{L}_l(G)]$, we get our result by induction on l. ◻

Example 4.2.9 (Virtual and welded braids)**.** Let G be a group such that $G^{\mathrm{ab}} \cong \mathbb{Z} \times (\mathbb{Z}/d)$, where the factors are generated, respectively, by u and v. Then $\mathcal{L}_2(G)$ is generated, as an abelian group, by $[u,v]$. Since $d \cdot [u,v] = [u, d \cdot v] = 0$, $\mathcal{L}_2(G)$ is of d-torsion, and then all the $\mathcal{L}_k(G)$, for $k \geqslant 2$, must be too. This applies, with $d = 2$, to the virtual and welded braid groups on two strands $\mathbf{vB}_2$ and $\mathbf{wB}_2$, which are both isomorphic to $\mathbb{Z} * \mathbb{Z}/2$.

4.2.2 Artin groups

Let S be a set, and $M = (m_{s,t})_{s \in S}$ be a Coxeter matrix, i.e. a symmetric matrix with coefficients in $\mathbb{N} \cup \{\infty\}$, with $m_{s,s} = 1$, and $m_{s,t} \geqslant 2$ if $s \neq t$. Let A_M be the associated Artin group, defined by the presentation:

$$A_M = \left\langle S \;\middle|\; \underbrace{stst s \cdots}_{m_{s,t}} = \underbrace{tstst \cdots}_{m_{s,t}} \quad (s,t \in S,\ m_{s,t} \neq \infty) \right\rangle.$$

Let us consider the graph $\mathcal{G}$ whose incidence matrix is M modulo 2. Namely, $\mathcal{G}$ is obtained by taking S as its set of vertices and by drawing an edge between s and t whenever $m_{s,t}$ is an odd integer.

Lemma 4.2.10. *The group A_M^{ab} is free abelian on the set $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$.*

Proof. This is clear from the presentation: in A_M^{ab}, the relation between s and t becomes $\bar{s} = \bar{t}$ if $m_{s,t}$ is odd, and it becomes trivial if $m_{s,t}$ is even or $m_{s,t} = \infty$. □

Let us now study the LCS of A_M. Suppose first that $\mathcal{G}$ is connected. Then A_M^{ab} is cyclic, and Corollary 4.2.2 applies: the LCS of A_M stops at Γ_2. This holds in particular for the classical braid group; see Example 4.2.3. Now, if $\mathcal{G}$ has several connected components, we need to study the interactions between the corresponding generators of the Lie ring of A_M. The simplest case happens when all the even $m_{s,t}$ are equal to 2. Then A_M splits into the (restricted) direct product of the Artin groups corresponding to the connected components of our graph, thus $\mathcal{L}(A_M)$ is a direct sum of copies of $\mathbb{Z}$ (concentrated in degree one). Thus, we obtain a first result (which recovers [BGG08, Prop. 1] for spherical Artin groups):

Proposition 4.2.11. *If all the coefficients of M are finite, and are either odd or equal to 2, then the LCS of A_M stops at Γ_2.*

In order to get a step further, we need to study more closely the interactions between the generators of the Lie ring of A_M corresponding to the connected components of $\mathcal{G}$.

Lemma 4.2.12. *Let $s, t \in S$. If $m_{s,t} = 2k$ for some integer k, then $k \cdot [\bar{s}, \bar{t}] = 0$ in $\mathcal{L}(A_M)$.*

Proof. If $m_{s,t} = 2k$, then $(st)^k = (ts)^k$ in A_M. We can write this relation as $(st)^k(ts)^{-k} = 1$. We recall that modulo $\Gamma_3(A_M)$, commutators commute with any other element. Hence, modulo $\Gamma_3(A_M)$, we have:

$$1 = (st)^k(ts)^{-k} = (st)^{k-1}[s,t](ts)^{-(k-1)} = [s,t](st)^{k-1}(ts)^{-(k-1)} = \cdots = [s,t]^k.$$

Thus, in $\mathcal{L}_2(A_M)$, we have $0 = k \cdot \overline{[s,t]} = k \cdot [\bar{s}, \bar{t}]$, as claimed. □

From this, we deduce that for $x, y \in \pi_0(\mathcal{G})$, we have $d_{x,y} \cdot [x,y] = 0$ in $\mathcal{L}_2(A_M)$, where $d_{x,x} = 1$ and $d_{x,y} = \gcd\left\{ \frac{m_{s,t}}{2} \;\middle|\; s,t \in S,\ \bar{s} = x \text{ and } \bar{t} = y \right\}$ if $x \neq y$. In particular, this proves:

Proposition 4.2.13. *If all the coefficients of M are finite, then $\Gamma_2(A_M)/\Gamma_3(A_M)$ is of d-torsion, where $d = \operatorname{lcm}\{d_{x,y} \mid x, y \in \pi_0(\mathcal{G})\}$.*

Remark 4.2.14. Proposition 4.2.13 is more general than Proposition 4.2.11, which is recovered as a particular case where $d = 1$. In fact, for d to be equal to 1 (which implies that the LCS stops at Γ_2), we do not need the even $m_{s,t}$ to be equal to 2, but only all the $d_{x,y}$ to be equal to 1.

Remark 4.2.15. Proposition 4.2.13 does not say anything when at least one of the $d_{x,y}$ is infinite. For instance, when all the $m_{s,t}$ are infinite, A_M is the free group on S, whose Lie ring is without torsion. More generally, Right-Angled Artin Groups (where all the $m_{s,t}$ are infinite or equal to 2) have torsion-free Lie rings [DK92]. Another example is the subgroup KB_n of the virtual braid group $\mathbf{vB}_n$ given by the normal closure of $\mathbf{B}_n \subset \mathbf{vB}_n$ (denoted by H_n in [BB09a]). This is an Artin group with all $m_{s,t}$ infinite or equal to 2 or 3 [BB09a, Prop. 17]; its abelianisation is free abelian and its LCS stops at Γ_2 for $n \geqslant 3$ [BB09a, Prop. 19].

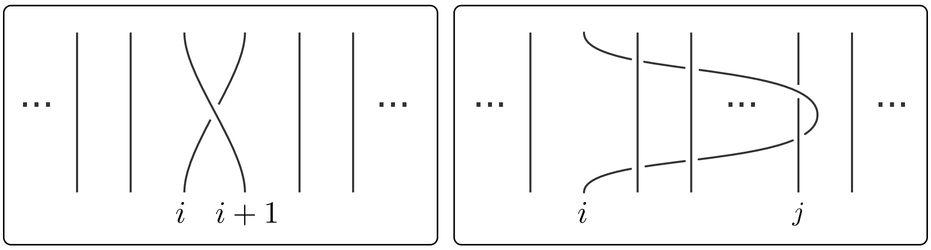


Figure 4.1 The standard generators σ_i and A_{ij} of, respectively, the braid group and the pure braid group.

4.3 Partitioned classical braids

This chapter is devoted to the study of the LCS of the group of *partitioned braids*; see Definition 4.3.2 below. Our main results are summarised in Theorem 4.3.6. The group of partitioned braids is a subgroup of $\mathbf{B}_n$, which has already been studied notably by Manfredini [Man97] and by Bellingeri, Godelle and Guaschi [BGG17]. The former gave a presentation of this group, using the Reidemeister-Schreier method, which may be applied to this finite-index subgroup of Artin's braid group.

Remark 4.3.1. One could use the aforementioned presentation of the partitioned braid group [Man97] to get generators, a calculation of the abelianisation, and (with a little bit of work) the stable case in the study of the LCS. However, we will avoid using this presentation altogether, for several reasons. Firstly, we want the present chapter to be as self-contained as possible. Secondly, even if a presentation is of some help in the study of the LCS of a group, there is only so much that one can deduce directly from it; in fact, not much more than a computation of the abelianisation, and that $\Gamma_2 = \Gamma_3$ when it holds. In particular, most of our non-stable results would not be simplified by using the presentation. Thirdly, and perhaps most importantly, to the best of our knowledge, nobody has written down a presentation of the other partitioned groups that we study later on. This could certainly be done using the Reidemeister-Schreier technique (at least when we have a presentation of the non-partitioned group), but this would require a fair amount of work, which we intend to avoid. We will do so precisely by generalising proofs that do not depend on the use of Manfredini's presentation.

4.3.1 Reminders: braids

We recall that the standard generators σ_i and A_{ij} of the braid group $\mathbf{B}_n$ and of the pure braid group $\mathbf{P}_n$ respectively, are the braids drawn in Figure 4.1. In Figure 4.2, they also appear in a "bird's eye view" as paths of configurations.

4.3.2 Basic theory of partitioned braids

For a partition $\lambda = (n_1, \dots, n_l)$ of an integer n, we denote by $\mathfrak{S}_\lambda$ the subgroup $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ of the symmetric group $\mathfrak{S}_n$. Let us now consider the braid group $\mathbf{B}_n$ on n strands, and the usual projection $\pi\colon \mathbf{B}_n \twoheadrightarrow \mathfrak{S}_n$.

Definition 4.3.2. Let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, \dots, n_l)$ be a partition of n. The corresponding *partitioned braid group* (also referred to as the group of λ-*partitioned braids*) is:

$$\mathbf{B}_\lambda := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right) \subseteq \mathbf{B}_n.$$

Lemma 4.3.3. *Let $\lambda = (n_1, \dots, n_l)$ be a partition of an integer $n \geqslant 1$. Then $\mathbf{B}_\lambda$ is the subgroup of $\mathbf{B}_n$ generated by:*

- *The σ_α for $1 \leqslant \alpha \leqslant n$ such that α and $\alpha + 1$ are in the same block of λ.*
- *The $A_{\alpha\beta}$ for $1 \leqslant \alpha < \beta \leqslant n$ such that α and β do not belong to the same block of λ.*

Proof. Consider the subgroup G of $\mathbf{B}_n$ generated by these elements. Clearly, $G \subseteq \mathbf{B}_\lambda$, and we need to show that G contains $\mathbf{B}_\lambda$. First, we see that G contains $\mathbf{B}_{n_1} \times \cdots \times \mathbf{B}_{n_l}$, which is generated by the chosen σ_α. As a consequence, $\pi(G) = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. Then, G also contains the $A_{\alpha\beta}$, for all $1 \leqslant \alpha < \beta \leqslant n$. Indeed, the $A_{\alpha\beta}$ missing in the list of the statement are the ones with α and β in the same block of λ, which are exactly the ones belonging to $\mathbf{B}_{n_1} \times \cdots \times \mathbf{B}_{n_l}$. Thus, the pure braid group $\mathbf{P}_n$ is contained in G. Now, if β is any λ-partitioned braid, then $\pi(\beta) \in \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$, hence we can choose $g \in G$ such that $\pi(g) = \pi(\beta)$. Then $g^{-1}\beta$ is a pure braid, thus $g^{-1}\beta \in G$, which implies that $\beta \in G$. □

Remark 4.3.4. The generating set of $\mathbf{B}_\lambda$ described in Lemma 4.3.3 is clearly redundant, since $A_{\alpha\beta}$ is conjugate to $A_{\gamma\delta}$ if α (resp. β) is in the same block as γ (resp. δ). For instance, in $\mathbf{B}_{2,2}$, we have $\sigma_1 A_{13} \sigma_1^{-1} = A_{23}$. Since it is more convenient for our purpose, we prefer to keep it that way. Notice however that it is easy to extract a minimal set of generators: using the computation of $\mathbf{B}_\lambda^{\mathrm{ab}}$ below, it is easy to show that keeping all the σ_α and choosing only one lift $A_{\alpha\beta}$ of each a_{ij} (that is, one $A_{\alpha\beta}$ for each pair of blocks) does give a minimal set of generators of $\mathbf{B}_\lambda$. A similar remark (with a similar method for choosing minimal sets of generators) applies to the generating sets described later in this chapter for variants of partitioned braid groups.

We now compute $\mathbf{B}_\lambda^{\mathrm{ab}}$, using the above generating set, together with the split projections corresponding to forgetting blocks of strands. These projections can be seen as particular cases of the projections from Proposition 4.4.15 below, applied to braids on the disc.

Proposition 4.3.5. *Let $\lambda = (n_1, \ldots, n_l)$ be a partition of an integer $n \geqslant 1$. The abelianisation $\mathbf{B}_\lambda^{\mathrm{ab}}$ is free abelian on the following basis:*

- *For each $i \in \{1, \ldots, l\}$ such that $n_i \geqslant 2$, one generator s_i: this is the common class of the σ_α for α and $\alpha + 1$ in the i-th block of λ.*
- *For each $i, j \in \{1, \ldots, l\}$ such that $i < j$, one generator a_{ij}: this is the common class of the $A_{\alpha\beta}$ for α in the i-th block of λ and β in the j-th one.*

Proof. The abelianisation $\mathbf{B}_\lambda^{\mathrm{ab}}$ is generated by the classes of the generators from Lemma 4.3.3. Moreover, we note that for any i, all the σ_α with α and $\alpha + 1$ in the i-th block of λ (which exist only if $n_i \geqslant 2$) are conjugate to one another: for instance, if $n_1 \geqslant 3$, then $\sigma_2 = (\sigma_2\sigma_1)^{-1}\sigma_1(\sigma_2\sigma_1)$ is a conjugation relation inside $\mathbf{B}_\lambda$. Similarly, for each choice of $i < j$, all the $A_{\alpha\beta}$ with α in the i-th block and β in the j-th one are conjugate to one another. Thus, the family described in the statement is well-defined and generates $\mathbf{B}_\lambda^{\mathrm{ab}}$. Let us show that it is linearly independent, by using the projections obtained by forgetting strands.

Suppose that $\sum k_i s_i + \sum k_{ij} a_{ij} = 0$ for some integers k_i ($i \leqslant l$) and k_{ij} ($i < j \leqslant l$). Let us fix i such that $n_i \geqslant 2$, and let us apply the canonical projection $\mathbf{B}_\lambda^{\mathrm{ab}} \twoheadrightarrow \mathbf{B}_{n_i}^{\mathrm{ab}} \cong \mathbb{Z}$ to the above relation. Since this projection kills all the generators save s_i, we get $k_i = 0$. This holds for all i, so we are left with the relation $\sum k_{ij} a_{ij} = 0$. To which, for any choice of $i < j$, we can apply the morphism $\mathbf{B}_\lambda^{\mathrm{ab}} \twoheadrightarrow \mathbf{B}_{n_i,n_j}^{\mathrm{ab}} \to \mathbf{B}_{n_i+n_j}^{\mathrm{ab}} \cong \mathbb{Z}$. This kills all the a_{kl}, save a_{ij} (which is sent to 2), hence $k_{ij} = 0$, whence the result. □

4.3.3 The lower central series

This section is devoted to the proof of the following result, which states exactly when the LCS of the partitioned braid group stops. The proof requires different techniques for a number of different cases. We consider each case in turn and then combine the individual results into a proof of Theorem 4.3.6 at the very end of this section.

Theorem 4.3.6. *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, \ldots, n_l)$ be a partition of n. The LCS of the partitioned braid group $\mathbf{B}_\lambda$:*

- stops at Γ_2 *if $n_i \geqslant 3$ for all i, save at most two indices for which $n_i = 1$.*
- does not stop *in all the other cases, except for $\mathbf{B}_2 \cong \mathbb{Z}$.*

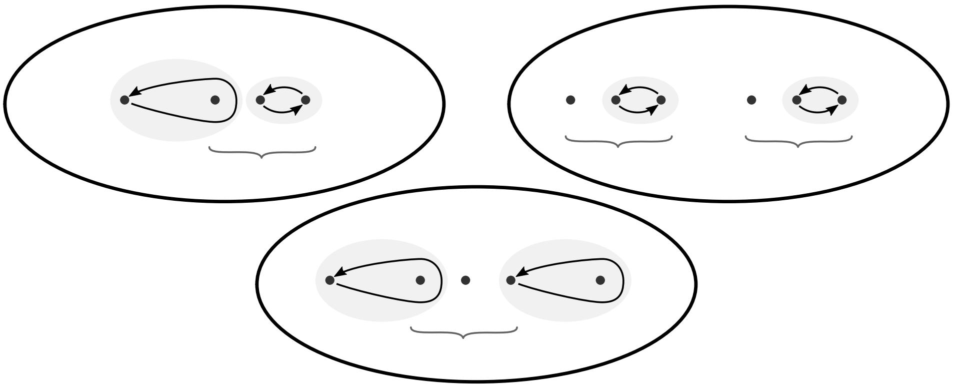


Figure 4.2 Choosing representatives with disjoint support for pairs of generators of the abelianisation of the partitioned braid group. Braces indicate points that lie in the same block of the partition.

The stable case: a disjoint support argument

Proposition 4.3.7. *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, \dots, n_l)$ be a partition of n. Suppose that all the n_i are at least 3. Then the LCS of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. Consider the generating set for $\mathbf{B}_\lambda^{\mathrm{ab}}$ described in Corollary 4.3.5. For any pair of such generators, it is possible to find lifts in $\mathbf{B}_\lambda$ having disjoint support (as mapping classes of the punctured disc), and thus commuting – see Figure 4.2 for the three cases that may arise. Then, we can apply Corollary 4.2.6 to prove our claim. □

Blocks of size 1

Lemma 4.3.8. *If at least three blocks of the partition λ are of size 1, then the LCS of $\mathbf{B}_\lambda$ does not stop.*

Proof. Under the hypothesis, there is a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{1,1,1} \cong \mathbf{P}_3$ obtained by forgetting all the blocks save three blocks of size 1. The LCS of $\mathbf{P}_3 \cong \mathbf{F}_2 \rtimes \mathbb{Z}$ does not stop, since it is an almost-direct product, and the LCS of $\mathbf{F}_2$ does not stop; see for instance [FR85]. As a consequence, the one of $\mathbf{B}_\lambda$ does not either by Lemma 4.1.1. □

The cases with one or two blocks of size 1 (and still no blocks of size 2) are more difficult to handle. In both cases, we will have to use the following observation:

Lemma 4.3.9. *The quotient of $\mathbf{P}_3$ by the relation $[A_{13}, A_{23}] = 1$ is $\mathbf{P}_3^{\mathrm{ab}} \cong \mathbb{Z}^3$.*

Proof. Let N be the normal closure of $[A_{13}, A_{23}]$ in $\mathbf{P}_3$. We want to show that $N = \Gamma_2(\mathbf{P}_3)$. Clearly, $N \subseteq \Gamma_2(\mathbf{P}_3)$. To show the converse inclusion, we need to show that $\mathbf{P}_3/N$ is abelian. We check that the relations $A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}$ and $A_{12}^{-1}A_{23}A_{12} = A_{13}A_{23}A_{13}^{-1}$ hold in $\mathbf{P}_3$. As a consequence, modulo the relation $[A_{13}, A_{23}] = 1$ (that is, modulo N), we get $A_{12}A_{13}A_{12}^{-1} \equiv A_{13}$ and $A_{12}^{-1}A_{23}A_{12} \equiv A_{23}$. Thus A_{12}, A_{13} and A_{23} commute modulo N and therefore $\mathbf{P}_3/N$ is abelian. □

Proposition 4.3.10. *Let $n \geqslant 1$ be an integer, let $\lambda = (1, n_2, \dots, n_l)$ be a partition of n, with $n_i \geqslant 3$ for $i \geqslant 2$. Then the LCS of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. The case $l = 2$ (i.e. $\lambda = (1, m)$ for some integer $m \geqslant 3$) works exactly like the stable case of Proposition 4.3.7. Namely, the two generators of the abelianisation do have lifts with disjoint supports, so that $\Gamma_2(\mathbf{B}_{1,m}) = \Gamma_3(\mathbf{B}_{1,m})$ by Corollary 4.2.6. This however does not work for $l > 2$.

Let us denote by μ the partition $(n_2, \dots, n_l)$ of $n-1$, so that $\lambda = (1, \mu)$. We are going to show that $\mathbf{B}_\lambda/\Gamma_\infty$ is abelian, which implies that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_\lambda$.

It follows from Proposition 4.3.7 that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_\mu$. As a consequence, the obvious morphism $\mathbf{B}_\mu \to \mathbf{B}_\lambda/\Gamma_\infty$ factors through $\mathbf{B}_\mu/\Gamma_2 \cong \mathbf{B}_\mu^{\mathrm{ab}}$. Given the description of $\mathbf{B}_\mu^{\mathrm{ab}}$ from Proposition 4.3.5, we have that modulo Γ_∞, $\sigma_\alpha \equiv \sigma'_\alpha$ and $A_{\alpha\beta} \equiv A_{\alpha'\beta'}$ if α and α' (resp. β and β') are in the same block of μ. Moreover, the corresponding classes s_i and a_{ij} commute with one another. In the same way, we can deduce from the case of $\mathbf{B}_{1,m}$ for $m \geqslant 3$ treated above that the class of a generator $A_{1\alpha}$ depends only on the block of α. More precisely, we use the fact that the obvious morphism $\mathbf{B}_{1,n_i} \to \mathbf{B}_{1,\mu}/\Gamma_\infty$ factors through $\mathbf{B}_{1,n_i}^{\mathrm{ab}}$, where all the $A_{1\alpha}$ are identified. Moreover, the corresponding a_{1i} commutes with all the s_p and a_{pq} coming from $\mathbf{B}_\mu$, since they have lifts in $\mathbf{B}_\lambda$ with disjoint support.

We are left with showing that a_{1i} commutes with a_{1j} when $i < j$. This uses Lemma 4.3.9. Let us choose any α in the i-th block and any β in the j-th one, and consider the morphism $\mathbf{P}_3 \to \mathbf{B}_\lambda$ induced by $1 \mapsto 1$, $2 \mapsto \alpha$ and $3 \mapsto \beta$. If we compose it with the projection onto $\mathbf{B}_\lambda/\Gamma_\infty$, we get a morphism f sending A_{12} to a_{1i}, A_{13} to a_{1j}, and A_{23} to a_{ij}. Since a_{ij} commutes with a_{1j}, f sends $[A_{13}, A_{23}]$ to 1. Lemma 4.3.9 then implies that it factors through $\mathbf{P}_3^{\mathrm{ab}}$, so that all the elements in its image commute, including a_{1i} and a_{1j}. This finishes the proof that $\mathbf{B}_\lambda/\Gamma_\infty$ is abelian, whence equal to $\mathbf{B}_\lambda/\Gamma_2$. ◻

Proposition 4.3.11. *Let $n \geqslant 1$ be an integer, let $\lambda = (1, 1, n_3, \dots, n_l)$ be a partition of n, with $n_i \geqslant 3$ for $i \geqslant 3$. Then the LCS of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. Let μ denote the partition $(n_3, \dots, n_l)$ of $n - 2$, so that $\lambda = (1, 1, \mu)$. It follows from Proposition 4.3.10 that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_{1,\mu}$, hence both the obvious maps $\mathbf{B}_{1,\mu} \to \mathbf{B}_\lambda/\Gamma_\infty$ factor through $\mathbf{B}_{1,\mu}^{\mathrm{ab}}$. From this, we deduce that $\mathbf{B}_\lambda/\Gamma_\infty$ is generated by elements s_i ($3 \leqslant i \leqslant l$) and a_{ij} ($1 \leqslant i < j \leqslant l$), and that the a_{1j} (resp. the a_{2j}), the s_p ($p \geqslant 1$) and the a_{pq} ($q > p \geqslant 3$) commute with one another. Moreover, for all $i, j \geqslant 3$, a_{1i} and a_{2j} have lifts with disjoint support in $\mathbf{B}_\lambda$, so they commute (even when $i = j$).

We are left with showing that the class a_{12} of A_{12} commutes with all the other generators. A disjoint support argument shows that it commutes with the s_k and a_{kl} for $l > k \geqslant 3$. Now, for $i \geqslant 3$, let us choose α in the i-th block, and let us consider the morphism $\mathbf{P}_3 \to \mathbf{B}_\lambda$ induced by $1 \mapsto 1$, $2 \mapsto 2$ and $3 \mapsto \alpha$. If we compose it with the projection onto $\mathbf{B}_\lambda/\Gamma_\infty$, we get a morphism f sending A_{12} to a_{12}, A_{13} to a_{1i}, and A_{23} to a_{2i}. Since a_{1i} commutes with a_{2i}, f sends $[A_{13}, A_{23}]$ to 1. Lemma 4.3.9 then implies that it factors through $\mathbf{P}_3^{\mathrm{ab}}$, so that all the elements in its image commute, showing that a_{12} commutes with a_{1i} and with a_{2i}. Thus, we have proved that $\mathbf{B}_\lambda/\Gamma_\infty$ is abelian, whence $\Gamma_\infty(\mathbf{B}_\lambda) = \Gamma_2(\mathbf{B}_\lambda)$. ◻

Blocks of size 2

When there is exactly one block of size 2 and no block of size 1, we get a complete description of the quotient of $\mathbf{B}_\lambda$ by its residue:

Proposition 4.3.12. *Let $n \geqslant 1$ be an integer, let $\lambda = (2, n_2, \dots, n_l)$ be a partition of n, with $n_i \geqslant 3$ if $i \geqslant 2$. Then $\mathbf{B}_\lambda/\Gamma_\infty$ decomposes as a direct product of $l(l-1)/2$ copies of $\mathbb{Z}$ with $\mathbb{Z}^{2(l-1)} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts via the involution exchanging the elements e_{2i} and e_{2i+1} of a basis of $\mathbb{Z}^{2(l-1)}$. In particular, if $l \geqslant 2$, then the LCS of $\mathbf{B}_\lambda$ does not stop.*

Proof. Let μ denote the partition $(n_2, \dots, n_l)$ of $n - 2$, so that $\lambda = (2, \mu)$. Then the canonical projection $\mathbf{B}_{2,\mu} \twoheadrightarrow \mathfrak{S}_{2,\mu} \twoheadrightarrow \mathfrak{S}_2$ has $\mathbf{B}_{1,1,\mu}$ as its kernel. Moreover, $\Gamma_\infty(\mathbf{B}_{2,\mu})$ contains $\Gamma_\infty(\mathbf{B}_{1,1,\mu})$, which is equal to $\Gamma_2(\mathbf{B}_{1,1,\mu})$ by Proposition 4.3.11. We show that these are in fact equal. In order to do this, it is enough to show that $\mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$ is residually nilpotent. In fact, we are going to compute it completely.

First, let us remark that it makes sense to consider this quotient: $\Gamma_2(\mathbf{B}_{1,1,\mu})$ is a characteristic subgroup of $\mathbf{B}_{1,1,\mu}$, which is normal (of index 2) in $\mathbf{B}_{2,\mu}$, hence it is a normal subgroup of $\mathbf{B}_{2,\mu}$. Next, we can write $\mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$ as an extension:

$$\mathbf{B}_{1,1,\mu}^{\mathrm{ab}} \hookrightarrow \mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu}) \twoheadrightarrow \mathfrak{S}_2. \tag{4.1}$$

We can use the method from Appendix 4.6 to get a presentation of the group $G := \mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$. Namely, we have a presentation of the kernel: $\mathbf{B}_{1,1,\mu}^{\mathrm{ab}}$ is free abelian on the s_i and the a_{ij} indexed by the blocks of $(1, 1, \mu)$. We also know the action of $\mathfrak{S}_2$ on $\mathbf{B}_{1,1,\mu}^{\mathrm{ab}}$ induced by conjugation by σ_1 in $\mathbf{B}_{2,\mu}$: it exchanges the a_{1j} with the a_{2j} with $j \geqslant 3$ and it acts trivially on all the other generators. Finally, we can lift the only relation defining $\mathfrak{S}_2$ to $\sigma_1^2 = A_{12}$ in $\mathbf{B}_{2,\mu}$. As a consequence, we get the presentation:

$$G = \left\langle \begin{array}{l} s, \\ s_i \ (3 \leqslant i \leqslant l+1), \\ a_{ij} \ (1 \leqslant i < j \leqslant l+1). \end{array} \middle| \begin{array}{ll} \forall i,j,p,q,u,v, & [s_i, s_j] = [s_i, a_{pq}] = [a_{pq}, a_{uv}] = 1, \\ \forall i \geqslant 1, & [s, s_i] = 1, \\ \forall j > i \geqslant 3, & [s, a_{ij}] = 1, \\ \forall j \geqslant 3, & s a_{1j} s^{-1} = a_{2j} \text{ and } s a_{2j} s^{-1} = a_{1j}, \\ & s^2 = a_{12}. \end{array} \right\rangle$$

One can deduce from this presentation that G decomposes as $\mathbb{Z}^N \times (\mathbb{Z}^{2(l-1)} \rtimes \mathbb{Z})$ where the first factor is free abelian on the s_i $(i \geqslant 3)$ and the a_{ij} $(j > i \geqslant 3)$ (hence $N = l(l-1)/2$), and the action of $\mathbb{Z}$ (free on s) on $\mathbb{Z}^{2(l-1)}$ (free abelian on the a_{1j} and the a_{2j}) is given by s exchanging the a_{1j} and the a_{2j}. Checking that this holds is a matter of writing the appropriate split projections from the presentations of G_l and its factors.

Finally, the decomposition of G allows us to apply Proposition 4.5.10 to compute its LCS. Namely, we apply it to $A = \langle a_{1j}, a_{2j} \rangle_{j \geqslant 3}$ (which is free abelian on these generators) endowed with the involution τ exchanging a_{1j} with a_{2j} for all j. Then $V = \mathrm{Im}(\tau - 1)$ is the free abelian subgroup generated by the $a_{1j} - a_{2j}$, and for $k \geqslant 2$, we have $\Gamma_k(G_l) = 2^{k-1}V$. In particular, this LCS does not stop. However its intersection is trivial: the group G_l is residually nilpotent, which implies that $\Gamma_\infty(\mathbf{B}_{2,\mu}) = \Gamma_\infty(\mathbf{B}_{1,1,\mu})$, and finishes our proof. ◻

Remark 4.3.13. The Lie ring of G (which identifies with the Lie ring of $\mathbf{B}_{2,\mu}$) can be completely computed, using Corollary 4.5.11. Namely, it identifies with $\mathbb{Z}^N \times (L \rtimes \mathbb{Z})$, where $L = \mathbb{Z}^{l-1} \oplus (\mathbb{Z}/2)^{l-1} \oplus (\mathbb{Z}/2)^{l-1} \oplus \cdots$ and the action of the generator t of $\mathbb{Z}$ on L is via the degree-one map $\mathbb{Z}^{l-1} \twoheadrightarrow (\mathbb{Z}/2)^{l-1} \cong (\mathbb{Z}/2)^{l-1} \cong \cdots$. In other words, as a Lie ring, $\mathcal{L}(\mathbf{B}_{2,\mu})$ admits the presentation via generators $t, X_1, \ldots, X_{l-1}, Y_1, \ldots, Y_N$ and relations:

$$\begin{cases} [Y_i, Y_j] = [Y_i, X_k] = [Y_i, t] = [X_k, X_l] = 0, \\ 2[t, X_i] = 0. \end{cases}$$

Let us now turn our attention to the case when there are two blocks of size 2 in the partition.

Proposition 4.3.14. *The LCS of $\mathbf{B}_{2,2}$ does not stop.*

Proof. Since $\mathbf{B}_{2,2}$ surjects onto $\mathbf{B}_{2,2}(\mathbb{S}^2)$, this is a direct consequence of Proposition 4.4.76 below, by an application of Lemma 4.1.1. Alternatively, one can adapt the proof of Proposition 4.4.76 to this case, getting that:

$$\mathbf{B}_{2,2}/\langle A_{12}, A_{34}, \Gamma_2(\mathbf{P}_4) \rangle \cong (\mathbb{Z}^2)^{\otimes 2} \rtimes (\mathfrak{S}_2)^2,$$

where $\mathfrak{S}_2$ acts on $\mathbb{Z}^2$ by permutation of the factors. Then the methods of Appendix 4.6 can be used to compute completely the LCS of this group. ◻

Corollary 4.3.15. *If at least two blocks of the partition λ are of size 2, then the LCS of $\mathbf{B}_\lambda$ does not stop.*

Proof. Under the hypothesis, there is a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{2,2}$. Thus, this corollary is obtained from a direct application of Lemma 4.1.1. ◻

Blocks of size 1 and 2: study of $\mathbf{B}_{1,2}$

We use that $\mathbf{B}_{1,2}$ is isomorphic to the Artin group of type $\mathbf{B}_2$, a classical fact of which we give a proof, for the sake of completeness.

Lemma 4.3.16. *The group $\mathbf{B}_{1,2}$ is isomorphic to the Artin group of type B_2, that is, to $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$. As a consequence, it is residually nilpotent, but not nilpotent. In particular, its LCS does not stop.*

Proof. On the one hand, we can re-write the presentation of G as:

$$G = \langle \sigma, x, y \mid (\sigma x)^2 = (x\sigma)^2,\ y = \sigma x \sigma^{-1} \rangle.$$

Then, modulo the second relation (which we conveniently rewrite $y\sigma = \sigma x$), the first one is equivalent to $(y\sigma)^2 = x(y\sigma)\sigma$, and in turn to $\sigma y \sigma^{-1} = y^{-1}xy$. Thus:

$$G = \langle \sigma, x, y \mid \sigma x \sigma^{-1} = y,\ \sigma y \sigma^{-1} = y^{-1}xy \rangle.$$

On the other hand, the projection $\mathbf{B}_{1,2} \twoheadrightarrow \mathbf{B}_2 \cong \mathbb{Z}$ splits, and its kernel identifies with the fundamental group of the disc minus two points, which is free on two generators x and y. In fact, the corresponding action of $\mathbf{B}_2 \cong \langle \sigma \rangle$ on $\mathbf{F}_2$ is the usual Artin action. Which means exactly that the above relations are true in $\mathbf{B}_{1,2}$. Whence a surjection of G onto $\mathbf{B}_{1,2}$, which induces a diagram with (split) short exact rows:

$$\begin{array}{ccccc}
\langle x, y \rangle & \hookrightarrow & G & \twoheadrightarrow & \mathbb{Z} \\
\downarrow\!\!\!\downarrow & & \downarrow & & \downarrow\!\!\!\downarrow \\
\mathbf{F}_2 & \hookrightarrow & \mathbf{B}_{1,2} & \twoheadrightarrow & \mathbf{B}_2 \cong \mathbb{Z}.
\end{array}$$

Using the freeness of their target, one sees that the left and right vertical maps must be isomorphisms, hence so is the middle map, by the Five Lemma. The rest of the statement is then a reformulation of Proposition 4.5.21. ☐

Remark 4.3.17. A more precise result is given by Proposition 4.5.23, which describes the Lie ring of the group G (hence of $\mathbf{B}_{1,2}$). Notice, however, that this difficult calculation is not needed if one only wants to see that its LCS does not stop; see Remarks 4.5.20 and 4.5.22.

Corollary 4.3.18. *If the partition λ has both a block of size 1 and a block of size 2, then the LCS of $\mathbf{B}_\lambda$ does not stop.*

Proof. Apply Lemma 4.1.1 to a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{1,2}$. ☐

Proof of Theorem 4.3.6. The first statement consists of Propositions 4.3.7, 4.3.10 and 4.3.11. The second one consists of the cases where λ has at least three blocks of size 1 (Lemma 4.3.8), exactly one block of size 2 together with blocks of size at least 3 (Proposition 4.3.12), at least two blocks of size 2 (Corollary 4.3.15) or at least one block of size 1 and one block of size 2 (Corollary 4.3.18). ☐

4.4 Braids on surfaces

In this chapter, we study the LCS of surface braid groups and their partitioned versions. This may be seen as a generalisation of results from §4.3, where we studied classical Artin braids, that is, braids on the disc. Contrary to what is usually done in the literature, we choose to consider braids on any surface. In particular, our surfaces may be non-compact, and they may have (countably) infinite genus or boundary components. For braids, which are always compactly supported and can be pushed away from the boundary, this level of generality does not really complicate things. We also do not assume that our surfaces are orientable, because the techniques that we use work very similarly for orientable surfaces and for non-orientable ones.

We first recall what we need from Richards' classification of surfaces (§4.4.1), then we introduce the tools that we need from the general theory of braids on surfaces (§4.4.2) and we review presentations of braid groups on compact surfaces (§4.4.3). We then find ourselves ready to tackle the study of LCS. We do this first for the whole braid group $\mathbf{B}_n(S)$: we show that the LCS stops if $n \geqslant 3$, and we completely compute the Lie ring in this case (§4.4.4). We then generalise these results to partitioned surface braid groups $\mathbf{B}_\lambda(S)$ in §4.4.5, whose LCS stops if the blocks of the partition have size at least 3, a stability hypothesis under which we can compute the associated Lie ring. Finally, we study the unstable cases in §4.4.6. There, four cases stand out from the crowd: the braid groups on the sphere $\mathbb{S}^2$, on the torus $\mathbb{T}^2$, on the Möbius strip $\mathbb{M}^2$ and on the projective plane $\mathbb{P}^2$.

4.4.1 Surfaces

Recall that a *surface* is a (second countable) 2-manifold, which we will *not* suppose compact or orientable in general. Such manifolds are well-understood: the classification of surfaces without boundary has been achieved by Richards [Ric63], and the classification of surfaces with boundary, which is more complicated, was completed more than fifteen years later by Brown and Messer [BM79].

Remark 4.4.1. By a *manifold*, we mean a *locally Euclidean, Hausdorff* space. Moreover, all our manifolds are assumed *second countable*. For connected manifolds, this is equivalent to either metrisability or paracompactness, and it implies triangulability (see [Rad25] for the latter implication).

For studying braids, we will in fact only need to consider Richards' classification. Indeed, let S be any connected surface (possibly with boundary). If we glue a copy of $\partial S \times [0, \infty)$ to S by identifying $\partial S \times 0$ with ∂S, we obtain a surface S' without boundary, and it is easy to show that the canonical injection $S \hookrightarrow S'$ is an isotopy equivalence. Moreover, one can show that an isotopy equivalence induces homotopy equivalences between configuration spaces and, in turn, isomorphisms between braid groups. In the sequel, we thus identify braids on S with braids on S' (or on $S - \partial S$). For instance, braids on the closed disc are identified with braids on the plane. This holds in particular for braids on one strand, that is, for fundamental groups, whose computation is recalled below in Proposition 4.4.4.

Let us briefly recall Richards' construction of all surfaces, up to homeomorphism; for a detailed account, the reader is referred to [Ric63], in particular to §5 and §6, especially Theorem 3, therein.

Proposition 4.4.2. *Let S be a connected surface. Then S is homeomorphic to a surface constructed in the following way:*

- *Consider the Cantor set K embedded in the sphere $\mathbb{S}^2$ in the usual manner. Choose some closed subset X of K, and remove it from $\mathbb{S}^2$.*
- *Choose a finite or countably infinite sequence of pairwise disjoint closed 2-discs in $\mathbb{S}^2 - X$, which has no accumulation point outside of X.*
- *Along each of these discs, perform a connected sum operation with either $\mathbb{T}^2$ or $\mathbb{P}^2$.*

Remark 4.4.3. At the second step, one can in fact choose an explicit sequence of discs depending only on X together with the subset $X_{np} \subseteq X$ of accumulation points of the sequence of discs; see [Ric63].

As a direct corollary of Richards' classification, one can compute fundamental groups of surfaces:

Proposition 4.4.4. *Let S be a connected surface without boundary. Then $\pi_1(S)$ is a free group, except when S is closed. Moreover, it is of finite type if and only if S is obtained (up to homeomorphism) from a closed surface by removing a finite number of points.*

Remark 4.4.5. We see that closed surfaces (that is, compact surfaces without boundary) are singled out here, as will also be the case later in our study of braid groups; see for instance Proposition 4.4.15 and Theorem 4.4.53.

Fundamental groups are in fact almost all homotopy groups of surfaces:

Corollary 4.4.6. *Let S be a connected surface without boundary. Then $\pi_{>2}(S)$ is trivial, except when S is the sphere or the projective plane.*

Proof. The universal covering $\widetilde{S}$ of S is a simply connected surface without boundary. Proposition 4.4.4 implies that such a surface must be of finite type, and the explicit computation of fundamental groups of surfaces of finite type shows that $\widetilde{S}$ must be homeomorphic either to the sphere or to the plane. If $\widetilde{S} \cong \mathbb{S}^2$, then S is compact, and the fibers in the covering must be finite, so $\pi_1(S)$ is finite, and S is either the sphere itself, or the projective plane. In all the other cases, $\pi_{>2}(S) \cong \pi_{>2}(\widetilde{S}) \cong \pi_{>2}(\mathbb{R}^2) = 0$. ◻

Another immediate corollary of Richards' classification is the following dichotomy:

Proposition 4.4.7. *Let S be a connected surface without boundary. Then either S can be embedded into the sphere $\mathbb{S}^2$, or it contains the 1-punctured torus or the Möbius strip as an embedded subsurface.*

In our study of braids, this appears as a trichotomy, between the following cases:

- S is *planar*, i.e. it embeds into the plane;
- S is the sphere $\mathbb{S}^2$;
- S contains an embedded 1-punctured torus (a *handle*) or an embedded Möbius strip (a *crosscap*).

In this regard, see in particular Proposition 4.4.19.

Convention 4.4.8. In the usual way, an embedded Möbius strip on a surface will be indicated in our figures by a *crosscap*, that is, a circle drawn on the surface, bounding a disc on which is drawn a cross (see for instance Figure 4.3a). In order to obtain the surface that is meant from the surface on which the crosscaps are drawn, one must, for each crosscap, remove the interior of the corresponding disc, and then glue together opposite points of the remaining circle.

The reader not familiar with this classical representation should check, as a good exercise, that a disc with one crosscap is a Möbius strip, that a sphere with one crosscap is a projective plane (so that adding a crosscap on a surface is the same as taking the connected sum with a projective plane), and that a sphere with two crosscaps is a Klein bottle. They should also keep in mind Dyck's theorem: three crosscaps on a surface is the same as a handle and a crosscap (see for instance [FW99]).

4.4.2 Braids on surfaces: general theory

We gather here some fundamental results in the theory of braids on surfaces. The main tools that we need are Goldberg's theorem (§4.4.2), the Fadell-Neuwirth exact sequences (§4.4.2) and a little calculation showing that the usual pure braid generators become commutators on non-planar surfaces (§4.4.2). Goldberg's theorem [Gol73, Th. 1] says that surface braid groups are generated by braids on the disc, together with braids obtained from loops on the surface; we give a new simple proof of this result, incidentally extending it to possibly non-compact surfaces. As for the Fadell-Neuwirth exact sequences, which are traditionally stated for pure braid groups, since they involve forgetting strands, we generalise them easily to partitioned braid groups on possibly non-compact surfaces, where the projections forget blocks of strands.

Definitions, notations and conventions

Let S be a connected surface. Let us consider the configuration space

$$F_n(S) = \{(x_1, \ldots, x_n) \in S^n \mid \forall i \neq j,\ x_i \neq x_j\} \subset S^n.$$

The *braid group on the surface* S on n strings is the fundamental group $\mathbf{B}_n(S)$ of the unordered configuration space $C_n(S) = F_n(S)/\mathfrak{S}_n$. When S is the 2-disc $\mathbb{D}$, this group is exactly Artin's braid group, that is $\mathbf{B}_n = \mathbf{B}_n(\mathbb{D})$.

Let us fix an embedded closed disc $\mathbb{D} \subset S$, together with a base configuration $c = (c_1, \ldots, c_n) \in F_n(S)$ of points $c_i \in \mathbb{D}$. Since the assignment $S \mapsto \mathbf{B}_n(S)$ is functorial with respect to embeddings of surfaces, we have a (not necessarily injective) group morphism:

$$\varphi\colon \mathbf{B}_n = \mathbf{B}_n(\mathbb{D}) \to \mathbf{B}_n(S).$$

In the sequel, we omit most mentions of φ: if $\beta \in \mathbf{B}_n$, we still denote by β its image in $\mathbf{B}_n(S)$ (although we should denote it by $\varphi(\beta)$).

We can also construct surface braids from curves on the surface. Precisely, for any $i \leqslant n$, let us define $\iota_i\colon S - \mathbb{D} \hookrightarrow F_n(S)$ by sending x to the configuration $(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n)$. This induces a morphism between fundamental groups:

$$\psi_i\colon \pi_1(S - \mathbb{D}) \to \mathbf{P}_n(S).$$

Remark 4.4.9. The map ι_i cannot preserve basepoints, since the basepoint of $F_n(S)$ is not in its image. However, the induced map between fundamental groups can easily be defined using a chosen fixed path between c_i and a point not in $\mathbb{D}$. The choice of such a path (for each i) is implicit in the sequel, and should be made as simply as possible. For instance, one can fix a segment from c_i to a point on $\partial\mathbb{D}$, and extend it slightly outside $\mathbb{D}$, such that these paths are disjoint for different values of i.

The canonical projection $\pi\colon \mathbf{B}_n(S) \twoheadrightarrow \mathfrak{S}_n$, corresponding to the covering of $F_n(S)/\mathfrak{S}_n$ by $F_n(S)$, can be enhanced to a projection:

$$\pi_S\colon \mathbf{B}_n(S) \twoheadrightarrow \pi_1(S) \wr \mathfrak{S}_n$$

as follows. Given a braid $\beta \in \mathbf{B}_n(S)$, let us lift it to a path $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $F_n(S)$ from $(c_i)_i$ to $(c_{\sigma^{-1}(i)})_i$ where $\sigma = \pi(\beta)$. Then send β to $((\overline{\gamma}_1, \ldots, \overline{\gamma}_n), \sigma)$, where $\overline{\gamma}_i$ is the image of γ_i in $\pi_1(S/\mathbb{D}) \cong \pi_1(S)$. We note that π_S is clearly surjective, since its image contains $\mathfrak{S}_n$ (which is the image of $\varphi(\mathbf{B}_n)$ by π_S), and all the factors $\pi_1(S)$ (which are the images of the $\psi_i(\pi_1(S))$).

The kernel of π_S, which is contained in $\mathbf{P}_n(S)$, obviously contains the group $\mathbf{P}_n$. We denote it by $\mathbf{P}_n^\circ(S)$ and we call its elements *geometrically pure braids.*

Generators of surface braid groups

The following result generalises one of Goldberg [Gol73, Th. 1] to any connected surface.

Proposition 4.4.10. *The following statements hold for any connected surface S and any integer $n \geqslant 1$:*

- *For any $i \leqslant n$, the group $\mathbf{B}_n(S)$ is generated by the images of φ and ψ_i.*
- *Its subgroup $\mathbf{P}_n(S)$ is generated by (the image of) $\mathbf{P}_n$ and the images of $\psi_1, \ldots, \psi_n$.*
- *The subgroup $\mathbf{P}_n^\circ(S)$ of $\mathbf{P}_n(S)$ is the normal closure of $\mathbf{P}_n$. Since $\mathbf{P}_n^\circ(S)$ is normal in $\mathbf{B}_n(S)$, it is also the normal closure of $\mathbf{P}_n$ in $\mathbf{B}_n(S)$.*

Proof. Let us first remark that the ψ_i are conjugate to each other by elements of the image of φ. Hence, the second statement implies the first one.

We prove both the second and the third statement by induction on n. Both proofs use the tools that we introduce now. Consider the following commutative diagram of spaces:

$$\begin{array}{ccccc} S - Q_n & \xhookrightarrow{\iota} & F_{n+1}(S) & \xrightarrow{p} & F_n(S) \\ \downarrow & & \downarrow & & \downarrow \\ S & \xhookrightarrow{\iota} & S^{n+1} & \xrightarrow{p} & S^n, \end{array}$$

where $Q_n = \{c_1, \ldots, c_n\}$, ι sends x to $(c_1, \ldots, c_n, x)$ and p send $(x_1, \ldots, x_{n+1})$ to $(x_1, \ldots, x_n)$. It induces a commutative diagram of morphisms between fundamental groups:

$$\begin{array}{ccccc} \pi_1(S - Q_n) & \xrightarrow{\iota_*} & \mathbf{P}_{n+1}(S) & \xrightarrow{p_*} & \mathbf{P}_n(S) \\ \downarrow u & & \downarrow v & & \downarrow w \\ \pi_1(S) & \hookrightarrow & \pi_1(S)^{n+1} & \twoheadrightarrow & \pi_1(S)^n, \end{array} \tag{4.2}$$

whose bottom line is obviously exact. The map p is a (locally trivial) fibration by [FN62, Th. 3] (see also [Bir74, Th. 1.2]), and the first line is part of its exact sequence in homotopy. Thus it is exact (but ι_* need not be injective in general).

Let us prove our second statement. For $n = 1$, ψ_1 is the canonical isomorphism $\pi_1(S) \cong \mathbf{P}_1(S) = \mathbf{B}_1(S)$, and there is nothing to prove.

Let us now suppose that the conclusion holds for some $n \geqslant 1$. By applying the Seifert-van Kampen theorem, we see that $\pi_1(S - Q_n) \cong \mathbf{F}_n *_{\mathbb{Z}} \pi_1(S - \mathbb{D})$, where $\mathbf{F}_n = \pi_1(\mathbb{D} - Q_n)$ is free on n generators. Then $\iota_* \colon \mathbf{F}_n *_{\mathbb{Z}} \pi_1(S - \mathbb{D}) \to \mathbf{P}_n(S)$ identifies with the map induced by $\mathbf{F}_n \hookrightarrow \mathbf{P}_n$ (the kernel of $\mathbf{P}_n \twoheadrightarrow \mathbf{P}_{n-1}$) and $\psi_{n+1} \colon \pi_1(S - \mathbb{D}) \to \mathbf{P}_n(S)$.

Now let G be the subgroup of $\mathbf{P}_{n+1}(S)$ generated by $\mathbf{P}_{n+1}$ and the images of the ψ_i for $i \leqslant n + 1$. It contains the image of ι_*, which is the kernel of p_*. Moreover, its image by p_* contains the images of $\psi_1, \ldots, \psi_n$, and $\mathbf{P}_n$, hence all of $\mathbf{P}_n(S)$, by the induction hypothesis. As a consequence, $G = \mathbf{P}_{n+1}(S)$, which was the desired conclusion.

Let us prove our third statement. For $n = 1$, π_S is the canonical isomorphism $\mathbf{B}_1(S) = \mathbf{P}_1(S) \cong \pi_1(S)$ (inverse to ψ_1). Then $\mathbf{P}_n^\circ(S)$ and $\mathbf{P}_n$ are trivial, and there is nothing to prove.

Let us now suppose that the conclusion holds for some $n \geqslant 1$. Consider the induced maps between kernels of the vertical morphisms in (4.2). By definition, the kernels of v and w are respectively $\mathbf{P}_{n+1}^\circ(S)$ and $\mathbf{P}_n^\circ(S)$. Let us denote by K the kernel of u. We get induced maps:

$$K \xrightarrow{\iota_\#} \mathbf{P}_{n+1}^\circ(S) \xrightarrow{p_\#} \mathbf{P}_n^\circ(S)$$

such that $p_\# \circ \iota_\# = 1$. An easy chase in the diagram (or an application of the Snake Lemma) shows that we can lift any element in the kernel of $p_\#$ to an element of K: the above sequence is exact.

The morphism u identifies with the projection $\mathbf{F}_n *_{\mathbb{Z}} \pi_1(S - \mathbb{D}) \twoheadrightarrow \{1\} *_{\mathbb{Z}} \pi_1(S - \mathbb{D}) \cong \pi_1(S)$ killing the first factor, hence K identifies with the normal closure of $\mathbf{F}_n$ in $\mathbf{F}_n *_{\mathbb{Z}} \pi_1(S)$. Moreover, ι_* sends $\mathbf{F}_n = \pi_1(\mathbb{D} - Q_n)$ to a subgroup of $\mathbf{P}_{n+1}$, so the image of K in $\mathbf{P}_{n+1}(S)$ is contained in the normal closure of $\mathbf{P}_{n+1}$.

Now, let N be the normal closure of $\mathbf{P}_{n+1}$ in $\mathbf{P}_{n+1}(S)$. Since $v(\mathbf{P}_{n+1}) = \{1\}$, we have $v(N) = \{1\}$, which means that $N \subseteq \mathbf{P}_{n+1}^\circ(S)$. By the induction hypothesis, $p_\#(N) = \mathbf{P}_n^\circ(S)$. Moreover, N contains the image of $i_\#$, which is the kernel of $p_\#$. Thus $N = \mathbf{P}_{n+1}^\circ(S)$, which was the desired conclusion. □

Remark 4.4.11. For a closed, oriented surface, the group K appearing in the proof is exactly the group F_{n+1} from [GP04, page 227], where they give a precise description of it in this particular case. However, in their paper, they were using the very result that we are recovering and generalising here, quoting [Bir74] for it [GP04, page 225].

The proof of Proposition 4.4.10 also works for manifolds in higher dimension, allowing us to recover the classical [Bir69, Th. 1]:

Proposition 4.4.12. *For any manifold M of dimension at least 3, the morphism $\pi_M \colon \mathbf{B}_n(M) \twoheadrightarrow \pi_1(M) \wr \mathfrak{S}_n$ is an isomorphism.*

Proof. One can directly check that the proof of Proposition 4.4.10 works if we replace the surface S with a connected manifold M of any dimension $d \geqslant 2$ and the disc $\mathbb{D}$ with a d-disc $\mathbb{D}^d$. Then $\mathbf{P}_n$ gets replaced with $\mathbf{P}_n(\mathbb{D}^d)$, which is trivial whenever $d \geqslant 3$ (the configuration space $F_n(\mathbb{D}^d) \cong F_n(\mathbb{R}^d)$ is obtained from $\mathbb{R}^{nd}$ by removing subspaces of codimension $d \geqslant 3$, so it is simply connected). Thus, the normal closure $\mathbf{P}_n^\circ(M)$ of $\mathbf{P}_n(\mathbb{D}^d)$ is trivial too, and the latter is exactly the kernel of π_M. ◻

Remark 4.4.13. This means that *braid groups on manifolds of dimension at least* 3 *are exactly wreath products*, whose LCS is studied in Appendix 4.5.4; see in particular Corollaries 4.5.27 and 4.5.29.

The Fadell-Neuwirth exact sequences

The locally trivial fibrations used in the proof of Proposition 4.4.10 are particular instances of the Fadell-Neuwirth fibrations. These induce exact sequences between pure braid groups, that are in fact exact sequences between partitioned braid groups. We now recall how these work, and when these exact sequences split.

Definition 4.4.14. Let S be a surface, let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, \ldots, n_l)$ be a partition of n. The corresponding *partitioned surface braid group* is:

$$\mathbf{B}_\lambda(S) := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right) \subseteq \mathbf{B}_n(S).$$

There are canonical surjections between partitioned braid groups, obtained by forgetting blocks. For most surfaces, these projections behave exactly as they do for the disc. However, their behaviour for closed surfaces is somewhat trickier, especially when it comes to the sphere and the projective plane. The latter are in fact the only ones whose braid groups admit non-trivial torsion elements, a fact that can be seen as a consequence of their second homotopy group being non-trivial.

Proposition 4.4.15 (Fadell-Neuwirth exact sequences)**.** *Let S be a connected surface, let μ be a partition of an integer $m \geqslant 1$, ν be a partition of an integer $n \geqslant 1$, and let us denote by $\mu\nu$ their concatenation, which is a partition of $m+n$. The following sequence of canonical maps is exact:*

$$\mathbf{B}_\mu(S - \{n \text{ pts}\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

Moreover, except when $S = \mathbb{S}^2$ and $n = 1, 2$ or $S = \mathbb{P}^2$ and $n = 1$, this is in fact a short exact sequence:

$$1 \longrightarrow \mathbf{B}_\mu(S - \{n \text{ pts}\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

Furthermore, if S is not closed, the surjection $\mathbf{B}_{\mu\nu}(S) \twoheadrightarrow \mathbf{B}_\nu(S)$ splits.

Remark 4.4.16. Recall that base configurations, hence also the n points removed from S, must not be on the boundary of S, if S has a non-trivial boundary.

Proof of Proposition 4.4.15. Let us first recall that in considering braid groups, we consider surfaces up to isotopy equivalence, so we can remove the boundary of S if it is non-trivial, and assume that $\partial S = \varnothing$. Recall that if λ is a partition of N, we denote by $C_\lambda(S)$ the configuration space $F_N(S)/\mathfrak{S}_\lambda$. Forgetting the first n points induces a map of configuration spaces $C_{\mu\nu}(S) \to C_\nu(S)$, which is a locally trivial fibration with fibres homeomorphic to $C_\mu(S - \{n \text{ pts}\})$, by a slight adaptation of [FN62, Th. 3] (or [Bir74, Th. 1.2]), which works for any manifold (without boundary). Since its fibres are path-connected, part of its long exact sequence of homotopy groups is:

$$\pi_2(C_\nu(S)) \to \mathbf{B}_\mu(S - \{n \text{ pts}\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

The map $F_n(S) \twoheadrightarrow C_\nu(S)$ is a covering, so that $\pi_2(C_\nu(S)) \cong \pi_2(F_n(S))$, which is trivial except when $S = \mathbb{S}^2$ and $n = 1, 2$ or $S = \mathbb{P}^2$ and $n = 1$. When S is not the sphere or the projective plane, this follows from [Bir74, Prop. 1.3], using the fact that higher homotopy groups of surfaces different from $\mathbb{S}^2$ and $\mathbb{P}^2$ are trivial (see Corollary 4.4.6). When $S = \mathbb{S}^2$ this is [FB62, Cor. p. 244] and when $S = \mathbb{P}^2$ it is [Bus66, Cor. p. 82].

If S is not closed, then there is an isotopy equivalence between S and a proper subsurface S' of S. Then one can choose a configuration of m points in $S - S'$ and add them to each configuration of n points of S', getting a map $F_n(S') \to F_{m+n}(S)$. The induced map $\pi_1(C_\nu(S)) \cong \pi_1(C_\nu(S')) \to \pi_1(C_{\mu\nu}(S))$ is the required section. ☐

Remark 4.4.17. A weaker form of the asphericity statement for $\pi_2(F_n(S))$ used in the above proof may be found [FN62, Cor. 2.2]. They ask that the surface be compact, but this is only in order to be able to use the classification of compact surfaces, which we easily replaced by the classification of all surfaces in our proof.

Remark 4.4.18. When S is not closed, the construction of the splitting in the proof of Proposition 4.4.15 can also be used to get a morphism ι from $\mathbf{B}_n(S)$ to $\mathbf{B}_{n+1}(S)$, corresponding to adding an $(n+1)$-st strand near a boundary or an end of S. The restriction of this morphism to pure braid groups is a split injection (it is the section in the particular case $\mu = (1)$ and $\nu = (1, \dots, 1)$ in the above proof). Since it also induces an injection from $\mathbf{B}_n(S)/\mathbf{P}_n(S) \cong \mathfrak{S}_n$ into $\mathbf{B}_{n+1}(S)/\mathbf{P}_{n+1}(S) \cong \mathfrak{S}_{n+1}$, the morphism ι itself is injective. Notice that, thanks to the connectedness of S, ι depends on the choices made in its construction only up to conjugation. By contrast, if S is closed, there is no obvious construction of such a map, and in fact it does not exist in general [GG04a; GG10].

Pure braid generators and commutators.

Some of the results below will hold for all surfaces S. However, in order to get more precise results, we need to get more specific and use the classification of surfaces recalled in §4.4.1. Recall that all the generators σ_i of $\mathbf{B}_n$ are identified in $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$ (see Example 4.2.3), hence also in $\mathbf{B}_n(S)^{\mathrm{ab}}$. The next proposition deals notably with the order of their common class σ. The trichotomy that appears here, which comes from Proposition 4.4.7, will play an important role in all that follows.

Proposition 4.4.19. *Let $n \geqslant 2$. Let us consider the generator A_{ij} of $\mathbf{P}_n$ as an element of $\mathbf{B}_n(S)$.*

- *If S is planar, then the class $\overline{A}_{ij} \in \mathbf{B}_n(S)^{\mathrm{ab}}$ has infinite order.*
- *If $S \cong \mathbb{S}^2$, then the class $\overline{A}_{ij} \in \mathbf{B}_n(S)^{\mathrm{ab}}$ has order $n-1$. However, its class in $\mathbf{P}_n(S)^{\mathrm{ab}}$ has infinite order.*
- *In all the other cases, A_{ij} is the commutator of two elements of $\mathbf{P}_n(S)$.*

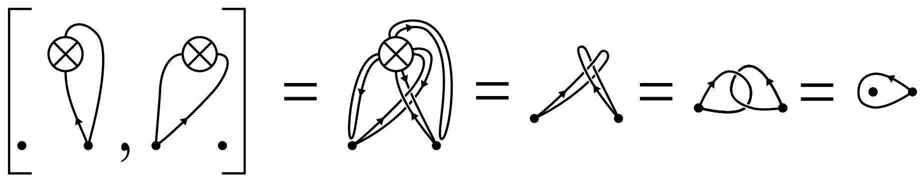
Proof. <u>If S is planar:</u> then S can be embedded in a disc. Such an embedding induces a morphism $\mathbf{B}_n(S) \to \mathbf{B}_n(\mathbb{D}) = \mathbf{B}_n$, which in turn induces a morphism from $\mathbf{B}_n(S)^{\mathrm{ab}}$ to $\mathbf{B}_n^{\mathrm{ab}}$. The latter is infinite cyclic, generated by σ. Our element $\overline{A}_{ij}$ is sent to σ^2, hence it cannot be of finite order.

<u>If S is the sphere:</u> then from the usual presentation of $\mathbf{B}_n(\mathbb{S}^2)$ (see for instance Corollary 4.4.27 below), we get that $\mathbf{B}_n(\mathbb{S}^2)^{\mathrm{ab}} \cong \mathbb{Z}/(2(n-1))$, generated by σ. Again, $\overline{A}_{ij} = \sigma^2$, whose order is $n-1$.

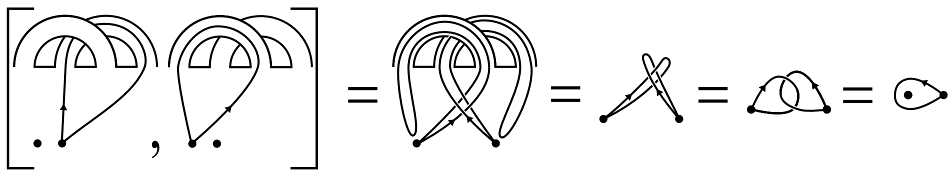
<u>If S cannot be embedded in the sphere:</u> then S contains a handle or a crosscap; see Proposition 4.4.7 and the remark following it. We can then use the explicit isotopies drawn in Figures 4.3a and 4.3b to show that A_{ij} is a bracket of two pure braids (which are respectively in the image of ψ_j and in the image of ψ_i). ☐

4.4.3 Presentations of surface braid groups

In order to prove some of the results below, in particular to determine completely the Lie rings of partitioned braid groups in the stable case, we will need to use presentations of braids groups on compact surfaces. The main tool for determining presentations of surface braid groups (including braids on the disc, which are usual Artin braids) are the Fadell-Neuwirth exact sequences; see Proposition 4.4.15. These were already used in the course of the proof of Proposition 4.4.10 to obtain generators of these groups. Let us now briefly explain how they may be used in order to determine defining relations on these generators. For S a non-closed surface, these exact sequences



(a) Pure braid generator as a commutator on a surface with a crosscap.



(b) Pure braid generator as a commutator on a surface with a handle.

Figure 4.3 Explicit isotopies expressing a pure braid generator as a commutator. We use them in many different guises throughout the chapter, each time with respect to a different embedding of the crosscap or the handle in our surface, whose image contains exactly two points of the base configuration. For instance, with the notations from Figure 4.4 below, for any choice of $1 \leqslant i < j \leqslant n$ and of $1 \leqslant r \leqslant g$, the first one gives $A_{ij} = [c_j^{(r)}, (c_i^{(r)})^{-1}]$ in $\mathbf{B}_n(\mathcal{N}_g)$.

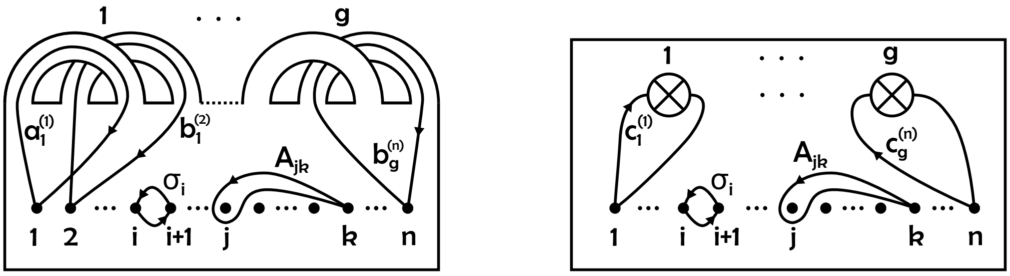
give a decomposition of $\mathbf{B}_{1,n-1}(S)$ as a semi-direct product of $\mathbf{B}_{n-1}(S)$ with a free group F. Then suppose that one has a set of relations satisfied in $\mathbf{B}_n(S)$, defining a group G_n and a well-defined surjection $\pi\colon G_n \twoheadrightarrow \mathbf{B}_n(S)$. One can consider the subgroup $G_{1,n-1} = \pi^{-1}(\mathbf{B}_{1,n-1}(S))$, use the relations to show that it is of index at most n in G_n (which implies that the induced surjection of $G_n/G_{1,n-1}$ onto $\mathbf{B}_n(S)/\mathbf{B}_{1,n-1}(S)$ is a bijection – but of course not a group morphism, since these quotients do not bear a group structure), and determine a presentation of $G_{1,n-1}$ by using the Reidemeister-Schreier method. Then one shows that $G_{1,n-1}$ decomposes as a semi-direct product of a quotient isomorphic to G_{n-1} with a kernel K generated by a family of elements sent by π to a basis of the free group F. The latter fact implies that this family must be a free basis of K, which means that $\pi\colon K \to F$ is an isomorphism. By induction, $\pi\colon G_{n-1} \to \mathbf{B}_{n-1}(S)$ is an isomorphism. Then $\pi\colon G_{1,n-1} \to \mathbf{B}_{1,n-1}(S)$ must be an isomorphism too. And since the induced surjection of $G_n/G_{1,n-1}$ onto $\mathbf{B}_n(S)/\mathbf{B}_{1,n-1}(S)$ is a bijection, $\pi\colon G_n \to \mathbf{B}_n(S)$ is an isomorphism (the reader can easily convince themselves that the latter implication works regardless of the existence of a group structure on the quotients).

This method can be used to get presentations of the braid groups of every non-closed surface of finite type; see [Bel04] for instance. It can also be adapted to the case of closed surfaces, replacing semi-direct product decompositions by non-split extensions, with some care for the exceptional cases where this is not even an extension. However, we prefer to deduce the case of closed surfaces from the non-closed one: we give here a direct general argument presenting $\mathbf{B}_n(S)$ as the quotient of $\mathbf{B}_n(S-\mathrm{pt})$ by one explicit relation, by applying the Seifert-van Kampen theorem to configuration spaces; see Proposition 4.4.26.

Surfaces with one boundary component

Presentations of braid groups of compact surfaces with one boundary component can be found in [HL02, §4] and in [Bel04, Th. 1.1 and A.2]. We re-write them with our own conventions, which we now explain.

Let us denote by $\Sigma_{g,1}$ the orientable connected compact surface of genus g with one boundary component, and by $\mathcal{N}_{g,1}$ the non-orientable connected compact surface of genus g with one boundary component. We draw $\Sigma_{g,1}$ as a rectangle with $2g$ handles attached to it, and $\mathcal{N}_{g,1}$ as a rectangle with g crosscaps (see Convention 4.4.8). Our notations for braid generators are detailed in Figure 4.4. Our drawings of braids are to be thought of as seen from above, and the left-to-right direction in products corresponds to the foreground-to-background direction in our drawings. For instance, with the notations of Figure 4.4, we have that $\sigma_1 a_k^{(i)} \sigma_1^{-1} = a_k^{(i+1)}$. As an illustration of



(a) Generators of $\mathbf{B}_n(\Sigma_{g,1})$. (b) Generators of $\mathbf{B}_n(\mathcal{N}_{g,1})$.

Figure 4.4 Generators of surface braid groups. For each generator (except for σ_i, where the points i and $i+1$ move), only one point of the configuration moves, and the others stay put. We often denote $x_k^{(1)}$ by x_k, for $x = a, b, c$.

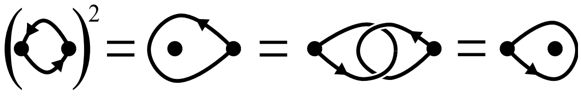


Figure 4.5 The classical pure braid generator $A_{i,i+1} = \sigma_i^2$.

these conventions, we draw different representations of the classical pure braid generator $A_{i,i+1}$ in Figure 4.5.

Notation 4.4.20. For clarity, in group presentations, we write $a \rightleftarrows b$ for the relation saying that a and b commute.

Proposition 4.4.21 ([Bel04, Th. 1.1])**.** *Let $g \geqslant 0$. A presentation of the braid group on $\Sigma_{g,1}$, generated by $\sigma_1, \ldots, \sigma_{n-1}$, $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, is given by the braid relations for $\sigma_1, \ldots, \sigma_{n-1}$, to which are added the following four families of relations (where x and y denote either a or b, and $1 \leqslant r, s \leqslant g$):*

$$\begin{cases} (BS1) & \sigma_i \rightleftarrows x_r & \textit{for all } r \textit{ and all } i \geqslant 2\ ; \\ (BS2) & x_r \rightleftarrows \sigma_1 y_s \sigma_1^{-1} & \textit{for } r < s\ ; \\ (BS3) & (\sigma_1 x_r)^2 = (x_r \sigma_1)^2 & \textit{for all } r\ ; \\ (BS4) & [\sigma_1 b_r \sigma_1^{-1}, a_r^{-1}] = \sigma_1^2 & \textit{for all } r. \end{cases} \tag{4.3}$$

Notice that it is easy to check that these relations hold in $\mathbf{B}_n(\Sigma_{g,1})$ by drawing explicit isotopies. See for instance Figure 4.3b for a drawing of $(BS4)$ (which generalises to $[b_s^{(j)}, (a_r^{(j)})^{-1}] = A_{r,s}$ if $r < s$). The translation between Bellingeri's conventions and ours is as follows:

- Our statement is the case $p = 1$ of [Bel04, Th. 1.1], whence the absence of the z_k, and of the relations involving them.
- Our σ_i is his σ_i^{-1}, our a_r is his b_r^{-1}, and our b_r is his a_r^{-1}.
- Our $(BS1) - (BS4)$ are his $(R1) - (R4)$, with 2 and 3 exchanged.

Remark 4.4.22. Although [Bel04, Th. 1.1] is stated for $g \geqslant 1$, the proof works equally well if $g = 0$. In fact, the case $g = 0$ of our statement is just the usual presentation of braid groups on the disc.

Proposition 4.4.23 ([Bel04, Th. A.2])**.** *Let $g \geqslant 1$. A presentation of the braid group on $\mathcal{N}_{g,1}$, generated by $\sigma_1, \ldots, \sigma_{n-1}$, $c_1, \ldots, c_g$ is given by the braid relations for $\sigma_1, \ldots, \sigma_{n-1}$, to which are added the following three families of relations (where $1 \leqslant r, s \leqslant g$):*

$$\begin{cases} (BN1) & \sigma_i \rightleftarrows c_r & \textit{for all } r \textit{ and all } i \geqslant 2\ ; \\ (BN2) & c_r \rightleftarrows \sigma_1 c_s \sigma_1^{-1} & \textit{for } r < s\ ; \\ (BN3) & [\sigma_1 c_r \sigma_1^{-1}, c_r^{-1}] = \sigma_1^2 & \textit{for all } r. \end{cases} \tag{4.4}$$

Here again, it is easy to check these relations explicitly. See for instance Figure 4.3a for a drawing of $(BN3)$ (which generalises to $[c_s^{(j)}, (c_r^{(j)})^{-1}] = A_{rs}$ if $r < s$). Also, it is the case $p = 1$ of

Bellingeri's statement so, again, the z_k and the corresponding relations are irrelevant. Moreover, our σ_i is his σ_i^{-1}, our c_i are his a_i^{-1}, and the indexation of our relations is the same as his.

Remark 4.4.24. The statement of [Bel04, Th. A.2] is for $g \geqslant 2$, but the proof works equally well if $g = 1$.

In the case of non-orientable surfaces, we will also need the case $p > 1$ of [Bel04, Th. A.2], which will be used twice, in the proofs of Proposition 4.4.59 and Proposition 4.4.85. Let us denote by $\mathcal{N}_{g,n+1}$ the non-orientable connected compact surface of genus g with either $n+1$ punctures or $n+1$ boundary components (recall that, up to isotopy, removing a point and removing an open disc are equivalent). The braid group $\mathbf{B}_m(\mathcal{N}_{g,n+1})$ may be seen as a subgroup of $\mathbf{B}_{m+n}(\mathcal{N}_{g,1})$; see Proposition 4.4.15. Namely, this subgroup is generated by $\sigma_1, \ldots, \sigma_{n-1}, c_1, \ldots, c_g$ and $z_j := A_{1,m+j}$ for all $1 \leqslant j \leqslant n$.

Proposition 4.4.25 ([Bel04, Th. A.2])**.** *Let $g \geqslant 0$ and $m \geqslant 1$. A presentation of the braid group $\mathbf{B}_m(\mathcal{N}_{g,n+1})$, generated by $\sigma_1, \ldots, \sigma_{m-1}, c_1, \ldots, c_g, z_1, \ldots, z_n$ is given by the relations from Proposition 4.4.23, together with the following four families of relations:*

$$\begin{cases} (BN4) & z_j \rightleftarrows \sigma_i & \text{for all } j \leqslant n \text{ and all } i \in \{2, \ldots, m-1\}\ ; \\ (BN5) & c_r \rightleftarrows \sigma_1 z_j \sigma_1^{-1} & \text{for all } j \leqslant n \text{ and all } r \leqslant g\ ; \\ (BN6) & z_i \rightleftarrows \sigma_1 z_j \sigma_1^{-1} & \text{for } i > j\ ; \\ (BN7) & (\sigma_1 z_j)^2 = (z_j \sigma_1)^2 & \text{for all } j \leqslant n. \end{cases}$$

Once more, these relations are easy to check explicitly. The translation between [Bel04, Th. A.2] and our statement is the same as above, our z_j being the same as Bellingeri's.

Closed surfaces

When S is a closed surface, one needs to add a single relation to a presentation of $\mathbf{B}_n(S - \mathrm{pt})$ to get a presentation of $\mathbf{B}_n(S)$. In fact, this is a very general fact, which does not require any hypothesis on the surface.

Proposition 4.4.26. *Let S be a connected surface and $x \in S$ any point in its interior. The inclusion of $S - x$ into S induces a surjective homomorphism*

$$\mathbf{B}_n(S - x) \twoheadrightarrow \mathbf{B}_n(S)$$

whose kernel is normally generated by a single element β. Explicitly, β is a braid with $n-1$ trivial strands, whose remaining strand loops once around the puncture x.

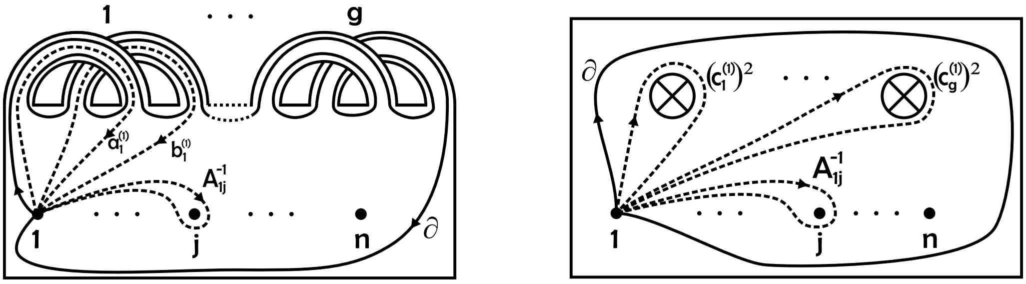
Proof. Choose a subdisc $D \subset S$ containing x in its interior, and a metric on D. Write $\mathcal{U}_n(S, x)$ for the subspace of $C_n(S) = F_n(S)/\mathfrak{S}_n$ of (unordered) configurations that have a unique closest point in D to x (which may be x itself). Together with $C_n(S - x)$, this forms an open cover

$$\{\mathcal{U}_n(S, x), C_n(S - x)\}$$

of $C_n(S)$, with intersection $\mathcal{U}_n(S, x) \cap C_n(S - x) = \mathcal{U}_n(S - x, x)$ the space of n-point configurations in $S - x$ that have a unique closest point in $D - x$ to x. Note that these subspaces of $C_n(S)$ are all path-connected. Let us choose a basepoint for $C_n(S)$ that lies in $\mathcal{U}_n(S - x, x)$. The Seifert-van Kampen theorem then gives us a pushout square of groups:

$$\begin{array}{ccc} \pi_1(\mathcal{U}_n(S - x, x)) & \longrightarrow & \mathbf{B}_n(S - x) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{U}_n(S, x)) & \longrightarrow & \mathbf{B}_n(S). \end{array} \tag{4.5}$$

There is a well-defined projection $\mathcal{U}_n(S, x) \twoheadrightarrow D$ given by remembering just the unique closest point in D to x, which restricts to a projection $\mathcal{U}_n(S - x, x) \twoheadrightarrow D - x$. These are both locally



(a) $\partial = [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] A_{1n}^{-1} \cdots A_{12}^{-1}$ (b) $\partial = c_1^2 \cdots c_g^2 A_{1n}^{-1} \cdots A_{12}^{-1}$

Figure 4.6 The boundary elements in $\mathbf{B}_n(\Sigma_{g,1})$ and $\mathbf{B}_n(\mathcal{N}_{g,1})$.

trivial fibrations with fibres canonically homeomorphic to $C_{n-1}(S - x)$. (Over a point $p \in D$, the homeomorphism is induced by the evident homeomorphism between $S - x$ and $S - \bar{B}_{d(p,x)}(x)$, where $\bar{B}_r(x)$ denotes the closed ball of radius r in D centred at x and $d(-,-)$ is the metric that we chose on D.) The inclusion of the latter into the former is therefore a map of locally trivial fibrations which is the identity on fibres. Considering the induced long exact sequences of homotopy groups, we obtain a map of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{B}_{n-1}(S-x) & \longrightarrow & \pi_1(\mathcal{U}_n(S-x,x)) & \longrightarrow & \pi_1(D-x) \cong \mathbb{Z} & \longrightarrow & 1 \\
 & & \cong \downarrow \nwarrow & \overset{r}{\text{- - - - -}} & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{B}_{n-1}(S-x) & \xrightarrow[\cong]{} & \pi_1(\mathcal{U}_n(S,x)) & \longrightarrow & \pi_1(D) = 1. & &
\end{array} \tag{4.6}$$

The map r obtained from this diagram is a retraction for the upper short exact sequence, whose existence implies that $\pi_1(\mathcal{U}_n(S-x,x))$ is the direct product of $\pi_1(\mathcal{U}_n(S,x))$ and $\mathbb{Z}$. Then, the left-hand vertical map in (4.5) identifies with the projection that forgets the $\mathbb{Z}$ factor. Together with the fact that (4.5) is a pushout square, this implies that the right-hand vertical map in (4.5) is the quotient of $\mathbf{B}_n(S-x)$ by (the subgroup normally generated by) the image of the $\mathbb{Z}$ factor of $\pi_1(\mathcal{U}_n(S-x,x))$ in $\mathbf{B}_n(S-x)$. We may choose for a generator of this $\mathbb{Z}$ factor any element of $\pi_1(\mathcal{U}_n(S-x,x))$ that projects to a generator of $\pi_1(D-x)$, for example the braid described in the proposition. ◻

Let us make this explicit:

Corollary 4.4.27. *For all $g \geqslant 0$, the braid group $\mathbf{B}_n(\Sigma_g)$ is the quotient of $\mathbf{B}_n(\Sigma_{g,1})$ by the relation:*

$$\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 \;=\; \prod_{r=1}^{g} [a_r, b_r^{-1}].$$

Similarly, the braid group $\mathbf{B}_n(\mathcal{N}_g)$ is the quotient of $\mathbf{B}_n(\mathcal{N}_{g,1})$ by the relation:

$$\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 \;=\; c_1^2 \cdots c_g^2.$$

We note that we recover as a particular case the usual presentations of the braid group on the sphere (see [FB62] or [Bir74, Th. 1.11]) and of the projective plane (see [Bus66, §III, page 83]). Not having to treat these as exceptional cases is one of the great advantages of the present method.

Proof of Corollary 4.4.27. This is a direct application of Proposition 4.4.26, using the fact that the punctured surface $\Sigma_g - \mathrm{pt}$ (resp. $\mathcal{N}_g - \mathrm{pt}$) is isotopy equivalent to $\Sigma_{g,1}$ (resp. to $\mathcal{N}_{g,1}$). We note that $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = A_{12} \cdots A_{1n}$ is the (pure) braid obtained by making the first strand turn once around all the other ones; see Figure 4.6 for the relevant drawings. ◻

Partitioned braids on closed surfaces

The above proof of Proposition 4.4.26 generalises to partitioned braid groups without much difficulty; see Remark 4.4.32. However, we prefer to deduce these generalisations directly from Proposition 4.4.26 itself. We begin with the case of pure braids by describing a direct equivalence between Proposition 4.4.26 and the following statement:

Proposition 4.4.28. *Let S be a connected surface and $x \in S$ any point in its interior. The inclusion of $S - x$ into S induces a surjective homomorphism*

$$\mathbf{P}_n(S-x) \twoheadrightarrow \mathbf{P}_n(S)$$

whose kernel is normally generated by n elements $\beta_1, \ldots, \beta_n$. Explicitly, β_i is a braid whose i-th strand loops once around the puncture x, the other strands being trivial.

Equivalence between Proposition 4.4.26 and Proposition 4.4.28. Let N be the subgroup of $\mathbf{B}_n(S-x)$ normally generated by the braid β from Proposition 4.4.26. Note that β is a pure braid, hence we have $N \subseteq \mathbf{P}_n(S-x)$. Now, let N' be the subgroup of $\mathbf{P}_n(S-x)$ normally generated by the β_i. We will show that $N = N'$. This implies the equivalence of Propositions 4.4.26 and 4.4.28 by considering the diagram

$$\begin{array}{ccccccc}
N' & \hookrightarrow & \ker(\pi_{\mathbf{P}}) & \hookrightarrow & \mathbf{P}_n(S-x) & \xrightarrow{\pi_{\mathbf{P}}} & \mathbf{P}_n(S) \\
 & & \| & & \downarrow & & \downarrow \\
N & \hookrightarrow & \ker(\pi_{\mathbf{B}}) & \hookrightarrow & \mathbf{B}_n(S-x) & \xrightarrow{\pi_{\mathbf{B}}} & \mathbf{B}_n(S) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathfrak{S}_n & = & \mathfrak{S}_n
\end{array}$$

and noting that the two propositions are equivalent, respectively, to the statements $N = \ker(\pi_{\mathbf{B}})$ and $N' = \ker(\pi_{\mathbf{P}})$.

Our definitions of β and the β_i are up to some choices, but all these choices give elements conjugate to each other (in $\mathbf{B}_n(S-x)$ or in $\mathbf{P}_n(S-x)$, respectively) or each other's inverses, which does not affect the definition of N and N'. We can make these choices so that:

- the only moving strand of β is the first one,
- β commutes with every element of the subgroup $\mathbf{B}_{1,n-1} = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$ of $\mathbf{B}_n(S-x)$ (which consists of braids in a fixed disc $\mathbb{D} \subset S - x$ involving only the strands 2 to n),
- for each i, $\beta_i = (\sigma_1 \cdots \sigma_{i-1})^{-1} \beta (\sigma_1 \cdots \sigma_{i-1})$.

See Figure 4.7 for an example of such choices. The latter relations imply $N' \subseteq N$.

We now show that N' contains all the conjugates of β by elements of $\mathbf{B}_n(S-x)$, which implies $N' \supseteq N$. In order to do this, we need only show that it contains $t^{-1}\beta t$ for t in a set of representatives of classes modulo $\mathbf{P}_n(S-x)$: then every element of $\mathbf{B}_n(S-x)$ is of the form $t\alpha$ for some such t and some $\alpha \in \mathbf{P}_n(S-x)$, and $(t\alpha)^{-1}\beta(t\alpha) = \alpha^{-1}(t^{-1}\beta t)\alpha$ must be in N'.

Every element $\tau \in \mathfrak{S}_n \cong \mathbf{B}_n(S-x)/\mathbf{P}_n(S-x)$ is the product of an element τ' fixing 1 with some cycle $\tau_1 \cdots \tau_{i-1}$ (precisely, $i = \tau(1)$). Since β commutes with every element of the subgroup $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle \subset \mathbf{B}_n(S-x)$, and since this subgroup surjects onto permutations fixing 1, we can choose a lift t' of τ' commuting with β, so that the lift $t = t'\sigma_1 \cdots \sigma_{i-1}$ of τ to $\mathbf{B}_n(S-x)$ satisfies:

$$\begin{aligned}
t^{-1}\beta t &= (\sigma_1 \cdots \sigma_{i-1})^{-1} t'^{-1} \beta t' (\sigma_1 \cdots \sigma_{i-1}) \\
&= (\sigma_1 \cdots \sigma_{i-1})^{-1} \beta (\sigma_1 \cdots \sigma_{i-1}) = \beta_i \in N',
\end{aligned}$$

whence our result. □

It is not difficult to generalise this to any partitioned braid group:

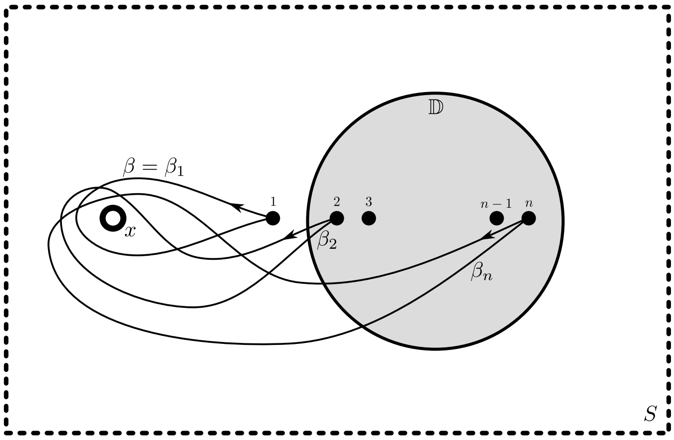


Figure 4.7 Braids β and $\beta_1, \ldots, \beta_n$ from the proof of the equivalence of Propositions 4.4.26 and 4.4.28.

Corollary 4.4.29. *Let S be a connected surface, $x \in S$ any point in its interior and $\lambda = (n_1, \ldots, n_l)$ be a partition of n of length l. The inclusion of $S - x$ into S induces a surjective homomorphism*

$$\mathbf{B}_\lambda(S - x) \twoheadrightarrow \mathbf{B}_\lambda(S)$$

whose kernel is normally generated by l elements $\beta_1, \ldots, \beta_l$. Explicitly, β_i is a braid with $n-1$ trivial strands, except one block in the i-th block which loops once around the puncture x.

Proof. We have $\mathbf{P}_n(S-x) \subset \mathbf{B}_\lambda(S-x) \subset \mathbf{B}_n(S-x)$. As a consequence, if N is a normal subgroup of $\mathbf{P}_n(S-x)$ which is normal in $\mathbf{B}_n(S-x)$, we have $\mathbf{P}_n(S-x)/N \subset \mathbf{B}_\lambda(S-x)/N \subset \mathbf{B}_n(S-x)/N$. Moreover, normal generators for N in $\mathbf{P}_n(S-x)$ are also normal generators for N in $\mathbf{B}_\lambda(S-x)$ and, in fact, fewer generators are needed to normally generate N in $\mathbf{B}_\lambda(S-x)$. Namely, using the notations from the previous proof, we have $\beta_{\alpha+1} = \sigma_\alpha^{-1}\beta_\alpha\sigma_\alpha$, so that $\beta_{\alpha+1}$ and β_α are conjugate in $\mathbf{B}_\lambda(S-x)$ whenever $\sigma_\alpha \in \mathbf{B}_\lambda(S-x)$, which happens when α and $\alpha+1$ are in the same block of λ. As a consequence, we need to pick only one index α in each block of λ in order for the β_α to normally generate N in $\mathbf{B}_\lambda(S-x)$. ☐

Remark 4.4.30. This boils down to considering representatives modulo $\mathbf{B}_\lambda(S-x)$ instead of modulo $\mathbf{P}_n(S-x)$ (that is, elements of $\mathfrak{S}_n/\mathfrak{S}_\lambda$ instead of $\mathfrak{S}_n$) in the previous reasoning. In fact, one can see that there are straightforward equivalences between all these statements for the different partitions of n.

Let us make these statements explicit. We use the usual convention $A_{ji} = A_{ij}$ and $A_{ii} = 1$:

Corollary 4.4.31. *For all $g \geqslant 0$ and any partition $\lambda = (n_1, \ldots, n_l)$, the braid group $\mathbf{B}_\lambda(\Sigma_g)$ is the quotient of $\mathbf{B}_\lambda(\Sigma_{g,1})$ by the relations:*

$$A_{\alpha 1} \cdots A_{\alpha n} \; = \; \prod_{r=1}^{g} [a_r^{(\alpha)}, (b_r^{(\alpha)})^{-1}].$$

Similarly, the braid group $\mathbf{B}_\lambda(\mathcal{N}_g)$ is the quotient of $\mathbf{B}_\lambda(\mathcal{N}_{g,1})$ by the relations:

$$A_{\alpha 1} \cdots A_{\alpha n} \; = \; (c_1^{(\alpha)})^2 \cdots (c_g^{(\alpha)})^2.$$

In both cases, α runs through any set of representatives of the blocks of λ.

Remark 4.4.32. Instead of the reasoning above, one could adapt the proof of Proposition 4.4.26 to partitioned configuration spaces. Precisely, we get an open cover $\{\mathcal{U}_\lambda(S,x), C_\lambda(S-x)\}$ of $C_\lambda(S) = F_n(S)/\mathfrak{S}_\lambda$ by an obvious adaptation of the definition of $\mathcal{U}_n(S,x)$. However, this time $\mathcal{U}_\lambda(S,x)$ is disconnected, with one path-component for each block of the partition λ. As a consequence, one needs to apply the Seifert-van Kampen theorem once for each path-component of $\mathcal{U}_\lambda(S,x)$, resulting in taking the quotient of the fundamental group by one additional relation for each application of the theorem. In doing so, one needs to be careful about basepoints, since one obviously cannot choose a common basepoint in the different path-components of $\mathcal{U}_\lambda(S,x)$.

4.4.4 The lower central series of the whole group

We now turn to the study of the LCS of $\mathbf{B}_n(S)$, which we completely determine for any $n \geqslant 3$ and any surface S. We begin by computing the abelianisation of this group; see Proposition 4.4.33. Then we study $\mathbf{B}_n(S)/\varGamma_\infty$, and we show that when $n \geqslant 3$, it is nilpotent of class at most 2, which means that the LCS of $\mathbf{B}_n(S)$ stops at most at $\varGamma_3$; see Theorem 4.4.35 and Corollary 4.4.37. Finally, we compute the Lie ring, generalising a result of [BGG08]; see Theorem 4.4.43.

The abelianisation

We first compute the abelianisation of $\mathbf{B}_n(S)$, for any n and any surface S. In order to do this, recall that the group morphism $\varphi\colon \mathbf{B}_n \to \mathbf{B}_n(S)$ induces a map $\mathbf{B}_n^{\mathrm{ab}} \to \mathbf{B}_n(S)^{\mathrm{ab}}$, and that, since all the σ_i are identified in $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$, they are so also in $\mathbf{B}_n(S)^{\mathrm{ab}}$. We denote by σ their common image in $\mathbf{B}_n(S)^{\mathrm{ab}}$.

Proposition 4.4.33. *In general, for all $n \geqslant 2$, we have:*

$$\mathbf{B}_n(S)^{\mathrm{ab}} \cong \pi_1(S)^{\mathrm{ab}} \times \langle \sigma \rangle.$$

Moreover, σ is:

- *of infinite order if S is planar,*
- *of order $2(n-1)$ if $S \cong \mathbb{S}^2$,*
- *of order 2 in all the other cases.*

Proof. Consider the short exact sequence $\mathbf{P}_n^{\circ}(S) \hookrightarrow \mathbf{B}_n(S) \overset{\pi_S}{\twoheadrightarrow} \pi_1(S) \wr \mathfrak{S}_n$. By Lemma 4.1.11, it induces an exact sequence of abelian groups:

$$(\mathbf{P}_n^{\circ}(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)} \longrightarrow \mathbf{B}_n(S)^{\mathrm{ab}} \longrightarrow\!\!\!\!\to (\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}} \longrightarrow 0.$$

On the one hand, the quotient $(\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}}$ is isomorphic to $\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2$ (Lemma 4.5.24). On the other hand, it follows from Proposition 4.4.10 that $\mathbf{P}_n^{\circ}(S)$ is generated by $\mathbf{P}_n$ under the action of $\mathbf{B}_n(S)$. As a consequence, the map $\mathbf{P}_n^{\mathrm{ab}} \to (\mathbf{P}_n^{\circ}(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)}$ induced by φ is surjective. Moreover, it factors through $(\mathbf{P}_n^{\mathrm{ab}})_{\mathbf{B}_n} = (\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_n} \cong \mathbb{Z}$. Thus $(\mathbf{P}_n^{\circ}(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)}$ is cyclic, and its image in $\mathbf{B}_n(S)^{\mathrm{ab}}$ is generated by σ^2, which is the image of any pure braid generator.

All of this implies that $\mathbf{B}_n(S)^{\mathrm{ab}}/\langle \sigma \rangle \cong \pi_1(S)^{\mathrm{ab}}$. Moreover, the corresponding projection map $\mathbf{B}_n(S)^{\mathrm{ab}} \twoheadrightarrow \pi_1(S)^{\mathrm{ab}}$ splits, a splitting being induced by any of the ψ_i. As a consequence, $\mathbf{B}_n(S)^{\mathrm{ab}}$ identifies with $\pi_1(S)^{\mathrm{ab}} \times \langle \sigma \rangle$, and we can use Proposition 4.4.19 to get a complete calculation. ◻

The lower central series

Let us now turn to the study of the LCS of $\mathbf{B}_n(S)$. Our main tool for studying it is a decomposition theorem (Theorem 4.4.35 below), whose proof relies on the following:

Lemma 4.4.34. *Let $n \geqslant 3$. The image of $\mathbf{B}_n$ in $\mathbf{B}_n(S)/\varGamma_\infty$ is cyclic, and it is central. Namely, it is generated by the common class σ of the usual generators σ_i of $\mathbf{B}_n$.*

Proof. The morphism φ sends $\varGamma_\infty(\mathbf{B}_n)$ to $\varGamma_\infty(\mathbf{B}_n(S))$. We know that $\sigma_i\sigma_j^{-1} \in \varGamma_\infty(\mathbf{B}_n)$ for all $i, j < n$, so $\sigma_i \equiv \sigma_j \pmod{\varGamma_\infty(\mathbf{B}_n(S))}$. Let us denote by $\sigma \in \mathbf{B}_n(S)/\varGamma_\infty$ the common image of the σ_i. Since the σ_i generate $\mathbf{B}_n$, the image of $\mathbf{B}_n$ in $\mathbf{B}_n(S)/\varGamma_\infty$ is the cyclic subgroup generated by σ. In particular, its elements commute with σ. Moreover, for all $\gamma \in \pi_1(S - \mathbb{D})$, the braids σ_2 and $\psi_1(\gamma)$ have disjoint support, hence σ also commutes with the image of ψ_1. Since the images of φ and ψ_1 generate $\mathbf{B}_n(S)$ by Proposition 4.4.10, this means that σ is a central element of $\mathbf{B}_n(S)/\varGamma_\infty$. ◻

We can now state our main decomposition theorem:

Theorem 4.4.35. *For all $n \geqslant 3$, there is a central extension:*

$$\langle \sigma^2 \rangle \hookrightarrow \mathbf{B}_n(S)/\varGamma_\infty \twoheadrightarrow \pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2.$$

Proof. Since π_S is surjective, it sends $\varGamma_\infty(\mathbf{B}_n(S))$ onto a normal subgroup of $\pi_1(S) \wr \mathfrak{S}_n$. This normal subgroup is contained in $\varGamma_\infty(\pi_1(S) \wr \mathfrak{S}_n)$ and contains the $\tau_i\tau_j^{-1}$, which are the images of the $\sigma_i\sigma_j^{-1} \in \varGamma_\infty(\mathbf{B}_n(S))$. By Lemma 4.5.26, it is equal to $\varGamma_2(\pi_1(S) \wr \mathfrak{S}_n)$. Then, we can apply the Nine Lemma to the diagram:

$$\begin{array}{ccccc}
\varGamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S) & \hookrightarrow & \varGamma_\infty(\mathbf{B}_n(S)) & \twoheadrightarrow & \varGamma_2(\pi_1(S) \wr \mathfrak{S}_n). \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}_n^\circ(S) & \hookrightarrow & \mathbf{B}_n(S) & \overset{\pi_S}{\twoheadrightarrow} & \pi_1(S) \wr \mathfrak{S}_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}_n^\circ(S)/\varGamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S) & \dashrightarrow & \mathbf{B}_n(S)/\varGamma_\infty & \overset{\overline{\pi}_S}{\dashrightarrow} & (\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}}.
\end{array}$$

Then recall that $(\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong \pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2$. Thus, we are left with analysing the kernel $\mathbf{P}_n^\circ(S)/\varGamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S)$ of $\overline{\pi}_S$, which is the image of $\mathbf{P}_n^\circ(S)$ in $\mathbf{B}_n(S)/\varGamma_\infty$. Since $\mathbf{P}_n^\circ(S)$ is the normal closure of $\mathbf{P}_n$ in $\mathbf{B}_n(S)$ by Proposition 4.4.10, its image in $\mathbf{B}_n(S)/\varGamma_\infty$ is the normal closure of the image of $\mathbf{P}_n$. But $\mathbf{P}_n$ is sent to $\langle \sigma^2 \rangle$ which is central (and, in particular, normal) in $\mathbf{B}_n(S)/\varGamma_\infty$, whence the result. □

Remark 4.4.36. We also have a central extension:

$$\langle \sigma \rangle \hookrightarrow \mathbf{B}_n(S)/\varGamma_\infty \twoheadrightarrow \pi_1(S)^{\mathrm{ab}}.$$

This slightly different statement tells us slightly different things: it implies that σ is central, whereas the statement of the theorem says that $\overline{\sigma}$ is not trivial in $\mathbf{B}_n(S)^{\mathrm{ab}}$.

Corollary 4.4.37. *For any $n \geqslant 3$, we have $\varGamma_3(\mathbf{B}_n(S)) = \varGamma_4(\mathbf{B}_n(S))$.*

Proof. Proposition 4.4.35 implies that $\mathbf{B}_n(S)/\varGamma_\infty$ is 2-nilpotent, which means exactly that its $\varGamma_3$ is trivial. In other words, $\varGamma_3 \subseteq \varGamma_\infty$ for $\mathbf{B}_n(S)$. □

The remaining cases consist in the cases when $n = 1$ and when $n = 2$.

Proposition 4.4.38. *For any connected surface S, either $\mathbf{B}_1(S) = \pi_1(S)$ is abelian, which occurs precisely when $S \in \{\mathbb{D} - pt, \mathbb{D}, \mathbb{S}^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{M}^2\}$ up to isotopy equivalence, or its LCS does not stop.*

Proof. If S is closed and not in $\{\mathbb{S}^2, \mathbb{P}^2, \mathbb{T}^2\}$ then the LCS of $\pi_1(S)$ does not stop. To see this, note that it admits a presentation of the form $\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$ or $\langle c_1, \ldots, c_g \mid c_1^2 \cdots c_g^2 = 1 \rangle$ for $g \geqslant 2$, and these both project onto $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$ by sending a_1, b_1, c_1 to x and all other generators to y; the LCS of $\mathbb{Z}/2 * \mathbb{Z}/2$ does not stop by Proposition 4.5.16. If S is not closed, its fundamental group is free by Proposition 4.4.4. If it is also not in $\{\mathbb{D} - pt, \mathbb{D}, \mathbb{M}^2\}$ then its fundamental group is moreover *non-abelian* free, and hence its LCS does not stop by [Mag35] (see also [MKS04, Chap. 5]). □

Remark 4.4.39. The fundamental group of a surface is known to always be residually nilpotent, and almost always residually torsion-free nilpotent (the only exceptions to the latter being the projective plane and the Klein bottle). This follows from [MKS04, Chap. 5] for non-closed surfaces, whose fundamental group is free. For closed surfaces, it follows from [Bau10] if S contains a handle: then the Seifert-van Kampen theorem applied to a decomposition $S = \mathbb{T}^2 \# S'$ gives a presentation of the form of Theorem 1 therein. The remaining surfaces are the sphere, the projective plane and the Klein bottle, whose LCS are easily computed explicitly (see Proposition 4.5.4 for the last one).

Proposition 4.4.40. *When S is not $\mathbb{D}$, $\mathbb{S}^2$ or $\mathbb{P}^2$, up to isotopy equivalence, the LCS of $\mathbf{B}_2(S)$ does not stop.*

Proof. The group $\mathbf{B}_2(S)$ surjects onto $\pi_1(S) \wr \mathfrak{S}_2$. Since $\pi_1(S)^{\mathrm{ab}}$ surjects onto $\mathbb{Z}$ except in the three excluded cases of the statement, we can apply Corollary 4.5.29 to see that the LCS of $\pi_1(S) \wr \mathfrak{S}_2$ does not stop. Then the one of $\mathbf{B}_n(S)$ does not either by Lemma 4.1.1. ☐

Remark 4.4.41. For orientable surfaces, $\mathbf{B}_2(S)$ is residually nilpotent [BB09b, Cor. 10]. This result can in fact be extended to non-orientable surfaces, with the same method, using the results quoted in Remark 4.4.39.

Remark 4.4.42. Clearly, the LCS of $\mathbf{B}_2(\mathbb{D}) \cong \mathbb{Z}$ and of $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$ both stop at $\varGamma_2$. The fact that $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$ is the case $n = 2$ and $g = 0$ of Corollary 4.4.27. On the other hand, the group $\mathbf{B}_2(\mathbb{P}^2)$ is the dicyclic group of order 16 (Corollary 4.4.83), which is 3-nilpotent, so its LCS stops at $\varGamma_4$.

The Lie ring

We can be more precise about our description of the LCS of surface braid groups. In particular, Corollary 4.4.37 says that, for $n \geqslant 3$, the LCS stops at most at $\varGamma_3$, but it does not say when it stops at $\varGamma_2$ (then the associated Lie ring consists only of the abelianisation, which has already been computed in Proposition 4.4.33), or when it stops at $\varGamma_3$ (in which case the Lie ring is 2-nilpotent but not abelian). We now show that the latter holds only for non-planar orientable surfaces, and we compute precisely $\mathcal{L}(\mathbf{B}_n(S))$, generalising results of [BGG08].

Theorem 4.4.43. *Let $n \geqslant 3$ be an integer and S be a connected surface. The LCS of $\mathbf{B}_n(S)$:*

- stops at $\varGamma_2$ *if S is either planar or non-orientable, or if $S \cong \mathbb{S}^2$.*
- stops at $\varGamma_3$ *in the other cases. Then $\mathcal{L}_2(\mathbf{B}_n(S))$ is cyclic, generated by the common class σ^2 of the pure braid generators A_{ij}.*

Moreover, in the second case, σ^2 is of finite order if and only if S is closed, in which case its order is $n + g - 1$, where g is the genus of S.

Proof. Since the LCS of $\mathbf{B}_n(S)$ stops at most at $\varGamma_3$, $\mathcal{L}_2(\mathbf{B}_n(S))$ identifies with $\varGamma_2\left(\mathbf{B}_n(S)/\varGamma_\infty\right)$ (the latter being $\varGamma_2/\varGamma_\infty(\mathbf{B}_n(S)) = \varGamma_2/\varGamma_3(\mathbf{B}_n(S))$). Moreover, using the central extension of Theorem 4.4.35, we see that in $\mathbf{B}_n(S)/\varGamma_\infty$, the subgroup $\varGamma_2$ must be contained in $\langle\sigma^2\rangle$, which implies that it is cyclic, generated by a power of σ^2.

<u>Planar surfaces:</u> if S is planar, then the common class σ of the σ_i in $\mathbf{B}_n(S)^{\mathrm{ab}} = (\mathbf{B}_n(S)/\varGamma_\infty)^{\mathrm{ab}}$ is of infinite order by Proposition 4.4.33. Hence $\varGamma_2(\mathbf{B}_n(S)/\varGamma_\infty)$ does not contain any power of $\sigma = \overline{\sigma_1}$. But $\varGamma_2(\mathbf{B}_n(S)/\varGamma_\infty)$ is contained in $\langle\sigma^2\rangle$, so it must be trivial, which means that $\varGamma_2 = \varGamma_\infty$ for $\mathbf{B}_n(S)$.

<u>The sphere:</u> if $S = \mathbb{S}^2$, then $\mathbf{B}_n(\mathbb{S}^2)$ is a quotient of $\mathbf{B}_n$ (see Corollary 4.4.27), which implies that its LCS also stops at $\varGamma_2$.

<u>Non-orientable surfaces:</u> the surface S is non-orientable if and only if it contains an embedded Möbius band. Then $\sigma_1^2 = [\sigma_1 c \sigma_1^{-1}, c^{-1}]$ for some $c \in \mathbf{B}_n(S)$: precisely, c is a braid whose first strand goes around the Möbius strip once, that is, through the crosscap; see Figure 4.3a. Since σ_1 is sent to the central element σ of $\mathbf{B}_n(S)/\varGamma_\infty$, this relation implies that $\sigma^2 = [\bar{c}, \bar{c}^{-1}] = 1$. Thus $\langle\sigma^2\rangle$ is trivial, and so is its subgroup $\varGamma_2\left(\mathbf{B}_n(S)/\varGamma_\infty\right) \cong \mathcal{L}_2(\mathbf{B}_n(S))$.

<u>Non-planar orientable surfaces:</u> if S has a handle, then $\sigma_1^2 = [a^{-1}, \sigma_1 b \sigma_1^{-1}]$ for some $a, b \in \mathbf{B}_n(S)$. Precisely, a and b are braids whose first strands go around a handle; see Figure 4.3b. Hence $\sigma^2 \in \varGamma_2\left(\mathbf{B}_n(S)/\varGamma_\infty\right)$, which implies that $\mathcal{L}_2(\mathbf{B}_n(S))$ is generated by σ^2.

If S is a <u>non-planar compact orientable surface with at least one boundary component</u>, then S can be embedded in some $\Sigma_{g,1}$ for an arbitrary large g, by attaching a disc to each boundary component save one. The induced map $\mathcal{L}_2(\mathbf{B}_n(S)) \to \mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1}))$ sends σ^2 to σ^2, and the latter is of infinite order (Proposition 4.4.44).

If S is a <u>non-compact non-planar orientable surface</u>, let us suppose that σ^2 is a torsion element in $\mathbf{B}_n(S)/\varGamma_\infty = \mathbf{B}_n(S)/\varGamma_3$. Then for some integer k, σ_1^{2k} is equal to some product of commutators of length at least 3 in $\mathbf{B}_n(S)$. Such a formula involves only a finite number of braids. Let us choose a representative of each of these isotopy classes. These involve a finite number of paths on the surface. If moreover we choose an isotopy realising the aforementioned equality of braids (using the concatenation of the chosen representatives as a representative of the right-hand side), the image of this isotopy is contained in a compact subsurface S' of S. Thus, our formula also holds in $\mathbf{B}_n(S')$: σ_1^{2k} is equal to some product of commutators of length at least 3 in there. Then $\sigma^{2k} = 1$ in $\mathbf{B}_n(S')/\varGamma_3$. However, this contradicts the previous case, since S' cannot be closed (nor planar). We conclude that σ^2 cannot be a torsion element in $\mathbf{B}_n(S)/\varGamma_\infty$; equivalently, $\mathcal{L}_2(\mathbf{B}_n(S)) \cong \mathbb{Z}$.

If S is a closed orientable surface Σ_g of genus $g \geqslant 1$, then $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g)) \cong \mathbb{Z}/(n+g-1)$. This is [BGG08, Th. 1], which can be deduced from Proposition 4.4.44 below. We will do so in a more general context later; see Proposition 4.4.57. □

The proof of the following result is inspired from [BGG08]. In fact, it is equivalent to [BGG08, Th. 1] in a quite straightforward way; see Remark 4.4.45.

Proposition 4.4.44. *Let $g \geqslant 1$, and let $\Sigma_{g,1}$ denote the compact surface of genus g with one boundary component. For every $n \geqslant 1$, $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1})) \cong \mathbb{Z}$, generated by the common class σ^2 of the pure braid generators.*

Proof. We compute completely $\mathbf{B}_n(\Sigma_{g,1})/\varGamma_3$. Let us consider the quotient of $\mathbf{B}_n(\Sigma_{g,1})$ by the relations $\sigma_i = \sigma_{i+1}$, that is, by the normal closure N of the $\sigma_i\sigma_{i+1}^{-1}$ for $1 \leqslant i \leqslant n-1$. We already know that, modulo $\varGamma_3$, the braid generators σ_i have a common class σ. In particular, $N \subseteq \varGamma_3$, and we will show that $N = \varGamma_3$ by showing that $G := \mathbf{B}_n(\Sigma_{g,1})/N$ is 2-nilpotent. Thanks to Proposition 4.4.21, we can compute a presentation of this quotient. It is generated by σ, together with a_r and b_r for $1 \leqslant r \leqslant g$. The braid relations on the σ_i become trivial there. $(BS1)$ says that σ is central. $(BS2)$ says that the a_r and the b_s commute with one another, except a_r and b_r (for each r). Since σ is central, $(BS3)$ becomes trivial. Finally, $(BS4)$ can be written as $b_r a_r b_r^{-1} = a_r\sigma^{-2}$. From this presentation, one can see that $G \cong (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, where the three factors are free abelian on σ, the a_r and the b_r respectively; the action of each b_r is trivial on σ and the a_s if $s \neq r$, and $b_r \cdot a_r = a_r - 2\sigma$. Since this group is 2-nilpotent, we have $N = \varGamma_3(\mathbf{B}_n(\Sigma_{g,1}))$, as announced. Moreover, the Lie ring of G, which is the Lie ring of $\mathbf{B}_n(\Sigma_{g,1})$ by Corollary 4.4.37, is easy to compute. Namely, $\mathcal{L}_1(G) = G^{\mathrm{ab}} \cong (\mathbb{Z}/2)^2 \times \mathbb{Z}^{2g}$, $\mathcal{L}_2(G) = \varGamma_2(G) \cong \mathbb{Z}$ is generated by $\overline{\sigma^2}$, and the only non-trivial brackets of generators are $[\overline{a}_r, \overline{b}_r] = \overline{\sigma^2}$. □

Remark 4.4.45. Here we choose to recover [BGG08, Th. 1] from Proposition 4.4.44. However, one can also *deduce* Proposition 4.4.44 from [BGG08, Th. 1]. Namely, one can embed $\Sigma_{g,1}$ into $\Sigma_{g'}$ for any $g' \geqslant g$, by attaching $\Sigma_{g'-g,1}$ along the boundary component. The induced map $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1})) \to \mathcal{L}_2(\mathbf{B}_n(\Sigma_{g'}))$ sends σ^2 to σ^2, and the latter is of order $n + g' - 1$ by [BGG08, Th. 1]. Thus, by varying g', we see that σ^2 cannot be of finite order inside $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1}))$.

Remark 4.4.46. The group $(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$ appearing in the proof of Proposition 4.4.44, which is then the maximal nilpotent quotient of $\mathbf{B}_n(\Sigma_{g,1})$, has a nice interpretation as a matrix group resembling the Heisenberg group of a symplectic vector space. Namely, it is the subgroup of $\mathrm{GL}_{g+2}(\mathbb{Q})$ given by:

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ & 1 & 0 & \cdots & 0 & \mathbb{Z} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & \mathbb{Z} \\ & & & & 1 & \mathbb{Z} \\ & & & & & 1 \end{pmatrix}$$

Remark 4.4.47 (About the Riemann-Hurwitz formula)**.** The result [BGG08, Th. 1] quoted above and recovered below (Proposition 4.4.57) says that when S is an orientable closed surface, the information encoded in $\mathcal{L}_2(\mathbf{B}_n(S))$ is essentially the genus of S, or its Euler characteristic. In fact, one can recover the Riemann-Hurwitz formula for (unramified) coverings of closed orientable surfaces from this computation. Indeed, let $p\colon \Sigma_h \twoheadrightarrow \Sigma_g$ be a k-sheeted covering. It induces a continuous map $p^*\colon C_n(\Sigma_g) \to C_{kn}(\Sigma_h)$ between unordered configuration spaces sending a configuration of n

points to the configuration of the kn preimages of these points by p. The induced map between the fundamental groups sends the pure braid generator A_{12} to the product of the $A_{j,j+1}$ for $j \in p^{-1}(1)$ (if the correct conventions are chosen). Thus the induced map from $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g))$ to $\mathcal{L}_2(\mathbf{B}_{kn}(\Sigma_h))$ sends σ^2 to $k\sigma^2$. Since σ^2 is of order $n+g-1$ in $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g))$, we obtain $(n+g-1)k\sigma^2 = 0$ in $\mathcal{L}_2(\mathbf{B}_{kn}(\Sigma_h))$. However, σ^2 is of order $kn+h-1$ there, implying that $kn+h-1$ divides $kn+k(g-1)$. This holds for all n. For n big enough, $kn+k(g-1) < 2(kn+h-1)$, so the only possibility is that $kn+k(g-1) = kn+h-1$, that is, $k(g-1) = h-1$, which is equivalent to the Riemann-Hurwitz formula for p.

4.4.5 Partitioned braids on surfaces

Let us now study the LCS of partitioned surface braid groups $\mathbf{B}_\lambda(S)$, generalising the results from §4.4.4, which can be seen as the case of the trivial partition $\lambda = (n)$ of n. We follow the same steps: we first compute the abelianisations of $\mathbf{B}_\lambda(S)$ in Proposition 4.4.48, before studying $\mathbf{B}_\lambda(S)/\varGamma_\infty$ and showing that the LCS of $\mathbf{B}_\lambda(S)$ stops at most at $\varGamma_3$ when the partition λ has only blocks of size at least 3, in Theorem 4.4.51 and Corollary 4.4.52. Finally, under the latter hypothesis, we compute the associated Lie rings in Theorem 4.4.53.

The abelianisation

Let $\lambda = (n_1, \ldots, n_l)$ be a partition of an integer n. A computation of $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ can be obtained by a quite straightforward generalisation of the computation of $\mathbf{B}_n(S)^{\mathrm{ab}}$ from Proposition 4.4.33.

Let us first recall that the morphism φ from §4.4.2 induces a map $\mathbf{B}_\lambda^{\mathrm{ab}} \to \mathbf{B}_\lambda(S)^{\mathrm{ab}}$. Then, from Proposition 4.3.5, we get that:

- For each $i \leqslant l$ such that $n_i \geqslant 2$, all the σ_α with α and $\alpha+1$ in the i-th block of λ have a common image in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$, called s_i.
- For each $i < j \leqslant l$, all the $A_{\alpha\beta}$ with α (resp. β) in the i-th (resp. the j-th) block of λ have a common image in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$, called a_{ij} (or a_{ji}).

Let us now consider the short exact sequence:

$$\mathbf{P}_n^\circ(S) \hookrightarrow \mathbf{B}_\lambda(S) \overset{\pi_S}{\twoheadrightarrow} \pi_1(S) \wr \mathfrak{S}_\lambda.$$

We can apply Lemma 4.1.11 to it, and we get an exact sequence of abelian groups:

$$(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)} \longrightarrow \mathbf{B}_\lambda(S)^{\mathrm{ab}} \longrightarrow (\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \longrightarrow 0.$$

On the one hand, the quotient $(\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}}$ is isomorphic to the product of the $(\pi_1(S) \wr \mathfrak{S}_{n_i})^{\mathrm{ab}}$, which is $(\pi_1(S)^{\mathrm{ab}})^l \times (\mathbb{Z}/2)^{l'}$, where l' is the number of indices $i \leqslant l$ such that $n_i \geqslant 2$ (see Corollary 4.5.25). On the other hand, it follows from Proposition 4.4.10 that $\mathbf{P}_n^\circ(S)$ is generated by $\mathbf{P}_n$ under the action of $\mathbf{P}_n(S)$, which is a subgroup of $\mathbf{B}_\lambda(S)$. As a consequence, the map $\mathbf{P}_n^{\mathrm{ab}} \to (\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)}$ induced by φ is surjective. Moreover, it factors through the quotient $(\mathbf{P}_n^{\mathrm{ab}})_{\mathbf{B}_\lambda} = (\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_\lambda} \cong \mathbb{Z}^{l(l-1)/2} \times \mathbb{Z}^{l'}$. Thus the image of $(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)}$ in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ is generated by the images of the pure braid generators, which are the elements $2s_i$ for $i \leqslant l$ such that $n_i \geqslant 2$, and a_{ij} for $i < j \leqslant l$.

Now, let H be the subgroup of $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ generated by the s_i and the a_{ij}. From the above, we get an isomorphism $\mathbf{B}_\lambda(S)^{\mathrm{ab}}/H \cong (\pi_1(S)^{\mathrm{ab}})^l$. Moreover, the corresponding projection $\mathbf{B}_\lambda(S)^{\mathrm{ab}} \twoheadrightarrow (\pi_1(S)^{\mathrm{ab}})^l$ splits: a splitting is defined by sending $(g_i)_i$ to $\sum \psi_i(g_i)$ where, for each i, ψ_i is induced by ψ_α for any α in the i-th block of λ. As a consequence, $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ identifies with $(\pi_1(S)^{\mathrm{ab}})^l \times H$, and we can use Proposition 4.4.19 to get a complete calculation:

Proposition 4.4.48. *Let $n \geqslant 1$ and $\lambda = (n_1, \ldots, n_l)$ be a partition of n. Then:*

$$\mathbf{B}_\lambda(S)^{\mathrm{ab}} \;\cong\; \left(\pi_1(S)^{\mathrm{ab}}\right)^l \times \left(\mathbf{B}_\lambda^{\mathrm{ab}}/R\right),$$

where $\mathbf{B}_\lambda^{\mathrm{ab}}$ is free abelian on the s_i and the a_{ij} from Proposition 4.3.5, and R is:

- *trivial if S is planar,*
- *generated by the relations $2(n_i-1)s_i+\sum_{j\neq i} n_j a_{ij}$ $(i=1,\dots,n)$ if $S\cong \mathbb{S}^2$,*
- *generated by the $2s_i$ and the a_{ij} in all the other cases.*

Explicitly, if l' denotes the number of indices $i\leqslant l$ such that $n_i\geqslant 2$, we have, for $S\ncong\mathbb{S}^2$:

$$\mathbf{B}_\lambda^{\mathrm{ab}}/R \;\cong\; \begin{cases} \mathbb{Z}^{l'}\times\mathbb{Z}^{l(l-1)/2} & \text{if } S \text{ is planar,}\\ (\mathbb{Z}/2)^{l'} & \text{if } S \text{ does not embed into the sphere.}\end{cases}$$

Proof. If $S\neq\mathbb{S}^2$, all the relations are direct consequences of Proposition 4.4.19, and it remains to prove that they are the only ones. In the first case, we can directly use the result of Proposition 4.3.5: an embedding of S into $\mathbb{R}^2\cong\mathbb{S}^2-pt$ induces a retraction $\mathbf{B}_\lambda(S)^{\mathrm{ab}}\twoheadrightarrow\mathbf{B}_\lambda^{\mathrm{ab}}$ of the morphism $\mathbf{B}_\lambda^{\mathrm{ab}}\to\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ induced by φ, whence the result in this case. In the third one, the same proof as the proof of Proposition 4.3.5 works: projections onto the factors are given by projections onto the $\mathfrak{S}_{n_i}^{\mathrm{ab}}\cong\mathbb{Z}/2$ for $n_i\geqslant 2$.

If $S=\mathbb{S}^2$, Corollary 4.4.31 describes $\mathbf{B}_\lambda(\mathbb{S}^2)$ as the quotient of $\mathbf{B}_\lambda$ by the collection of relations $A_{\alpha 1}A_{\alpha 2}\cdots A_{\alpha n}=1$ for all $1\leqslant\alpha\leqslant n$. The abelianisation $\mathbf{B}_\lambda(\mathbb{S}^2)^{\mathrm{ab}}$ is then the quotient of $\mathbf{B}_\lambda^{\mathrm{ab}}$ (described in Proposition 4.3.5) by the classes of the above relations. If α is in the i-th block of λ, then the class of $A_{\alpha\beta}$ is either $2s_i$ if β is also in the i-th block (which holds for n_i-1 values of β), or a_{ij} if β is in the j-th block for some $j\neq i$ (which happens for n_j values of β, for each $j\neq i$). Thus the classes of the $A_{\alpha 1}A_{\alpha 2}\cdots A_{\alpha n}$ are indeed the relations of the statement. □

Remark 4.4.49. The relation $2(n_i-1)s_i+\sum_{j\neq i}n_j a_{ij}$ makes sense even if $n_i=1$: in this case, with our conventions, there is no generator s_i, but this does not matter since its coefficient is $2(n_i-1)=0$ (alternatively, one could add a generator s_i and ask that $s_i=0$).

The lower central series

The work done above to show that the LCS of $\mathbf{B}_n(S)$ stops if $n\geqslant 3$ (see Corollary 4.4.37) generalises to the partitioned braid group when all the blocks of the partition are of size at least 3. First, here follows the generalisation of Lemma 4.4.34.

Lemma 4.4.50. *Let $\lambda=(n_1,\dots,n_l)$ be a partition of n, with $n_i\geqslant 3$ for all i. The image of $\mathbf{B}_\lambda$ in $\mathbf{B}_\lambda(S)/\Gamma_\infty$ is central. In particular, it is a quotient of $\mathbf{B}_\lambda^{\mathrm{ab}}$.*

Proof. It follows from Theorem 4.3.6 that $\Gamma_\infty(\mathbf{B}_\lambda)=\Gamma_2(\mathbf{B}_\lambda)$. As a consequence, the morphism $\varphi\colon\mathbf{B}_\lambda\to\mathbf{B}_\lambda(S)/\Gamma_\infty$ factors through $\mathbf{B}_\lambda/\Gamma_2=\mathbf{B}_\lambda^{\mathrm{ab}}$. Hence its image is abelian, generated by the images of the s_i and the a_{ij} from Proposition 4.3.5.

In order to show that it is central, we need to show that these elements commute with generators of $\mathbf{B}_\lambda(S)/\Gamma_\infty$. We deduce from Proposition 4.4.10 that $\mathbf{B}_\lambda(S)$ is generated by the images of φ and of the ψ_α. In fact, we can restrict to taking one α in each block of λ, since ψ_α and ψ_β are conjugated by elements of $\mathbf{B}_\lambda$ if α and β are in the same block. The s_i and the a_{ij} already commute with each other, so we only need to show that they commute with the images of the selected ψ_α. Since the blocks of λ have size at least 3, we can find representatives of the s_i and the a_{ij} having disjoint support with elements of all the chosen $\mathrm{Im}(\psi_\alpha)$. Thus, the s_i and the a_{ij} commute with a family of generators of $\mathbf{B}_\lambda(S)/\Gamma_\infty$, which proves our claim. □

We can now generalise our main decomposition theorem (Theorem 4.4.35) to partitioned braids:

Theorem 4.4.51. *Let $\lambda=(n_1,\dots,n_l)$ be a partition of n, with $n_i\geqslant 3$ for all i. There is a central extension:*

$$\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}\hookrightarrow\mathbf{B}_\lambda(S)/\Gamma_\infty\twoheadrightarrow(\pi_1(S)^{\mathrm{ab}}\times\mathbb{Z}/2)^l.$$

Proof. The proof is essentially the same as the proof of Theorem 4.4.35, so we only stress what changes. The element $\sigma_\alpha\sigma_\beta^{-1}$ is in $\Gamma_\infty(\mathbf{B}_n(S))$ only when α and β are in the same block of

λ. However, their images $\tau_\alpha\tau_\beta^{-1}$ in $\pi_1(S) \wr \mathfrak{S}_\lambda$ still normally generate $\Gamma_2(\pi_1(S) \wr \mathfrak{S}_\lambda)$, because $\pi_1(S) \wr \mathfrak{S}_\lambda$ is the product of the $\pi_1(S) \wr \mathfrak{S}_{n_i}$, whose Γ_2 is normally generated by the $\tau_\alpha\tau_\beta^{-1}$ for α and β in the i-th block of λ (Lemma 4.5.26). Thus, arguing exactly as in the proof of Theorem 4.4.35, we get a short exact sequence:

$$\mathbf{P}_n^\circ(S)/\Gamma_\infty(\mathbf{B}_\lambda(S)) \cap \mathbf{P}_n^\circ(S) \hookrightarrow \mathbf{B}_\lambda(S)/\Gamma_\infty \xrightarrow{\pi_S} (\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}}.$$

Recall that $(\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2)^l$. Moreover, the kernel is the normal closure of the image of $\mathbf{P}_n$, but this image is already normal, and even central by Lemma 4.4.50, generated by the a_{ij} and the squares of the s_i. ∎

Finally, Corollary 4.4.37 adapts readily to this context:

Corollary 4.4.52. *If all the blocks of λ have size at least 3, then:*

$$\Gamma_3(\mathbf{B}_\lambda(S)) = \Gamma_4(\mathbf{B}_\lambda(S)).$$

The Lie ring

Here again, in the stable case (that is, when all the blocks of the partition λ have size at least 3), we can be more precise about the description of the LCS. Namely, our next goal is to show the following generalisation of Theorem 4.4.43, whose proof occupies the rest of the present section.

Theorem 4.4.53. *Let $\lambda = (n_1, \ldots, n_l)$ be a partition of an integer n with $n_i \geqslant 3$ for all i, and let S be a connected surface. The LCS of $\mathbf{B}_\lambda(S)$:*

- stops at Γ_2 *if S is planar, if $S \cong \mathbb{S}^2$ or if $l = 1$ and S is non-orientable.*
- stops at Γ_3 *in the other cases.*

Remark 4.4.54. In both cases, the Lie ring can be computed completely. Namely, $\mathcal{L}_1(\mathbf{B}_\lambda(S)) = \mathbf{B}_\lambda(S)^{\mathrm{ab}}$ has already been computed in Proposition 4.4.48. In the first case, no further computation is required. In the second case, $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is described completely in Proposition 4.4.56 and Proposition 4.4.57 for orientable surfaces, and in Corollary 4.4.60 and Corollary 4.4.62 for non-orientable ones. Moreover, one can easily describe the Lie bracket from the computations given there. Precisely, the only non-trivial brackets come from the computations depicted in Figure 4.3.

The proof of Theorem 4.4.53 begins with the following observation, which will be of essence in our study of the Lie ring of $\mathbf{B}_\lambda(S)$:

Fact 4.4.55. *Let $\lambda = (n_1, \ldots, n_l)$ be a partition of n with all blocks of size at least 3. Then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$, which is included in the subgroup $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$ of $\mathbf{B}_\lambda(S)$ from Theorem 4.4.51. Furthermore, if S has a handle or a crosscap, then $\mathcal{L}_2(\mathbf{B}_\lambda(S)) \cong \langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$.*

Proof. We know that $\Gamma_3 = \Gamma_\infty$ for $\mathbf{B}_\lambda(S)$ by Corollary 4.4.52. Then, since $\mathcal{L}_2 = \Gamma_2/\Gamma_3$, $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$. Moreover, in $\mathbf{B}_\lambda(S)/\Gamma_\infty$, the subgroup Γ_2 must be contained in the kernel $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$ of the central extension of Theorem 4.4.51. Now, if S has a handle or a crosscap, then the quotient $(\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2)^l$ of $\mathbf{B}_\lambda(S)/\Gamma_\infty$ identifies with its abelianisation (see Proposition 4.4.48). As a consequence, the subgroup $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$ is the whole of $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$. ∎

Non-planar orientable surfaces. Let us turn our attention to the case where the surface is not closed.

Proposition 4.4.56. *If S is a non-planar orientable surface which is not closed, then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is free abelian on the a_{ij} and the s_i^2.*

Proof. We already know that these elements generate $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ by Fact 4.4.55, so we need to show that they are linearly independent. The argument from the proof of Proposition 4.3.5 works here, using the maps from $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ to $\mathcal{L}_2(\mathbf{B}_{n_i}(S)) \cong \mathbb{Z}$ and $\mathcal{L}_2(\mathbf{B}_{n_i+n_j}(S)) \cong \mathbb{Z}$ instead of the maps from $\mathbf{B}_\lambda^{\mathrm{ab}}$ to $\mathbf{B}_{n_i}^{\mathrm{ab}}$ and $\mathbf{B}_{n_i+n_j}^{\mathrm{ab}}$. This uses the fact that $\mathcal{L}_2(\mathbf{B}_n(S)) \cong \mathbb{Z}$ if $n \geqslant 3$, from Theorem 4.4.43. □

The case of closed orientable surfaces is a bit trickier. In fact, we first need to generalise [BGG08, Th. 1]:

Proposition 4.4.57. *Let $g \geqslant 1$. Then $\mathcal{L}_2(\mathbf{B}_\lambda(\Sigma_g))$ is the quotient of the free abelian group on the a_{ij} and the s_i^2 by the relations:*

$$(n_i + g - 1)s_i^2 + \sum_{j \neq i} n_j a_{ij} = 0, \qquad \text{for all } 1 \leqslant i \leqslant n.$$

Proof. Let us consider the subsurface $\Sigma_{g,1}$ of Σ_g obtained by removing an open disc. The corresponding embedding of $\Sigma_{g,1}$ into Σ_g induces the quotient map described in Corollary 4.4.31. In particular, $\mathbf{B}_\lambda(\Sigma_g)/\Gamma_3$ is the quotient of $\mathbf{B}_\lambda(\Sigma_{g,1})/\Gamma_3$ by the classes of the relations from Corollary 4.4.31. Since the $A_{\alpha\beta}$ are in $\Gamma_2(\mathbf{B}_\lambda(\Sigma_{g,1}))$, these relations are between elements of Γ_2, that is, elements of $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$. In order to write them as relations between the s_i^2 and the a_{ij}, we need to recall that if α is in the i-th block of λ, for any $r \leqslant g$, we have, in $\mathcal{L}_2(\mathbf{B}_\lambda(S))$:

$$\overline{\left[a_r^{(\alpha)}, (b_r^{(\alpha)})^{-1}\right]} = -\left[\overline{a_r^{(\alpha)}}, \overline{b_r^{(\alpha)}}\right] = -\overline{\left[\sigma_\alpha b_r^{(\alpha)} \sigma_\alpha^{-1}, (a_r^{(\alpha)})^{-1}\right]} = -\overline{\sigma_\alpha^2} = -s_i^2,$$

where the third equality comes from Figure 4.3b. Moreover, the class of $A_{\alpha\beta}$ modulo $\Gamma_3(\mathbf{B}_\lambda(S))$ is either s_i^2 if α and β are both in the i-th block of λ, or a_{ij} if α is in the i-th block and β is in the j-th block for some $j \neq i$. Finally, using additive notations in the central subgroup $\langle s_i^2, a_{ij}\rangle$, the classes of the relations from Corollary 4.4.31 are:

$$(n_i - 1)s_i^2 + \sum_{j \neq i} n_j a_{ij} = -g s_i^2 \qquad \text{for all } 1 \leqslant i \leqslant n.$$

Moreover, these relators are central in $\mathbf{B}_\lambda(\Sigma_g)/\Gamma_\infty$, so their normal closure is only the subgroup they generate, which implies that $\Gamma_2(\mathbf{B}_\lambda(\Sigma_g)/\Gamma_3)$ is the quotient of $\Gamma_2(\mathbf{B}_\lambda(\Sigma_{g,1})/\Gamma_3)$ by these relations. By Proposition 4.4.56, the latter is free on the s_i^2 and the a_{ij}. Whence our claim. □

Remark 4.4.58. Let us point out that this computation of $\mathcal{L}_2(\mathbf{B}_\lambda(\Sigma_g))$ is very similar to the computation of $\mathcal{L}_1(\mathbf{B}_\lambda(\mathbb{S}^2))$ in the proof of Proposition 4.4.48 (note that s_i^2 was equal to $2s_i$ there, but here s_i and s_i^2 do not live in the same part of the Lie ring). The only difference lies in the degree of our relations with respect to the LCS: if the surface has a handle, then the pure braid generators $A_{\alpha\beta}$ belong to the derived subgroup (see Figure 4.3b).

Non-orientable surfaces. For non-orientable surfaces, we already know that the s_i^2 vanish in the quotient by Γ_∞, as in the proof of Theorem 4.4.43. As a consequence, $\Gamma_2\left(\mathbf{B}_n(S)/\Gamma_\infty\right)$ is generated by the a_{ij}. The following proposition is an analogue of [BGG17, Prop. 3.7] in the non-orientable case:

Proposition 4.4.59. *Let $g \geqslant 0$ and $m, n \geqslant 3$. We have:*

$$\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\Gamma_\infty \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g,$$

where the factors are respectively generated by s_1 and s_2, a_{12}, c'_r and c_r. The action is given by $c_r c'_r c_r^{-1} = c'_r a_{12}$ (for all $r \leqslant g$) and all the other pairs of generators commuting. In particular, this group is 2-nilpotent, and $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$ is infinite cyclic, generated by a_{12}.

Proof. Recall that we have a split extension (from Proposition 4.4.15):

$$\mathbf{B}_m(\mathcal{N}_{g,1+n}) \hookrightarrow \mathbf{B}_{m,n}(\mathcal{N}_{g,1}) \twoheadrightarrow \mathbf{B}_n(\mathcal{N}_{g,1}).$$

Thus, we can get a presentation of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$ from the presentation of the quotient described in Proposition 4.4.23 and the presentation of the kernel from Proposition 4.4.25, using the method of Appendix 4.6. The generators of this presentation are the σ_i for $i \neq m$, the $c_r := c_r^{(1)}$, the $c'_r := c_r^{(m+1)}$ and the $z_j = A_{1,m+j}$ (where our conventions are those from §4.4.3).

We use this to get a presentation of the quotient of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$ by the normal closure N of the $\sigma_i\sigma_{i+1}^{-1}$ for $i < m+n$ and $i \notin \{m-1, m\}$, together with the $z_j z_{j+1}^{-1}$ for $j < n$. This quotient is generated by the common class s_1 (resp. s_2, resp. a_{12}) of $\sigma_1, \ldots, \sigma_{m-1}$, the common class s_2 of $\sigma_{m+1}, \ldots, \sigma_{m-1}$ and the common class a_{12} of $z_1, \ldots, z_n$, together with the c_r and the c'_r for $r \leqslant g$. They are subject to the following relations:

- Relations coming from those of $\mathbf{B}_n(\mathcal{N}_{g,1})$: s_2 commutes with the c'_r $(BN1)$, the c'_r commute with one another $(BN2)$ and $s_2^2 = 1$ $(BN3)$.
- Relations coming from those of $\mathbf{B}_m(\mathcal{N}_{g,1+n})$: s_1 commutes with the c_r $(BN1)$, the c_r commute with one another $(BN2)$, $s_1^2 = 1$ $(BN3)$, a_{12} commutes with s_1 $(BN4)$, and with the c_r $(BN5)$. $(BN6)$ and $(BN7)$ become trivial.
- Relations describing the action by conjugation of s_2 and the c'_r on s_1, a_{12} and the c_r. This action is easily seen to be trivial in most cases, since most of the pairs of elements involved come from elements having disjoint support in $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$, hence they commute. Namely, this holds for c_s and c'_r when $r < s$: using the fact that a_{12} commutes with c_s, we see that c_s is the class of $z_1 c_r^{(1)} z_1^{-1}$, whose support is disjoint from the support of $c_r^{(m+1)}$ (up to isotopy). This is again true for a_{12} and the c'_r, since s_2 commutes with the c'_r: c'_r is the class of $c_r^{(m+2)} = \sigma_{m+1} c_r^{(m+1)} \sigma_{m+1}^{-1}$, whose support is disjoint from that of z_1. Finally, the only pair of generators under scrutiny for which this does not hold are c_r and c'_r (for $r \leqslant g$). But for them, the situation is the one from Figure 4.3a: $[(c'_r)^{-1}, c_r] = \overline{A}_{1,m+1} = a_{12}$.

These relations can be summed up as:

- All the generators commute pairwise, except c_r and c'_r for $r \leqslant g$,
- $s_1^2 = s_2^2 = 1$,
- $c_r c'_r c_r^{-1} = c'_r a_{12}$ (for $r = 1, \ldots, g$).

This is a presentation of the group described in the statement. Its commutator subgroup is the factor $\mathbb{Z}$ of the decomposition, which is infinite cyclic, generated by a_{12}. It is also central, hence the group is 2-nilpotent. Since this group is $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/N$, N contains $\Gamma_3(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$. But the elements normally generating N are in $\Gamma_\infty(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$; see Lemma 4.4.50 and its proof. Thus $N = \Gamma_\infty(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$, and the Lie ring of the quotient $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/N$ identifies with the Lie ring of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$. In particular, since its Γ_3 is trivial, its $\mathcal{L}_2$ coincides with its Γ_2, which is infinite cyclic, generated by a_{12}. ◻

Corollary 4.4.60. *Let S be a non-orientable surface that is not closed. Let $\lambda = (n_1, \ldots, n_l)$ be a partition whose blocks are of size at least 3. Then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is free abelian on the a_{ij} for $1 \leqslant i < j \leqslant l$.*

Proof. We first show that if $m, n \geqslant 3$, $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ is infinite cyclic, generated by a_{12}. This is true for $S = \mathcal{N}_{g,1}$ by Proposition 4.4.59. Then we can follow the same method as in the proof of Theorem 4.4.43, to which the reader is referred for more details. Namely, we already know that $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ is generated by a_{12}, and we need to show that it has infinite order. If S is compact, we can embed S into some $\mathcal{N}_{g,1}$, and the image of a_{12} by the corresponding morphism from $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ to $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$ has infinite order, whence our claim in this case. Then, if S is not compact, a relation saying that a_{12} has finite order in $\mathbf{B}_{m,n}(S)/\Gamma_3$ would hold in a compact subsurface, which is impossible by the previous case.

From this, we can deduce the result for λ having more than two blocks, reasoning as in the proofs of Propositions 4.3.5 and 4.4.56. Indeed, each canonical map $\mathcal{L}_2(\mathbf{B}_\lambda(S)) \to \mathcal{L}_2(\mathbf{B}_{n_i,n_j}(S)) \cong \mathbb{Z}$ kills all the a_{kl}, except a_{ij}, which is sent to a generator of the target. Thus the a_{ij} must be linearly independent. ◻

Proposition 4.4.61. *Let $g \geqslant 0$ and $m, n \geqslant 3$. Then $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_g))$ is cyclic of order 2, generated by a_{12}.*

Proof. Let us consider $G := \mathbf{B}_{m,n}(\mathcal{N}_g)/\varGamma_3$. This group is the quotient of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\varGamma_3 = \mathbf{B}_{m,n}(\mathcal{N}_g)/\varGamma_\infty \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$ described in Proposition 4.4.59 by the images of the boundary relations from Corollary 4.4.31. Before considering these relations, we can already remark that the commutator subgroup of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\varGamma_3$ is cyclic generated by a_{12}, so the same holds for G. Since $\varGamma_2(G)$ identifies with $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_g))$, we only need to show that a_{12} has order 2 in G to prove our statement.

Recall that $A_{\alpha\beta}$ is sent to $s_1^2 = s_2^2 = 1$ if α and β are in the same block of the partition (m, n), and to a_{12} if they are not. As a consequence, these relations are:

$$c_1^2 \cdots c_g^2 = a_{12}^m \quad \text{and} \quad (c_1')^2 \cdots (c_g')^2 = a_{12}^n.$$

Since c_1 commutes with all the other generators except c_1', and $[c_1, c_1'] = a_{12}$, by applying the commutator with c_1 to the second relation, we get $a_{12}^2 = 1$. We can thus consider the intermediate quotient by the central element a_{12}^2, and see G as a quotient of $(\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$. Modulo a_{12}^2, the above relations (hence the structure of $G = \mathbf{B}_{m,n}(\mathcal{N}_g)/\varGamma_3$) depend only on the parity of m and n. Moreover, we now know that $a_{12}^2 = 1$ in G, so we are left with showing that a_{12} is not trivial in G.

Suppose first that m and n are both even. Then the relations become $c_1^2 \cdots c_g^2 = (c_1')^2 \cdots (c_g')^2 = 1$. Notice that after the quotient by a_{12}^2, all the c_r^2 and the $(c_r')^2$ have become central, so both $c_1^2 \cdots c_g^2$ and $(c_1')^2 \cdots (c_g')^2$ are central in $(\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$. Thus, if A denotes the abelian group $\mathbb{Z}^g/(2, 2, \ldots, 2)$ (which is also $\pi_1(\mathcal{N}_g)^{\mathrm{ab}}$), we see that $G \cong (\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times A) \rtimes A$, whose commutator subgroup is cyclic of order 2, generated by a_{12}.

If m and n have different parities, we can assume (by symmetry) that m is even and n is odd. Then the relations become $c_1^2 \cdots c_g^2 = 1$ and $(c_1')^2 \cdots (c_g')^2 = a_{12}$. In $(\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, both $c_1^2 \cdots c_g^2$ and $(c_1')^2 \cdots (c_g')^2 a_{12}^{-1}$ are central, so they general cyclic normal subgroups. Thus, if we denote by $\tilde{A}$ the quotient $\mathbb{Z}^g/(4, 4, \ldots, 4)$ of $(\mathbb{Z}/2) \times \mathbb{Z}^g$ by $(1, 2, \ldots, 2)$, we have $G \cong (\mathbb{Z}/2)^2 \times \tilde{A} \rtimes A$, whose commutator subgroup is cyclic of order 2, generated by a_{12} (the latter identifies with the class of $(2, 2, \ldots, 2)$ in $\tilde{A}$).

If m and n are both odd, then the relations become $c_1^2 \cdots c_g^2 = (c_1')^2 \cdots (c_g')^2 = a_{12}$. In this case, there is no obvious semi-direct product decomposition of G where a_{12} is clearly non-trivial, so we need another argument to show that $a_{12} \neq 1$. If $g = 1$, one can see that $G \cong (\mathbb{Z}/2)^2 \times Q_8$, where c_1 is sent to i, c_1' is sent to j, and a_{12} identifies with the central element -1 of the quaternion group Q_8. Indeed, one can easily check that the above correspondence defines a morphism from G to $(\mathbb{Z}/2)^2 \times Q_8$, and use the presentation $Q_8 = \langle i, j \mid i^2 = i^2 = (ij)^2 \rangle$ to construct its inverse. For $g \geqslant 1$, we can find a similar quotient of G, by considering the quotient $(Q_8)^g/H$, where H is the hyperplane of $\mathcal{Z}((Q_8)^g) \cong (\mathbb{Z}/2)^g$ defined by the vanishing of the sum of the coordinates. Since H is central in $(Q_8)^g$, it is normal. Moreover, $(Q_8)^g$ decomposes as a central extension of $(\mathbb{Z}/2)^{2g}$ by $(\mathbb{Z}/2)^g$, which induces a central extension:

$$(\mathbb{Z}/2)^g/H \cong \mathbb{Z}/2 \hookrightarrow (Q_8)^g/H \twoheadrightarrow (\mathbb{Z}/2)^{2g}.$$

Using the presentation of G, we can see that there is a well-defined projection from G onto $(Q_8)^g/H$ sending c_r to $(1, \ldots, 1, i, 1, \ldots, 1)$, c_r' to $(1, \ldots, 1, j, 1, \ldots, 1)$ (where the non-trivial coordinate is the r-th one in both cases), and s_1 and s_2 to 1. This projection sends $a_{12} = [c_1, c_1']$ to the generator of the centre $(\mathbb{Z}/2)^g/H \cong \mathbb{Z}/2$. Thus a_{12} is again not trivial in G, whence our claim. ∎

Corollary 4.4.62. *Let $g \geqslant 0$ and let $\lambda = (n_1, \ldots, n_l)$ be a partition whose blocks are of size at least 3. Then $\mathcal{L}_2(\mathbf{B}_\lambda(\mathcal{N}_g)) \cong (\mathbb{Z}/2)^{l(l-1)/2}$ is the free elementary abelian 2-group on the a_{ij} for $1 \leqslant i < j \leqslant l$.*

Proof. The proof is again the same as the proofs of Propositions 4.3.5 and 4.4.56, using the canonical maps $\mathcal{L}_2(\mathbf{B}_\lambda(\mathcal{N}_g)) \to \mathcal{L}_2(\mathbf{B}_{n_i,n_j}(\mathcal{N}_g)) \cong \mathbb{Z}/2$, where the latter is generated by the image of a_{ij}, by Proposition 4.4.61. ∎

We can now finish the proof of the main result of this section.

Proof of Theorem 4.4.53. Under our hypotheses, $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$, which is a subgroup of $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l}$ (see Fact 4.4.55).

Planar surfaces: if S is planar, then the canonical projection from $\mathbf{B}_\lambda(S)/\Gamma_\infty$ to $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ sends the s_i and the a_{ij} to their counterparts in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$. The latter is a linearly independent family by Proposition 4.4.48, so the restriction $\langle s_i^2, a_{ij}\rangle_{i,j\leqslant l} \to \mathbf{B}_\lambda(S)^{\mathrm{ab}}$ must be injective. But its kernel is $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty) = \mathcal{L}_2(\mathbf{B}_\lambda(S))$, which must then be trivial.

The sphere: if $S = \mathbb{S}^2$, then $\mathbf{B}_\lambda(\mathbb{S}^2)$ is a quotient of $\mathbf{B}_\lambda$ (see Corollary 4.4.31), which implies that its LCS also stops at Γ_2.

Non-planar orientable surfaces: Proposition 4.4.56 deals with most of them; the remaining ones are closed orientable surfaces, for which Proposition 4.4.57 is the relevant statement.

Non-orientable surfaces: these are dealt with in Corollary 4.4.60, except for the closed ones, whose Lie ring is studied separately, in Corollary 4.4.62. □

4.4.6 Partitions with small blocks

Now that we have a complete description of the LCS of $\mathbf{B}_\lambda(S)$ in the stable case, that is, when the blocks of the partition λ have size at least 3 (§4.4.5), we turn our attention to the cases where λ does have blocks of size 1 or 2. Then we ask ourselves: under this assumption, when does the LCS of $\mathbf{B}_\lambda(S)$ stop? For most surfaces, it does not; see Proposition 4.4.63. In fact, there are only six surfaces to which this result does not apply. One of them is the disc, for which an answer has already been given in Chapter 4.3. Another one is the cylinder, whose case can easily be deduced from the case of the disc. Four surfaces remain: the torus $\mathbb{T}^2$ (§4.4.6), the Möbius strip $\mathbb{M}^2$ (§4.4.6), the sphere $\mathbb{S}^2$ (§4.4.6) and the projective plane $\mathbb{P}^2$ (§4.4.6).

The generic cases

As a direct corollary of Propositions 4.4.38 and 4.4.40, we get the following result:

Proposition 4.4.63. *Let $\lambda = (n_1, \ldots, n_l)$ be a partition of an integer $n \geqslant 1$.*

- *If λ has at least one block of size 1 and $\pi_1(S)$ is not abelian (that is, if we suppose that $S \notin \{\mathbb{D} - pt, \mathbb{D}, \mathbb{S}^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{M}^2\}$ up to isotopy equivalence), then the LCS of $\mathbf{B}_\lambda(S)$ does not stop.*
- *If λ has at least one block of size 2 and $\pi_1(S)^{\mathrm{ab}}$ is not finite (that is, if we suppose that $S \notin \{\mathbb{D}, \mathbb{S}^2, \mathbb{P}^2\}$ up to isotopy equivalence), then the LCS of $\mathbf{B}_\lambda(S)$ does not stop.*

Proof. In the first case, there is a surjection $\mathbf{B}_\lambda(S) \twoheadrightarrow \mathbf{B}_1(S) \cong \pi_1(S)$, and in the second one, a surjection $\mathbf{B}_\lambda(S) \twoheadrightarrow \mathbf{B}_2(S)$. Propositions 4.4.38 and 4.4.40 say that, under our hypotheses, the LCS of the quotient does not stop in either case. The result then follows from Lemma 4.1.1. □

Thus the question of whether the LCS of $\mathbf{B}_\lambda(S)$ stops has been answered for every partition, except for the six surfaces $\mathbb{D} - pt$, $\mathbb{D}$, $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$ and $\mathbb{M}^2$. In fact, $\mathbf{B}_\lambda(\mathbb{D}) = \mathbf{B}_\lambda$ has already been considered in Chapter 4.3; see Theorem 4.3.6. Also, since $\mathbf{B}_1(\mathbb{D}) = \{1\}$, we have an isomorphism $\mathbf{B}_\lambda(\mathbb{D} - pt) \cong \mathbf{B}_{1,\lambda}(\mathbb{D})$ for every partition λ by Proposition 4.4.15, so we can deduce the remaining answer for $\mathbb{D} - pt$ from the answer for the disc. Namely, Lemma 4.3.8 and Proposition 4.3.11 imply:

Lemma 4.4.64. *If λ has at least two blocks of size 1, then the LCS of $\mathbf{B}_\lambda(\mathbb{D} - pt)$ does not stop. If $\lambda = (1, n_2, \ldots, n_l)$ where every $n_i \geqslant 3$, then its LCS stops at Γ_2.*

Therefore, we are left with four remaining cases: the torus, the Möbius strip, the sphere and the projective plane.

Partitioned braids on the torus

We know that the LCS of $\mathbf{B}_\lambda(\mathbb{T}^2)$ stops if λ has only blocks of size at least 3, and that it does not if there is at least one block of size 2. The remaining cases are dealt with using the following generalisation of [BGG08, Lem. 17]:

Proposition 4.4.65. *There is an isomorphism $\mathbf{B}_{1,\mu}(\mathbb{T}^2) \cong \mathbf{B}_\mu(\mathbb{T}^2 - pt) \times \mathbb{Z}^2$, for any partition μ.*

Proof. Proposition 4.4.15 gives a short exact sequence:

$$\mathbf{B}_\mu(\mathbb{T}^2 - pt) \hookrightarrow \mathbf{B}_{1,\mu}(\mathbb{T}^2) \twoheadrightarrow \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2.$$

Let $n \geqslant 2$ such that μ is a partition of $n-1$. The centre of $\mathbf{B}_n(\mathbb{T}^2)$ is generated by two braids α and β corresponding to rotating all the punctures along each factor of $\mathbb{S}^1 \times \mathbb{S}^1$ [PR00, Prop. 4.2]. These are pure braids, so they are in the subgroup $\mathbf{B}_{1,\mu}(\mathbb{T}^2)$ of $\mathbf{B}_n(S)$. The above projection (forgetting all strands but one) maps these two elements to a basis of $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$. As a consequence, it restricts to an isomorphism $\langle \alpha, \beta \rangle \cong \mathbb{Z}^2$. Thus the above short exact sequence splits, and the corresponding action of $\mathbb{Z}^2$ is trivial (α and β being central), hence it is in fact a direct product. □

It is then an easy task to finish proving the following summary of our results for partitioned torus braids:

Theorem 4.4.66. *Let λ be a partition of an integer $n \geqslant 1$. The LCS of $\mathbf{B}_\lambda(\mathbb{T}^2)$:*

- does not stop *if λ has at least two blocks of size* 1 *or one block of size* 2*.*
- stops *at Γ_3 in all the other cases, except for $\mathbf{B}_1(\mathbb{T}^2) \cong \mathbb{Z}^2$.*

Proof. If λ has at least two blocks of size 1, then $\mathbf{B}_\lambda(\mathbb{T}^2)$ surjects onto $\mathbf{B}_{1,1}(\mathbb{T}^2)$, which is isomorphic to $\mathbf{B}_1(\mathbb{T}^2 - pt) \times \mathbb{Z}^2 \cong \mathbf{F}_2 \times \mathbb{Z}^2$ by Proposition 4.4.65. The LCS of $\mathbf{F}_2$ does not stop, whence the result in this case, by Lemma 4.1.1.

If λ has exactly one block of size 1 and no block of size 2, then Proposition 4.4.65 gives an isomorphism $\mathbf{B}_\lambda(\mathbb{T}^2) \cong \mathbf{B}_\mu(\mathbb{T}^2 - pt) \times \mathbb{Z}^2$ where μ has only blocks of size at least 3. Then Corollary 4.4.52 implies that the LCS stops at most at Γ_3 in this case. In fact, if μ is non-trivial, then the LCS of $\mathbf{B}_\mu(\mathbb{T}^2 - pt)$ stops exactly at Γ_3; see Theorem 4.4.53.

The other cases have already been treated as part of Corollary 4.4.52, Theorem 4.4.53 and Proposition 4.4.63. □

Partitioned braids on the Möbius strip

As in the case of the torus, we know that the LCS of $\mathbf{B}_\lambda(\mathbb{M}^2)$ stops if λ has only blocks of size at least 3, and that it does not if there is at least one block of size 2. The only remaining cases are the ones when λ has some blocks of size 1, the other ones being of size at least 3. We begin by showing that the LCS does not stop when there are at least two blocks of size 1. In this case, $\mathbf{B}_\lambda(\mathbb{M}^2)$ surjects onto $\mathbf{B}_{1,1}(\mathbb{M}^2) = \mathbf{P}_2(\mathbb{M}^2)$ so, by Lemma 4.1.1, this case follows from the following study of $\mathbf{P}_2(\mathbb{M}^2)$:

Lemma 4.4.67. *A presentation of the pure braid group $\mathbf{P}_2(\mathbb{M}^2)$ is given by generators γ_1, γ_2 and A subject to the relations*

$$\begin{cases} \gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2 \\ \gamma_1 \gamma_2 \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2^2. \end{cases}$$

Recall that $\mathbb{M}^2$ is the surface $\mathcal{N}_{1,1}$ from §4.4.3. Here, we have changed the notations from §4.4.3 to lighter ones, more suited to the case with only one crosscap ($\gamma_i := c_1^{(i)}$) and only two strands ($A := A_{12}$).

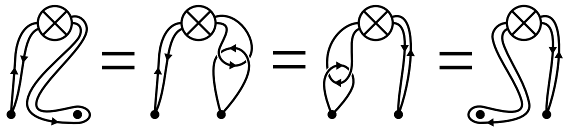


Figure 4.8 Relation in $\mathbf{P}_2(\mathbb{M}^2)$: $\gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2$.

Proof. Proposition 4.4.15 gives a decomposition $\mathbf{P}_2(\mathbb{M}^2) \cong \pi_1(\mathbb{M}^2 - pt) \rtimes \pi_1(\mathbb{M}^2) \cong \mathbf{F}_2 \rtimes \mathbb{Z}$, where the projection onto $\pi_1(\mathbb{M}^2) \cong \mathbb{Z}$ is given by forgetting one strand (say, the first one). Then the factor $\mathbb{Z}$ is generated by γ_1, and a free basis of the factor $\mathbf{F}_2$ is given by γ_2 and A (with the notations introduced just before the proof). Moreover, the action of γ_1 by conjugation on $\langle A, \gamma_2 \rangle$ is not difficult to compute. Namely, we have $\gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2$; see Figure 4.8. Then, from Figure 4.3a, we get that $[\gamma_2, \gamma_1^{-1}] = A$, which is equivalent to $\gamma_1 \gamma_2 \gamma_1^{-1} = \gamma_1 A \gamma_1^{-1} \gamma_2$. Using the previous relation, the latter equals $\gamma_2^{-1} A^{-1} \gamma_2^2$. These relations determine $\mathbf{P}_2(\mathbb{M}^2)$, since the group G that they define decomposes as $\langle A, \gamma_2 \rangle \rtimes \langle \gamma_1 \rangle$, and the obvious projection of G onto $\mathbf{P}_2(\mathbb{M}^2)$ must be an isomorphism on both the kernel and the quotient, hence it must be an isomorphism. ◻

Corollary 4.4.68. *The LCS of $\mathbf{P}_2(\mathbb{M}^2)$ does not stop.*

Proof. Let us consider the quotient of $\mathbf{P}_2(\mathbb{M}^2) = \mathbf{F}_2 \rtimes \mathbb{Z}$ by $\Gamma_2(\mathbf{F}_2)$. It is $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the generator $\overline{\gamma}_1$ of $\mathbb{Z}$ acts via the involution τ sending $\overline{A}$ to $-\overline{A}$ and $\overline{\gamma}_2$ to $\overline{\gamma}_2 - \overline{A}$. Then $V := \mathrm{Im}(\tau - 1)$ is $\mathbb{Z} \cdot \overline{A}$. By Proposition 4.5.10, for $i \geqslant 2$, we have $\Gamma_i(\mathbb{Z}^2 \rtimes \mathbb{Z}) = 2^{i-2} V$, so the LCS of $\mathbb{Z}^2 \rtimes \mathbb{Z}$ does not stop. Thus the LCS of $\mathbf{P}_2(\mathbb{M}^2)$ does not either. ◻

The answer for the remaining cases are consequences of the following result:

Proposition 4.4.69. *For every $m \geqslant 3$, the LCS of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ does not stop.*

Proof. We can use Proposition 4.4.15 to get a decomposition:

$$\mathbf{B}_{1,m}(\mathbb{M}^2) \cong \mathbf{B}_m(\mathbb{M}^2 - pt) \rtimes \pi_1(\mathbb{M}^2),$$

where $\pi_1(\mathbb{M}^2) \cong \mathbb{Z}$ is identified with the subgroup of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ generated by γ_1 (as above, we denote $c_1^{(i)}$ by γ_i). We know that $\Gamma_\infty(\mathbf{B}_{1,m}(\mathbb{M}^2))$ contains $\Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - pt))$. The latter is fully invariant in $\mathbf{B}_m(\mathbb{M}^2 - pt)$, hence normal in $\mathbf{B}_{1,m}(\mathbb{M}^2)$, and we can consider the quotient

$$G := \mathbf{B}_{1,m}(\mathbb{M}^2)/\Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - pt)) \cong (\mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty) \rtimes \mathbb{Z}.$$

Moreover, Lemma 4.4.34 and Remark 4.4.36 give a central extension:

$$\langle \sigma \rangle \hookrightarrow \mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty \twoheadrightarrow \pi_1(\mathbb{M}^2 - pt)^{\mathrm{ab}}.$$

The element σ is the common class of the usual generators σ_α of $\mathbf{B}_m$. Since these commute with γ_1 in $\mathbf{B}_{1,m}(\mathbb{M}^2)$, σ is central not only in $\mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty$, but in fact in G. In particular, $\langle \sigma \rangle$ is normal in G, and the quotient decomposes as:

$$G/\sigma \cong \left[(\mathbf{B}_\mu(\mathbb{M}^2 - pt)/\Gamma_\infty)/\sigma\right] \rtimes \pi_1(\mathbb{M}^2) \cong \pi_1(\mathbb{M}^2 - pt)^{\mathrm{ab}} \rtimes \pi_1(\mathbb{M}^2) \cong \mathbb{Z}^2 \rtimes \mathbb{Z}.$$

A basis of the $\mathbb{Z}^2$ factor is given by $\overline{\gamma_2}$ and $A = \overline{A_{12}}$. Consider the morphism $\mathbf{P}_2(\mathbb{M}^2) \to \mathbf{B}_{1,m}(\mathbb{M}^2)$ corresponding to adding $m-1$ trivial strands to the second block (constructed as in the proof of Proposition 4.4.15). Composing with the projection, we get a morphism from $\mathbf{P}_2(\mathbb{M}^2) = \mathbf{F}_2 \rtimes \mathbb{Z}$ to $G/\sigma = \mathbb{Z}^2 \rtimes \mathbb{Z}$. From the explicit description of both of these groups, one easily sees that it has to induce an isomorphism $\mathbf{P}_2(\mathbb{M}^2)/\Gamma_2(\mathbf{F}_2) \cong G/\sigma$. But the LCS of $\mathbf{P}_2(\mathbb{M}^2)/\Gamma_2(\mathbf{F}_2)$ does not stop (see the proof of Corollary 4.4.68), so we have found a quotient of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ whose LCS does not stop, whence the result. ◻

Let us sum up our results about the LCS of $\mathbf{B}_\lambda(\mathbb{M}^2)$:

Theorem 4.4.70. *Let $\lambda = (n_1, \ldots, n_l)$ be a partition of an integer $n \geqslant 1$. The LCS of $\mathbf{B}_\lambda(\mathbb{M}^2)$:*

- does not stop *if λ has at least one block of size 1 or 2, with the exception of $\mathbf{B}_1(\mathbb{M}^2) \cong \mathbb{Z}$.*
- stops *in all the other cases, at Γ_3 if $l \geqslant 2$ and at Γ_2 if $l = 1$.*

Proof. The first statement follows from Proposition 4.4.63 (for blocks of size 2) and from Corollary 4.4.68 and Proposition 4.4.69 (for blocks of size 1). The second one is part of the general results of Corollary 4.4.52 and Theorem 4.4.53. □

Partitioned braids on the sphere

For a partition $\lambda = (n_1, \ldots, n_l)$, the inclusion of the disc into the sphere induces a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_\lambda(\mathbb{S}^2)$ by Corollary 4.4.31. We can thus apply Lemma 4.1.1 to deduce that the LCS of $\mathbf{B}_\lambda(\mathbb{S}^2)$ stops whenever the one of $\mathbf{B}_\lambda$ does. Namely, by Theorem 4.3.6, it stops at Γ_2 if $n_i \geqslant 3$ for all i, save at most two indices for which $n_i = 1$. Since the abelianisation of $\mathbf{B}_\lambda(\mathbb{S}^2)$ has already been computed in Proposition 4.4.48, we have a complete description of the LCS in these cases.

The above argument can be extended somewhat if we remark that, because of Proposition 4.4.15, $\mathbf{B}_1(\mathbb{S}^2) = \pi_1(\mathbb{S}^2) = \{1\}$ is the cokernel of the canonical morphism $\mathbf{B}_\mu(\mathbb{D}) \to \mathbf{B}_{1,\mu}(\mathbb{S}^2)$ (for any partition μ), which is thus surjective. As a consequence, the LCS of $\mathbf{B}_\lambda(\mathbb{S}^2)$ also stops at Γ_2 when λ has three blocks of size 1, the other ones being of size at least 3. This proves the first half of the following theorem. The second half is proven in the remainder of this subsection in several different cases, which are synthesised into a proof of the theorem at the end of the subsection.

Theorem 4.4.71. *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, \ldots, n_l)$ be a partition of n. The LCS of $\mathbf{B}_\lambda(\mathbb{S}^2)$:*

- stops at Γ_2 *if $n_i \geqslant 3$ for all i, save at most three indices for which $n_i = 1$.*
- does not stop *in all the other cases, except for $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$, $\mathbf{B}_{2,1}(\mathbb{S}^2) \cong \mathbb{Z}/4$ and the $\mathbf{B}_{2,m}(\mathbb{S}^2)$ with $m \geqslant 3$, whose LCS stop either at $\Gamma_{v_2(m)+1}$ or at $\Gamma_{v_2(m)+2}$ when m is even, where v_2 is the 2-adic valuation, and either at Γ_2 or at Γ_3 when m is odd.*

In the case of $\mathbf{B}_{2,m}(\mathbb{S}^2)$ with $m \geqslant 3$, where the answer given above is ambiguous between two consecutive possibilities, we conjecture that its LCS stops at $\Gamma_{v_2(m)+2}$ for all $m \geqslant 3$; see Remark 4.4.75.

Blocks of size 1. We need to consider the case of the pure braid group on four strands:

Lemma 4.4.72. *The LCS of $\mathbf{P}_4(\mathbb{S}^2)$ does not stop.*

Proof. We sketch a proof of the decomposition of $\mathbf{P}_4(\mathbb{S}^2)$ from [GG04c, Th. 4]. Recall that we have a short exact sequence by Proposition 4.4.15:

$$\mathbf{P}_1(\mathbb{S}^2 - \{3 \text{ pts}\}) \hookrightarrow \mathbf{P}_4(\mathbb{S}^2) \twoheadrightarrow \mathbf{P}_3(\mathbb{S}^2).$$

It is known that $\mathbf{P}_3(\mathbb{S}^2) \cong \mathbb{Z}/2$. In fact, an isomorphism between $\pi_1(SO_3(\mathbb{R}))$ and $\mathbf{P}_3(\mathbb{S}^2)$ is induced by $\varphi \mapsto (\varphi(e_1), \varphi(e_2), \varphi(e_3))$ from $SO_3(\mathbb{R})$ to $F_3(\mathbb{S}^2)$. Moreover, a splitting of the above short exact sequence is given by sending the generator of $\mathbf{P}_3(\mathbb{S}^2)$ to the full twist. Since the latter is central, $\mathbf{P}_4(\mathbb{S}^2)$ is the direct product of $\mathbb{Z}/2$ with $\pi_1(\mathbb{S}^2 - \{3 \text{ pts}\}) \cong \mathbf{F}_2$. Thus $\mathbf{P}_4(\mathbb{S}^2)$ is residually nilpotent (but not nilpotent), whence the result. □

Blocks of size 2. Let us begin with the case where the partition has at least three blocks:

Proposition 4.4.73. *Let λ be a partition of n with at least three blocks, and at least one block of size 2. Then the LCS of $\mathbf{B}_\lambda(\mathbb{S}^2)$ does not stop.*

Proof. Let μ be any partition. Let us consider the quotient of $\mathbf{B}_{2,\mu}(\mathbb{S}^2)$ by $\Gamma_2(\mathbf{B}_{1,1,\mu}(\mathbb{S}^2))$, which is (clearly) an extension:

$$\mathbf{B}_{1,1,\mu}(\mathbb{S}^2)^{\mathrm{ab}} \hookrightarrow G = \mathbf{B}_{2,\mu}(\mathbb{S}^2)/\Gamma_2(\mathbf{B}_{1,1,\mu}(\mathbb{S}^2)) \twoheadrightarrow \mathfrak{S}_2. \tag{4.7}$$

Let us fix $(n_1, \ldots, n_l) := (1, 1, \mu)$. From a presentation of the kernel (from Proposition 4.4.33) and a presentation of the quotient, using the method from Appendix 4.6, we can write down a presentation of G. Precisely, G admits the presentation with generators s, s_i (for $1 \leqslant i \leqslant l$), and a_{ij} (for $1 \leqslant i < j \leqslant l$), subject to the following relations:

$$\begin{cases} (1) & s^2 = a_{12}, & \\ (2) & s_i = 1 & \text{if } n_i = 1, \\ (3) & [s_i, s_j] = [s_i, a_{pq}] = [a_{pq}, a_{uv}] = 1 & \forall i, j, p, q, u, v, \\ (4) & [s, s_i] = 1 & \forall i \geqslant 1, \\ (5) & [s, a_{ij}] = 1 & \forall j > i \geqslant 3, \\ (6) & s a_{1j} s^{-1} = a_{2j} \text{ and } s a_{2j} s^{-1} = a_{1j} & \forall j \geqslant 3, \\ (7) & 2(n_i - 1)s_i + \sum_{j \neq i} n_j a_{ij} = 0 & \forall i \geqslant 1, \end{cases}$$

where the last relation uses additive notation in the abelian subgroup generated by the s_i and the a_{ij}.

Consider the subgroup H of G generated by s^2 together with the s_i and the a_{ij} for $i, j \geqslant 3$. One can easily see that it is central in G. Moreover, we can deduce from the presentation of G a presentation of G/H, which turns out to be very simple. Indeed, it is generated by s, the a_{1i} and the a_{2i} for $i \geqslant 3$. Moreover, since $n_1 = n_2 = 1$, for $i \geqslant 3$ the last relation becomes $a_{1i} + a_{2i} = 0$, which means that a_{1i} and a_{2i} are inverse to each other. For $i \in \{1, 2\}$, we find the relations $\sum n_j a_{1j} = 0$ and $\sum n_j a_{2j} = 0$, which are equivalent to each other modulo the previous relations. Thus, if we denote by a_i the element $a_{1i} = a_{2i}^{-1}$ of G/H, we get that G/H is generated by $a_3, a_4, \ldots, a_l$ and s, subject to the relations:

$$\begin{cases} (1) & s^2 = 1 & \\ (3) & [a_i, a_j] = 1 & \forall i, j, \\ (6) & s a_j s^{-1} = a_j^{-1} & \forall j \geqslant 3, \\ (7) & \sum n_j a_j = 0. & \end{cases}$$

Finally, $G/H \cong A \rtimes (\mathbb{Z}/2)$, where A is the quotient of the free abelian group on $a_3, a_4, \ldots, a_l$ by the single relation $\sum n_j a_j = 0$, and $\mathbb{Z}/2$ acts on A via $-id$. We can use Corollary 4.5.8 to compute the LCS of G/H. Namely, since $A \cong \mathbb{Z}^{l-3} \times \mathbb{Z}/\gcd(n_i)$, it does not stop whenever $l > 3$. Thus, we have found a quotient of $\mathbf{B}_{2,\mu}(\mathbb{S}^2)$ having a non-stopping LCS whenever μ has at least two blocks. ◻

If λ has only two blocks, we first assume that the other block is large enough:

Proposition 4.4.74. *Let $m \geqslant 3$ be an integer. The LCS of $\mathbf{B}_{2,m}(\mathbb{S}^2)$ stops at $\Gamma_{v_2(m)+1}$ or at $\Gamma_{v_2(m)+2}$ when m is even, where v_2 is the 2-adic valuation, and at Γ_2 or at Γ_3 when m is odd.*

Proof. We have seen above that the LCS of $\mathbf{B}_{1,1,m}(\mathbb{S}^2)$ (which is a quotient of $\mathbf{B}_{1,1,m}$) stops at Γ_2 if $m \geqslant 3$. Then, as in the proof of Proposition 4.3.12, we have that $\Gamma_\infty(\mathbf{B}_{2,m}(\mathbb{S}^2))$ contains $\Gamma_\infty(\mathbf{B}_{1,1,m}(\mathbb{S}^2)) = \Gamma_2(\mathbf{B}_{1,1,m}(\mathbb{S}^2))$. Thus it is enough to understand when the LCS of $G := \mathbf{B}_{2,m}(\mathbb{S}^2)/\Gamma_2(\mathbf{B}_{1,1,m}(\mathbb{S}^2))$ stops. In order to do this, we use the calculations already done in the course of the proof of Proposition 4.4.73: the group G is a central extension of $(\mathbb{Z}/m) \rtimes (\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts via $-id$. Thanks to Corollary 4.5.8, we know that the LCS of the latter is given by $\Gamma_i = 2^{i-1}\mathbb{Z}/m$ for $i \geqslant 2$. Hence it stops at Γ_2 if m is odd and at $\Gamma_{v_2(m)+1}$ if m is even. Thus, the result will follow from Corollary 4.1.6, if we show that the kernel H of the central extension $H \hookrightarrow G \twoheadrightarrow (\mathbb{Z}/m) \rtimes (\mathbb{Z}/2)$ injects into G^{ab}. We can compute G^{ab} using the presentation of G from the proof of Proposition 4.4.73, or using Proposition 4.4.48 and the fact that $G^{\mathrm{ab}} \cong \mathbf{B}_{2,m}(\mathbb{S}^2)^{\mathrm{ab}}$ (by definition of G). Either way, we find that it is generated by three elements s $(= s_1)$, s_3 and a (the latter being the common image of a_{13} and $a_{23} = sa_{13}s^{-1}$ in G^{ab}), subject to the relations:

$$\begin{cases} 2s + ma = 0; \\ 2(m-1)s_3 + 2a = 0. \end{cases}$$

In other words, $G^{\mathrm{ab}} = \mathbb{Z}^3/R$ where, if (s, s_3, a) denotes a basis of $\mathbb{Z}^3$, R is generated by $2s+ma$ and $2(m-1)s_3+2a$. The image Q of H in G^{ab} is generated by $2s$ and s_3. An easy calculation shows that, in $\mathbb{Z}^3$, $\langle 2s, s_3\rangle \cap R$ is generated by $r_m := 2s + m(m-1)s_3$ if m is even, and $r_m := 4s + 2m(m-1)s_3$ if m is odd. This means that, as an abelian group, Q is presented as the quotient of the free abelian group $\mathbb{Z}^2$ over $2s$ and s_3 by the relation $r_m = 0$. But this relation already holds in H. Indeed, in $\mathbf{B}_{1,1,m}(\mathbb{S}^2)^{\mathrm{ab}} \subset G$, we have $s^2 = a_{12} - ma_{13} = -ma_{23}$ (see the proof of Proposition 4.4.73), so if we multiply the relation $2(m-1)s_3 + a_{13} + a_{23} = 0$ by $m/2$ if m is even (resp. by m if m is odd), we obtain $m(m-1)s_3 = s^2$ (resp. $2m(m-1)s_3 = 2s^2$), hence $r_m = 0$ in $H \subset G$. Finally, the projection $H \twoheadrightarrow Q$ must be an isomorphism (one can construct an inverse to it using the presentation of Q), whence our conclusion. ∎

Remark 4.4.75. It seems difficult to decide theoretically between the two possibilities, although our experimental calculations [DPS22a] using GAP [GAP] and the package NQ [Nic96] suggest that the LCS of G (and hence also the one of $\mathbf{B}_{2,m}(\mathbb{S}^2)$) always stops at $\Gamma_{v_2(m)+2}$ (for both even and odd m); we conjecture that this is the case.

Finally, let us show that the LCS of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ does not stop, using Lemma 4.1.1. Namely, we are looking for a quotient of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ whose LCS does not stop. As a manageable non-abelian quotient, one can think of $\mathbf{B}_{2,2}(\mathbb{S}^2)/\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$, which is an extension of $\mathfrak{S}_2 \times \mathfrak{S}_2$ by $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. In fact, in order to make it even more manageable, we take a further quotient, turning it into a split extension to which the methods of Appendix 4.5 apply.

Proposition 4.4.76. *The LCS of* $\mathbf{B}_{2,2}(\mathbb{S}^2)$ *does not stop.*

Proof. Recall that $\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$ is fully invariant in $\mathbf{P}_4(\mathbb{S}^2)$, hence normal in $\mathbf{B}_{2,2}(\mathbb{S}^2)$, so the quotient $G = \mathbf{B}_{2,2}(\mathbb{S}^2)/\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$ is a well-defined extension of $\mathfrak{S}_2 \times \mathfrak{S}_2$ by $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. The latter it is the abelian group generated by a_{ij} for $1 \leqslant i < j \leqslant 4$, subject to the four relations $\sum_{j \neq i} a_{ij} = 0$ (for each $i \leqslant 4$); this classical computation is part of Proposition 4.4.33.

The action of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ on $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ induced by conjugation factors through $\mathbf{B}_{2,2}(\mathbb{S}^2)/\mathbf{P}_4(\mathbb{S}^2) \cong \mathfrak{S}_2 \times \mathfrak{S}_2$. This action is by permutation of the indices of the generators a_{ij} of $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. In particular, it fixes a_{12} and a_{34}, which then generate a central subgroup H of G. Let us consider the quotient by this central subgroup, that is, the quotient of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ by its subgroup $\tilde{H} = \langle A_{12}, A_{34}, \Gamma_2(\mathbf{P}_4(\mathbb{S}^2))\rangle$. This quotient is an extension:

$$\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H \hookrightarrow G/H \twoheadrightarrow \mathfrak{S}_2 \times \mathfrak{S}_2.$$

By definition of $\tilde{H}$, we have $\overline{\sigma}_1^2 = \overline{\sigma}_3^2 = 1$ in $\mathbf{B}_{2,2}(\mathbb{S}^2)/\tilde{H} = G/H$, so this short exact sequence splits. Moreover, $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H$ is the quotient of $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ by the relations $a_{12} = a_{34} = 0$, modulo which the relations defining $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ become $a_{13} = -a_{14} = a_{13} = -a_{23}$. Thus $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H \cong \mathbb{Z}$, and $G/H \cong \mathbb{Z} \rtimes (\mathfrak{S}_2)^2$, where both transpositions act via a sign. This action factors through the signature $\varepsilon\colon \mathfrak{S}_2 \times \mathfrak{S}_2 \twoheadrightarrow \mathbb{Z}/2$, so that $\Gamma_*^{\mathfrak{S}_2 \times \mathfrak{S}_2}(\mathbb{Z}) = \Gamma_*^{\mathbb{Z}/2}(\mathbb{Z})$; see §4.5.1 for the definition of relative LCS. From Lemma 4.5.4 (whose proof could alternatively be repeated in this situation), we get that $\Gamma_i(\mathbb{Z} \rtimes (\mathfrak{S}_2)^2) = 2^{i-1}\mathbb{Z}$ for $i \geqslant 2$, so the LCS of $\mathbf{B}_{2,2}/\tilde{H} = G/H$ does not stop. ∎

Proof of Theorem 4.4.71. The first point of the theorem was proven just before its statement. We explain how to deduce the second point from the results above.

If λ has at least four blocks of size 1, then $\mathbf{B}_\lambda(\mathbb{S}^2)$ surjects onto the pure braid group $\mathbf{P}_4(\mathbb{S}^2)$ (by forgetting the other blocks). Using Lemma 4.1.1, the result in this case follows from Lemma 4.4.72.

The remaining cases are the ones where there is at least one block of size 2. If the partition has at least three blocks, the result follows from Proposition 4.4.73. If the partition has exactly two blocks, and if the size of the other block is at least 2, then we can apply either Proposition 4.4.74 or Proposition 4.4.76.

Finally, let us compute $\mathbf{B}_{2,1}(\mathbb{S}^2)$ and $\mathbf{B}_2(\mathbb{S}^2)$. As recalled in the proof of Lemma 4.4.72, $\mathbf{P}_3(\mathbb{S}^2) \cong \pi_1(SO_3(\mathbb{R})) \cong \mathbb{Z}/2$, so that $\mathbf{B}_{2,1}(\mathbb{S}^2)$ is an extension of $\mathfrak{S}_2 \cong \mathbb{Z}/2$ by $\mathbb{Z}/2$. Thus, it must be isomorphic to $(\mathbb{Z}/2)^2$ or to $\mathbb{Z}/4$. In order to decide between the two, one can use the computation of $\mathbf{B}_{2,1}^{\mathrm{ab}}$ from Proposition 4.4.33, and find that $\mathbf{B}_{2,1}(\mathbb{S}^2) \cong \mathbb{Z}/4$. As for $\mathbf{B}_2(\mathbb{S}^2)$, we use Proposition 4.4.15 to get an exact sequence $\pi_1(\mathbb{S}^2 - \{pt\}) \to \mathbf{P}_2(\mathbb{S}^2) \to \pi_1(\mathbb{S}^2)$, which implies that $\mathbf{P}_2(\mathbb{S}^2) = 1$, whence $\mathbf{B}_2(\mathbb{S}^2) \cong \mathfrak{S}_2$. □

Partitioned braids on the projective plane

As in §4.4.3, we see the projective plane $\mathbb{P}^2 = \mathcal{N}_1$ as a sphere with one crosscap, which is the quotient of the Möbius strip $\mathbb{M}^2 = \mathcal{N}_{1,1}$ by its boundary. This allows us to use the conventions of Figure 4.4b (with $g = 1$) to describe braids on the projective plane. We only modify slightly our notation, as we did before for braids on the Möbius strip:

Notation 4.4.77. We use the notation of §4.4.3 for braids on non-orientable surfaces, but since there is only one crosscap, we denote $c_1^{(\alpha)}$ by γ_α.

We now prove the following theorem, which describes when the LCS of $\mathbf{B}_\lambda(\mathbb{P}^2)$ stops, except for $\mathbf{B}_{2,m}(\mathbb{P}^2)$ with $m \geqslant 3$ (for this case, see Conjecture 4.4.95):

Theorem 4.4.78. *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, \dots, n_l)$ be a partition of n. The LCS of $\mathbf{B}_\lambda(\mathbb{P}^2)$:*

- stops at Γ_2 *if $l = 1$, except for $\mathbf{B}_2(\mathbb{P}^2)$, which is (strictly) 3-nilpotent.*
- stops at Γ_3 *if $l \geqslant 2$ and $n_i \geqslant 3$ for all i.*
- does not stop *in all the other cases where $l \geqslant 3$.*

Moreover, the LCS of $\mathbf{B}_{2,2}(\mathbb{P}^2)$ and of $\mathbf{B}_{1,2}(\mathbb{P}^2)$ do not stop, the LCS of $\mathbf{B}_{1,1}(\mathbb{P}^2)$ stops at Γ_3 and, for $m \geqslant 3$, the LCS of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ stops at $\Gamma_{v_2(m)+2}$ or at $\Gamma_{v_2(m)+3}$ when m is even, where v_2 is the 2-adic valuation, and at Γ_3 or Γ_4 when m is odd.

In the case of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ with $m \geqslant 3$, where the answer given above is ambiguous between two consecutive possibilities, we conjecture that its LCS stops at $\Gamma_{v_2(m)+3}$ for all $m \geqslant 3$; see Remark 4.4.82.

The proof splits into many different cases, which we consider individually below; they are synthesised into a proof of Theorem 4.4.78 immediately after Remark 4.4.94.

Blocks of size 1. In order to study partitioned braids with blocks of size 1 on the projective plane, we need to know what the Fadell-Neuwirth exact sequence becomes in the exceptional case $n = 1$ of Proposition 4.4.15.

Proposition 4.4.79. *For any partition μ of an integer m, we have a short exact sequence:*

$$\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2 \hookrightarrow \mathbf{B}_{1,\mu}(\mathbb{P}^2) \twoheadrightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2,$$

where ξ is the central element of $\mathbf{B}_\mu(\mathbb{M}^2)$ given by all the strands going once along $\partial\mathbb{M}^2$.

Sketch of proof. One needs to show that ξ^2 is the image of a generator of $\pi_2(\mathbb{P}^2)$ by the connecting morphism in the long exact sequence of the proof of Proposition 4.4.15. A generator of $\pi_2(\mathbb{P}^2)$ is given by the canonical projection $\mathbb{S}^2 \twoheadrightarrow \mathbb{P}^2$, which can be lifted to a map from the disc $\mathbb{D}$ to $F_{m+1}(\mathbb{P}^2)$ as follows. Let π denote the projection of $\mathbb{D}$ onto $\mathbb{S}^2 = \mathbb{D}/\partial\mathbb{D}$ sending $\partial\mathbb{D}$ to the south pole $P = -N$. Let $\rho\colon \mathbb{D} \to SO_3(\mathbb{R})$ be the unique continuous function sending each $x \in \mathrm{Int}(\mathbb{D})$ to the rotation of axis $N \times \pi(x)$ and angle $(N, \pi(x))$ (so that $\rho(x)(N) = \pi(x)$). Let $\mathrm{ev}\colon SO_3(\mathbb{R}) \to F_{m+1}(\mathbb{P}^2)$

be evaluation at a base configuration whose first element is $\pm N$. Then $\mathrm{ev}\circ\rho$ is the required lift. Moreover, when x goes once around $\partial\mathbb{D}$, then $\rho(x)$ goes twice around the circle of rotations of angle π with axis orthogonal to N. Since $\mathbb{P}^2$ is to be thought of as the quotient of the Möbius strip by its boundary (which goes to $\pm N$), this circle of rotations evaluates to the element ξ described in the statement. □

Proposition 4.4.80. *Let μ be a partition having at least two blocks. Then the LCS of $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ does not stop.*

Proof. We use the extension from Proposition 4.4.79:

$$\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2 \hookrightarrow \mathbf{B}_{1,\mu}(\mathbb{P}^2) \twoheadrightarrow \pi_1(\mathbb{P}^2)\cong\mathbb{Z}/2.$$

We denote the partition $\lambda := (1,\mu)$ of the integer n by $(n_1,\dots,n_l)$, and we denote by l' the number of indices i such that $n_i \geqslant 2$. Using notations from the §4.4.3 (changed to $\gamma_i := c_1^{(i)}$), we have that the quotient in the previous extension is generated by the class of γ_1, and we can write ξ as the following product of commuting braids:

$$\xi = \prod_{i=2}^{n}\gamma_i^2(A_{i,i+1}\cdots A_{i,n})^{-1}.$$

Let us consider the quotient G of $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ by $\Gamma_2(\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2)$. It is an extension of $\mathbb{Z}/2$ by $\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}}/2\overline{\xi}$. Recall from Proposition 4.4.48 that:

$$\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}} \cong \mathbb{Z}^{l-1}\times(\mathbb{Z}/2)^{l'}.$$

A basis of the first factor is given by $c_2,\dots,c_l$, where each c_i is the common class of the γ_α with α in the i-th block of λ. A $(\mathbb{Z}/2)$-basis of the second factor is given by the elements s_i described in Proposition 4.3.5. The images of the s_i in G commute with the class c_1 of γ_1, since they have lifts with disjoint support. As a consequence, they are central not only in $\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}}/2\overline{\xi}$, but in G. Let them generate the subgroup A of G, and consider G/A. There is an extension:

$$\mathbb{Z}^{l-1}/2\overline{\xi} \hookrightarrow G/A \twoheadrightarrow \mathbb{Z}/2,$$

where $\overline{\xi} = 2\sum_{i\geqslant 2} n_i c_i$. This extension is not split, but we can quotient further to get a split extension of abelian groups. Namely, we need to kill the element c_1^2 of G. In order to understand it, we use the relations from Corollary 4.4.31: $\gamma_\alpha^2 = A_{\alpha 1}\cdots A_{\alpha n}$ in $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$. In G/A, where the $A_{\alpha\beta}$ are killed if $\alpha,\beta\geqslant 2$, these relations give $\overline{A_{1\alpha}} = \overline{\gamma_\alpha}^2$ if $\alpha\geqslant 2$ and

$$c_1^2 \;=\; \overline{\gamma_1}^2 \;=\; \overline{A_{12}}\cdots\overline{A_{1n}} \;=\; \overline{\gamma_2}^2\cdots\overline{\gamma_n}^2 = 2\sum_{i\geqslant 2} n_i c_i \;=\; \overline{\xi}.$$

In particular, $\overline{\xi}$ commutes with c_1, which implies that it is central in G/A. The quotient H of G/A by $\overline{\xi}$ is thus a semi-direct product:

$$H \cong (\mathbb{Z}^{l-1}/\overline{\xi}) \rtimes \mathbb{Z}/2.$$

Finally, we need to compute the action of c_1 by conjugation on the other c_i. Recall that $[\gamma_\alpha,\gamma_1^{-1}] = A_{1,\alpha}$ in $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ (see Figure 4.3a), and $\overline{A_{1,\alpha}} = \overline{\gamma_\alpha}^2$ in H, so that $[c_i, c_1^{-1}] = c_i^2$ in H, whence $c_1^{-1}c_i^{-1}c_1 = c_i$, which implies that c_1 acts via $-id$ on $\langle c_2,\dots,c_l\rangle = \mathbb{Z}^{l-1}/\overline{\xi} \cong \mathbb{Z}^{l-2}\times(\mathbb{Z}/2)$. Since we have assumed that $l\geqslant 3$, the LCS of $H = (\mathbb{Z}^{l-2}\times(\mathbb{Z}/2))\rtimes(\mathbb{Z}/2)$ does not stop; see Corollary 4.5.6. □

Proposition 4.4.81. *For $m\geqslant 3$, the LCS of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ stops either at $\Gamma_{v_2(m)+2}$ or at $\Gamma_{v_2(m)+3}$ when m is even, where v_2 is the 2-adic valuation, and either at Γ_3 or at Γ_4 when m is odd.*

Proof. Again, we use the extension from Proposition 4.4.79:

$$\mathbf{B}_m(\mathbb{M}^2)/\xi^2 \hookrightarrow \mathbf{B}_{1,m}(\mathbb{P}^2) \twoheadrightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2.$$

Recall that, since $m \geqslant 3$, the LCS of $\mathbf{B}_m(\mathbb{M}^2)$ stops at $\varGamma_2$; see Theorem 4.4.43. Moreover, $\mathbf{B}_m(\mathbb{M}^2)/\varGamma_\infty = \mathbf{B}_m(\mathbb{M}^2)^{\mathrm{ab}} \cong \mathbb{Z} \times \mathbb{Z}/2$, where the first factor is generated by the common class γ of $\gamma_2, \ldots, \gamma_{m+1}$ and the second factor is generated by the common class σ of the σ_i; see Proposition 4.4.33. The image of the central element ξ of $\mathbf{B}_m(\mathbb{M}^2)^{\mathrm{ab}}$ is then:

$$\overline{\xi} = \sum_{i=2}^{m+1} \overline{\gamma_i^2 (A_{i,i+1} \cdots A_{i,m+1})^{-1}} = 2m\gamma - 2 \cdot \frac{m(m+1)}{2}\sigma = 2m\gamma,$$

so that $\mathbf{B}_m(\mathbb{M}^2)/(\xi^2 \varGamma_2) \cong \mathbb{Z}/4m \times \mathbb{Z}/2$. Let us consider the quotient G of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ by the image of $\Gamma_2(\mathbf{B}_m(\mathbb{M}^2))$. The image of $\varGamma_2(\mathbf{B}_m(\mathbb{M}^2)) = \varGamma_\infty(\mathbf{B}_m(\mathbb{M}^2))$ is inside $\varGamma_\infty(\mathbf{B}_{1,m}(\mathbb{P}^2))$, so that G and $\mathbf{B}_{1,m}(\mathbb{P}^2)$ have the same associated Lie ring. Now, G is an extension:

$$\mathbb{Z}/4m \times \mathbb{Z}/2 \hookrightarrow G \twoheadrightarrow \mathbb{Z}/2.$$

We can already see that G is finite, and deduce that its LCS stops, so that $\varGamma_*(\mathbf{B}_{1,m}(\mathbb{P}^2))$ stops too. In order to be more precise, let us recall that $G = \langle \gamma, \sigma, \gamma_1 \rangle$, where the class of γ_1 generates the quotient in the previous extension. Notice that γ_1^2 is in the kernel, so it commutes with γ and σ. Since it obviously commutes with γ_1, it is central in G. Moreover, as in the previous proof, the boundary relations in $\mathbf{B}_{1,m}(\mathbb{P}^2)$ give:

$$\gamma = \overline{\gamma_i^2} = \overline{A_{i,1} \cdots A_{i,m+1}} = \overline{A_{i,1}} \quad \text{for } i \geqslant 2 \quad \text{and} \quad \gamma_1^2 = \overline{A_{1,2} \cdots A_{1,m+1}} = \gamma^{2m} = \overline{\xi}.$$

Thus, the quotient of G by its central subgroup $A = \langle \gamma^{2m} \rangle$ is an extension of $\mathbb{Z}/2$ by $(\mathbb{Z}/2m) \times \mathbb{Z}/2$. Since $\gamma_1^2 = 1$ in G/A, this extension splits as a semi-direct product:

$$G/A \cong ((\mathbb{Z}/2m) \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2.$$

The element γ_1 of G/A commutes with σ, and, using once again the isotopy from Figure 4.3a:

$$[\gamma, \gamma_1^{-1}] = \overline{[\gamma_2, \gamma_1^{-1}]} = \overline{A_{12}} = \gamma^2,$$

which implies that $\gamma_1^{-1}\gamma^{-1}\gamma_1 = \gamma$. This means that $\mathbb{Z}/2$ acts trivially on the $\mathbb{Z}/2$ factor and via $-id$ on the $\mathbb{Z}/2m$ factor. This explicit description allows us to compute completely the LCS of $G/A \cong \mathbb{Z}/2 \times (\mathbb{Z}/2m \rtimes \mathbb{Z}/2)$ using Corollary 4.5.8. Precisely, $\varGamma_i(G/A) = 2^{i-1}\mathbb{Z}/(2m)$ for $i \geqslant 2$. Hence it stops at $\varGamma_3$ if m is odd and at $\varGamma_{v_2(m)+2}$ if m is even. Finally, since A is cyclic of order 2, we can apply Lemma 4.1.2 (with $l = 1$) to conclude that the LCS of G (whence the one of $\mathbf{B}_{1,m}(\mathbb{P}^2)$) stops at $\varGamma_{v_2(m)+2}$ or at $\varGamma_{v_2(m)+3}$ when m is even and at $\varGamma_3$ or $\varGamma_4$ when m is odd. □

Remark 4.4.82. Similarly to the situation of $\mathbf{B}_{2,m}(\mathbb{S}^2)$ (see Remark 4.4.75), it seems difficult to decide theoretically between the two possibilities in Proposition 4.4.81. However, based on experimental calculations [DPS22a] using GAP [GAP] and the package NQ [Nic96], we conjecture that the LCS of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ always stops at $\varGamma_{v_2(m)+3}$ (for both even and odd m).

We are left with two cases to consider where there is a block of size 1, namely $\mathbf{B}_{1,1}(\mathbb{P}^2)$ and $\mathbf{B}_{1,2}(\mathbb{P}^2)$. The group $\mathbf{B}_{1,1}(\mathbb{P}^2) = \mathbf{P}_2(\mathbb{P}^2)$ is isomorphic to the quaternion group Q_8 (see Corollary 4.4.83 below), which is 2-nilpotent, so its LCS stops at $\varGamma_3$. Notice that this means that the conclusion of Proposition 4.4.81 is correct also for $m = 1$, since that proposition would assert that the LCS of $\mathbf{B}_{1,1}(\mathbb{P}^2)$ stops at $\varGamma_3$ or $\varGamma_4$. This fact is also compatible with our conjecture in Remark 4.4.82. In contrast, we will show that the LCS of $\mathbf{B}_{1,2}(\mathbb{P}^2)$ does not stop (Proposition 4.4.89). The study of this latter case is postponed, and will be part of our study of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, of which a presentation will be computed in Proposition 4.4.85 for every $m \geqslant 1$.

Blocks of size 2. We now study the case where there is at least one block with exactly two strands.

Proposition 4.4.79 can be used to recover the following classical calculations from [Bus66, p. 87]:

Corollary 4.4.83. *The pure braid group $\mathbf{P}_2(\mathbb{P}^2)$ is isomorphic to the quaternion group Q_8 (which is 2-nilpotent), and $\mathbf{B}_2(\mathbb{P}^2)$ to the dicyclic group of order 16 (which is 3-nilpotent). Precisely, a presentation of the latter is:*

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \;\middle|\; \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle,$$

where σ_1 and γ_1 are the elements of $\mathbf{B}_2(\mathbb{P}^2)$ defined above (Notation 4.4.77).

Proof. Recall that the dicyclic group of order 16 can be defined by the presentation:

$$Dic_{16} = \left\langle s, x, y \;\middle|\; sx = ys,\ x^2 = y^2 = (xy)^2 = s^2 \right\rangle,$$

and that it contains the subgroup Q_8 of quaternions as the index-2 subgroup generated by x and y. Notice that, modulo the other relations, $(xy)^2 = s^2$ is equivalent to $s^2x^{-1}yxs^2 = s^2$, which is equivalent to $[y, x^{-1}] = s^2$ by passing to the inverses. We can also use the first relation to eliminate $y = sxs^{-1}$. As a consequence, we also have:

$$Dic_{16} = \left\langle s, x \;\middle|\; x^2 = [sxs^{-1}, x^{-1}] = s^2 \right\rangle.$$

It is easy to check that the elements σ_1, γ_1 and γ_2 of $\mathbf{B}_2(\mathbb{P}^2)$ satisfy the above relations, so that $s \mapsto \sigma_1$ and $x \mapsto \gamma_1$ define a morphism φ from Dic_{16} to $\mathbf{B}_2(\mathbb{P}^2)$. Indeed, the relation $\gamma_1^2 = \sigma_1^2$ is one of the boundary relations from Corollary 4.4.27, and an isotopy witnessing the last one is drawn in Figure 4.3a. Note that φ sends the element $y = sxs^{-1}$ to $\sigma_1\gamma_1\sigma_1^{-1} = \gamma_2$.

By taking $m = n = 1$, the short exact sequence of Proposition 4.4.79 specialises to:

$$\pi_1(\mathbb{M}^2)/\xi^2 \cong \mathbb{Z}/4 \hookrightarrow \mathbf{P}_2(\mathbb{P}^2) \twoheadrightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2.$$

Indeed, $\mathbf{B}_1(\mathbb{M}^2) = \pi_1(\mathbb{M}^2)$ is isomorphic to $\mathbb{Z}$, and the element ξ, which is a loop parallel to the boundary of the Möbius strip, is the square of a generator. From this, we deduce that $\mathbf{P}_2(\mathbb{P}^2)$ has eight elements, and that it is generated by γ_1 (the image of a generator of $\pi_1(\mathbb{M}^2)$) and γ_2 (a lift of the generator of $\pi_1(\mathbb{P}^2)$). As a consequence, φ must induce an isomorphism between $Q_8 = \langle x, y \rangle \subset Dic_{16}$ and $\mathbf{P}_2(\mathbb{P}^2)$. Then, we can use the usual extension

$$\mathbf{P}_2(\mathbb{P}^2) \hookrightarrow \mathbf{B}_2(\mathbb{P}^2) \twoheadrightarrow \mathfrak{S}_2$$

to deduce that σ_1, γ_1 and γ_2 generate $\mathbf{B}_2(\mathbb{P}^2)$, and that $\mathbf{B}_2(\mathbb{P}^2)$ has sixteen elements. Hence φ is an isomorphism. ◻

We first deal with the case where there are at least three blocks:

Proposition 4.4.84. *Let μ be a partition having at least two blocks. Then the LCS of $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$ does not stop.*

Proof. We are looking for a quotient whose LCS can be computed, and does not stop. Let us consider the Fadell-Neuwirth extension from Proposition 4.4.15:

$$\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\}) \hookrightarrow \mathbf{B}_{2,\mu}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

We use notations similar to the ones from the proof of Proposition 4.4.80: we denote the partition $\lambda := (1, 1, \mu)$ of the integer n by $(n_1, \ldots, n_l)$, and we denote by l' the number of indices i such that $n_i \geqslant 2$. We also use Notation 4.4.77 for braids on the projective plane.

In order to get a more manageable extension, we first take the quotient G of $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$ by $\varGamma_2(\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\}))$, getting an extension:

$$\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}} \hookrightarrow G \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

The kernel $\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}}$ is computed in Proposition 4.4.48: it is the product $(\pi_1(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}})^{l-2} \times (\mathbb{Z}/2)^{l'}$, where the first factor is generated by the classes c_i of γ_α, a_{1i} of $A_{1\alpha}$ and a_{2i} of $A_{2\alpha}$ (for α in the i-th block, with $i \geqslant 3$), subject to the relations $a_{1i} + a_{2i} = 2c_i$ (for each i, we get a copy of $\pi_1(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}} \cong \mathbb{Z}^2$), and the second one is generated by the classes s_i of the σ_α (for α and $\alpha+1$ in the i-th block of λ, which is possible only if $n_i \geqslant 2$). The group G is generated by these elements, together with the classes of σ_1 and γ_1 (whose images generate $\mathbf{B}_2(\mathbb{P}^2)$).

The elements s_i commute with the other elements of $\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}}$ (which is abelian), but also with σ_1 and γ_1 (for reasons of support). Hence they are central elements of G. Let us denote by $A \cong (\mathbb{Z}/2)^{l'}$ the central subgroup they generate, and let us consider the extension:

$$\mathbb{Z}^{2(l-2)} \hookrightarrow G/A \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

Corollary 4.4.83 gives a presentation of the quotient, namely:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \;\middle|\; \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

We now compute a presentation of G/A, using the method from Appendix 4.6. Generators are c_i, a_{1i} and a_{2i} (for $3 \leqslant i \leqslant l$), together with σ_1 and γ_1. Relations defining the kernel are the ones saying that the c_i, the a_{1i} and the a_{2i} commute with each other, together with $a_{1i} + a_{2i} = 2c_i$ (for each $i \geqslant 3$). The latter could be used to eliminate $a_{2i} = 2c_i - a_{1i}$. However, we will choose not to do so here, and to give a redundant, but more tractable presentation of G/A. Relations lifting the above presentation of the quotient are:

$$\begin{cases} \gamma_1^2 = \sigma_1^2 \cdot a_{13}^{n_3} \cdots a_{1l}^{n_l}, \\ [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2. \end{cases}$$

Indeed, these hold in $\mathbf{B}_{2,\mu}(\mathbb{P}^2) \subset \mathbf{B}_n(\mathbb{P}^2)$: the first one is one of the boundary relations from Corollary 4.4.27 and the second one is the one pictured in Figure 4.3a. Moreover, these are clearly lifts of the relations defining $\mathbf{B}_2(\mathbb{P}^2)$.

We are left with understanding how σ_1 and γ_1 (whence also $\gamma_2 = \sigma_1\gamma_1\sigma_1^{-1}$) act by conjugation on the c_i, the a_{1i} and the a_{2i}. We claim that the following relations hold in G/A:

$$\begin{cases} \sigma_1 a_{1i} = a_{2i}\sigma_1, \quad \sigma_1 a_{2i} = a_{1i}\sigma_1 \quad \text{and} \quad \sigma_1 \rightleftarrows c_i \\ \gamma_1 a_{ki}\gamma_1^{-1} = (-1)^{\delta_{1k}} a_{ki} \quad \text{and} \quad \gamma_1 c_i \gamma_1^{-1} = c_i - a_{1i}, \end{cases}$$

where we use additive notations in the (abelian) subgroup generated by the c_i, the a_{1i} and the a_{2i}. These are images of relations holding in $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$, which can be proved by drawing explicit isotopies. Precisely, the first one comes from $\sigma_1 A_{1i}\sigma_1^{-1} = A_{2i}$, the second one from $\sigma_1 A_{2i}\sigma_1^{-1} = A_{2i}^{-1}A_{1i}A_{2i}$ and the third one from the commutation of σ_1 with all the γ_α if $\alpha \geqslant 3$. The other relation comes from γ_1 commuting with A_{2i}, and from $[\gamma_i, \gamma_1^{-1}] = A_{1i}$ (Figure 4.3a), that is, $\gamma_1^{-1}\gamma_i\gamma_1 = A_{1i}^{-1}\gamma_i$. Notice that the relation involving $\gamma_1 a_{1i}\gamma_1^{-1}$ can be deduced from the other two, using $a_{1i} = 2c_i - a_{2i}$.

We now have a presentation of G/A. In order to get a simpler quotient, we quotient further by σ_1^2 and γ_1^2. That is, we add the relations $\sigma_1^2 = \gamma_1^2 = 1$ to the previous presentation. The result is a split extension:

$$\mathbb{Z}^{2(l-2)} / \left(\sum_{i=3}^{l} n_i a_{1i} = \sum_{i=3}^{l} n_i a_{2i} = 0 \right) \rtimes W_2.$$

If we quotient further by $\sum n_i c_i$ (which is central in the above semi-direct product, since it is fixed by the action of W_2), we obtain a semi-direct product $M \rtimes W_2$, where M has an explicit workable description. In fact, as a W_2-representation, $M \cong \Lambda \otimes B$, where Λ is the canonical representation of W_2 defined in §4.5.2 and B is the quotient of $\mathbb{Z}^{l-2}$ by the vector $(n_3, \dots, n_l)$, seen as a trivial W_2-representation. Precisely, if e_i $(i = 3, \dots, l)$ is the generating family of B obtained from the canonical basis of $\mathbb{Z}^2$, an isomorphism $\Lambda \otimes B \cong M$ is given, with the notations from Remark 4.5.15, by $a \otimes e_i \mapsto a_{2i}$, $b \otimes e_i \mapsto a_{1i}$ and $c \otimes e_i \mapsto c_i$ (with notations from Remark 4.5.15).

We finally use our hypothesis: since $l \geqslant 4$, the rank of B is not 0, so it surjects onto $\mathbb{Z}$. Thus, M surjects onto $\Lambda \otimes \mathbb{Z} = \Lambda$ (as a W_2-representation), and $M \rtimes W_2$ surjects onto $\Lambda \rtimes W_2$, whose LCS (computed in Proposition 4.5.12) does not stop. □

Now, we are left with the case where there are precisely two blocks, one of which has exactly two strands. First of all, we give an explicit presentation of the associated group:

Proposition 4.4.85. *Let $m \geqslant 1$ be an integer. The group $\mathbf{B}_{2,m}(\mathbb{P}^2)$ admits the presentation with generators $\sigma_1, \sigma_3, \sigma_4, \dots, \sigma_{m+1}$, γ_1, γ_3 and A_{23}, subject to the following relations:*

$$\begin{cases} (\mathbb{P}R1) & \sigma_4, \dots, \sigma_{m+1} \rightleftarrows \gamma_3, A_{23}; \\ (\mathbb{P}R2) & A_{23} \rightleftarrows \sigma_3\gamma_3\sigma_3^{-1}; \\ (\mathbb{P}R3) & (\sigma_3 A_{23})^2 = (A_{23}\sigma_3)^2; \\ (\mathbb{P}R4) & [\sigma_i\gamma_i\sigma_i^{-1}, \gamma_i^{-1}] = \sigma_i^2 \quad \text{for } i \in \{1,3\}; \\ (\mathbb{P}R5) & \gamma_1^2 = \sigma_1 \left(\prod_{k=2}^{m+1} (\sigma_k \cdots \sigma_3) A_{23} (\sigma_k \cdots \sigma_3)^{-1} \right) \sigma_1; \\ (\mathbb{P}R6) & \sigma_1 \rightleftarrows \sigma_3, \sigma_4, \dots, \sigma_{m+1}, \gamma_3; \\ (\mathbb{P}R7) & \gamma_1 \rightleftarrows \sigma_3, \sigma_4, \dots, \sigma_{m+1}, A_{23}; \\ (\mathbb{P}R8) & \gamma_3^2 = (\sigma_1^{-1} A_{23} \sigma_1) A_{23} \cdot \prod_{k=3}^{m+1} (\sigma_k \cdots \sigma_4) \sigma_3^2 (\sigma_k \cdots \sigma_4)^{-1}; \\ (\mathbb{P}R9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1} A_{23} \sigma_1. \end{cases}$$

As above, the choice of names for the generators is coherent with their geometric interpretation (see Notation 4.4.77 and Figure 4.4b).

Remark 4.4.86. This set of generators is not minimal, although it is close to being minimal. Indeed, we can eliminate A_{23} using $(\mathbb{P}R9)$. The set of generators thus obtained is then minimal, at least for $m = 1, 2$, since the classes of the generators are a $\mathbb{Z}/2$-basis of the abelianisation in this case. However, this elimination would render the relations much less tractable; the above seems a better compromise between the number of generators and readability of the relations.

Proof of Proposition 4.4.85. Let us begin by checking that these relations hold for the usual elements $\sigma_1, \sigma_3, \sigma_4, \dots, \sigma_{m+1}$, γ_1, γ_3 and A_{23} of $\mathbf{B}_{2,m}(\mathbb{P}^2)$. Let us first remark that we can express other usual elements in terms of these elements. Namely, $A_{13} = \sigma_1^{-1} A_{23} \sigma_1$ and, if $k \in \{3, \dots, m+2\}$, γ_k (resp. A_{2k}, A_{1k}) is obtained from γ_3 (resp. A_{23}, A_{13}) by conjugation by $\sigma_k \cdots \sigma_4$ on the left (by convention, this product equals 1 if $k = 3$). Then one can see that $(\mathbb{P}R5)$ and $(\mathbb{P}R8)$ are the boundary relations from Corollary 4.4.27, corresponding to the strands 1 and 3 (recall that $A_{12} = \sigma_1^2$ and that σ_1 commutes with $\sigma_3, \dots, \sigma_{m+1}$). All the other relations can be checked by drawing explicit isotopies. Precisely, the relations $(\mathbb{P}R1)$, $(\mathbb{P}R2)$, $(\mathbb{P}R6)$ and $(\mathbb{P}R7)$ are commutation relations between elements having disjoint support; $(\mathbb{P}R4)$ and $(\mathbb{P}R9)$ are instances of the relation drawn in Figure 4.3a; $(\mathbb{P}R3)$ can either be proved by drawing an explicit isotopy, or it can be deduced from the similar relation in $\mathbf{B}_{1,2}$ (see Lemma 4.3.16), by considering the morphism from $\mathbf{B}_{1,2}$ to $\mathbf{B}_{2,m}(\mathbb{P}^2)$ induced by a well-chosen embedding of $\mathbb{D}$ into $\mathbb{P}^2$.

In order to show that these relations describe the group, we now apply the methods of Appendix 4.6 to the Fadell-Neuwirth extension from Proposition 4.4.15:

$$\mathbf{B}_m(\mathbb{P}^2 - \{2 \text{ pts}\}) \hookrightarrow \mathbf{B}_{2,m}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

Since $\mathbb{P}^2 - \{2 \text{ pts}\} = \mathcal{N}_{1,2}$, a presentation of the kernel is given by Proposition 4.4.25, for $n = 1$. In order for it to identify with the right subgroup of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ (corresponding to braids on the strands $3, \dots, m+2$), indices are shifted by 2 (so that, for instance, c_1 becomes γ_3), and we take $z_1 = A_{23}$;

this last choice changes the presentation a little bit, but one can easily figure out the necessary modifications. Thus, $(BN1)$ and $(BN4)$ give $(\mathbb{P}R1)$; $(BN2)$ and $(BN6)$ are empty; $(BN3)$ is the case $i = 3$ of $(\mathbb{P}R4)$; $(BN5)$ is $(\mathbb{P}R2)$; finally, $(BN7)$ is the case $i = 3$ of $(\mathbb{P}R3)$.

A presentation of the quotient is given by Corollary 4.4.83 and its proof:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1, \gamma_2 \;\middle|\; \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

The elements $\gamma_1, \sigma_1 \in \mathbf{B}_{2,m}(\mathbb{P}^2)$ are lifts of the elements $\gamma_1, \sigma_1 \in \mathbf{B}_2(\mathbb{P}^2)$. The second relation holds without change for these lifts, giving the case $i = 1$ of $(\mathbb{P}R4)$. The relation $\gamma_1^2 = \sigma_1^2$ lifts to the boundary relation associated with the first strand, which is $(\mathbb{P}R5)$.

We are left with finding relations describing the action of σ_1 and γ_1 (or of σ_1^{-1} and γ_1^{-1} – see Remark 4.6.4) by conjugation on the other generators. The commutation relations $(\mathbb{P}R6)$ and $(\mathbb{P}R7)$ describe most of this action. Only two elements still need to be expressed in terms of the generators $\sigma_3, \ldots, \sigma_{m+1}$, γ_3 and A_{23} of the kernel, namely $\sigma_1^{-1}A_{23}\sigma_1$ and $\gamma_1^{-1}\gamma_3\gamma_1$. The boundary relation $(\mathbb{P}R8)$ deals with $\sigma_1^{-1}A_{23}\sigma_1$. Finally, $(\mathbb{P}R9)$ deals with $\gamma_1^{-1}\gamma_3\gamma_1$: it says that $\gamma_1^{-1}\gamma_3\gamma_1 = \gamma_3\sigma_1^{-1}A_{23}\sigma_1$, and the right-hand side can be expressed in terms of the generators of the kernel using the previous relation $(\mathbb{P}R8)$. This finishes the proof that the above relations are the ones obtained using the method of Appendix 4.6, whence the result. ◻

Let us now consider the case $m = 1$. In this case, there is no σ_i for $i \geqslant 3$, so the presentation is much simpler; in fact, in the extension of the proof, the kernel is just $\pi_1(\mathbb{P}^2 - \{2 \text{ pts}\})$, which is free on γ_3 and A_{23}. Thus, $(\mathbb{P}R1)$, $(\mathbb{P}R2)$ and $(\mathbb{P}R3)$ are empty in this case, and there is no case $i = 3$ in $(\mathbb{P}R4)$. The other relations reduce to:

Corollary 4.4.87. *The group $\mathbf{B}_{2,1}(\mathbb{P}^2)$ has a presentation with 4 generators σ_1, γ_1, γ_3 and A_{23} and 6 relations (indexed as above):*

$$\begin{cases} (4) & [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2; \\ (5) & \gamma_1^2 = \sigma_1 A_{23}\sigma_1; \\ (6) & \sigma_1 \rightleftarrows \gamma_3; \end{cases} \qquad \begin{cases} (7) & \gamma_1 \rightleftarrows A_{23}; \\ (8) & \gamma_3^2 = \sigma_1^{-1}A_{23}\sigma_1 A_{23}; \\ (9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1}A_{23}\sigma_1. \end{cases}$$

Remark 4.4.88. This presentation is smaller than van Buskirk's presentation [Bus66, Lem. p. 84], which has 6 generators and 13 relations. Note that in our language, his generators are as follows, where the right-hand side of each equation uses our notation and the left-hand side uses his:

$$\begin{cases} \sigma_2 = \sigma_1, \\ a_2 = \sigma_1^{-1}A_{23}\sigma_1, \\ a_3 = A_{23}, \end{cases} \qquad \begin{cases} \rho_1 = \gamma_3, \\ \rho_2 = \gamma_1\sigma_1^{-1}A_{23}^{-1}\sigma_1, \\ \rho_3 = \sigma_1^{-1}\gamma_1\sigma_1. \end{cases}$$

Proposition 4.4.89. *The LCS of $\mathbf{B}_{2,1}(\mathbb{P}^2)$ does not stop.*

Proof. Let G be the quotient of $\mathbf{B}_{2,1}(\mathbb{P}^2)$ by the single relation $\sigma_1\gamma_1\sigma_1^{-1} = \gamma_1^{-1}$. Let us consider the presentation of G given by the presentation of Corollary 4.4.87, to which this relation is added. Then relation (4) becomes $\sigma_1^2 = 1$, that is, $\sigma_1^{-1} = \sigma_1$. Relation (5) becomes $A_{23} = \sigma_1\gamma_1^2\sigma_1 = \gamma_1^{-2}$, hence (7) becomes redundant. Relation (8) is then equivalent to $\gamma_3^2 = 1$. Relation (9) becomes $\gamma_3\gamma_1^{-1}\gamma_3^{-1} = \gamma_1$. If we add to this relation (6), which says that σ_1 commutes with γ_3, we get a presentation of $\mathbb{Z} \rtimes (\mathbb{Z}/2)^2$ (where both elements of a basis of $(\mathbb{Z}/2)^2$ act via $-id$). Thus, $G \cong \mathbb{Z} \rtimes (\mathbb{Z}/2)^2$, whose LCS does not stop (one can either compute it with the method of Appendix 4.5, or quotient further by $\sigma_1 = \gamma_3$ to get $\mathbb{Z} \rtimes (\mathbb{Z}/2)$ as a quotient and apply Corollary 4.5.8). ◻

Remark 4.4.90. Even if the imposed relation looks very much like a relation defining the infinite dihedral group $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, it is not at all clear *a priori* why adding this one relation should work. Much experimentation has been needed before ending up here.

Next, let us consider the case $m = 2$. In this case, there is no σ_i for $i \geqslant 4$. Thus, $(\mathbb{P}R1)$ is empty, and the boundary relations $(\mathbb{P}R5)$ and $(\mathbb{P}R8)$ become much simpler. Let us spell out the result in this case:

Corollary 4.4.91. *The group* $\mathbf{B}_{2,2}(\mathbb{P}^2)$ *has a presentation with* 5 *generators* σ_1, γ_1, γ_3 *and* A_{23} *and* 8 *relations (indexed as above):*

$$\begin{cases} (2) & A_{23} \rightleftarrows \sigma_3\gamma_3\sigma_3^{-1}; \\ (3) & (\sigma_3 A_{23})^2 = (A_{23}\sigma_3)^2; \\ (4) & [\sigma_i\gamma_i\sigma_i^{-1}, \gamma_i^{-1}] = \sigma_i^2 \ \textit{ for } i \in \{1,3\}; \\ (5) & \gamma_1^2 = \sigma_1 A_{23}\sigma_3 A_{23}\sigma_3^{-1}\sigma_1; \end{cases} \qquad \begin{cases} (6) & \sigma_1 \rightleftarrows \sigma_3, \gamma_3; \\ (7) & \gamma_1 \rightleftarrows \sigma_3, A_{23}; \\ (8) & \gamma_3^2 = \sigma_1^{-1}A_{23}\sigma_1 A_{23}\sigma_3^2; \\ (9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1}A_{23}\sigma_1. \end{cases}$$

Proposition 4.4.92. *The LCS of* $\mathbf{B}_{2,2}(\mathbb{P}^2)$ *does not stop.*

Proof. Let us consider, as above, the projection $p\colon \mathbf{B}_{2,2}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2) \cong Dic_{16}$ induced by forgetting the last two strands. Recall that Corollary 4.4.83 gives a presentation of this quotient:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \;\middle|\; \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

Since the second relation is already true in $\mathbf{B}_{2,2}(\mathbb{P}^2)$ (it is the case $i = 1$ of relation (4) in Corollary 4.4.91), the projection p becomes split if we impose the relation $\sigma_1^2 = \gamma_1^2$. We will in fact consider the quotient G of $\mathbf{B}_{2,2}(\mathbb{P}^2)$ by the two relations:

$$\begin{cases} (Q1) & \sigma_1^2 = \gamma_1^2; \\ (Q2) & \sigma_3\gamma_3\sigma_3^{-1} = \gamma_3^{-1}. \end{cases}$$

Since the second relator also sits in the kernel of p, we get an induced split projection $\overline{p}\colon G \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2)$.

Let us consider the presentation of G given by the presentation of Corollary 4.4.91, together with the two relations $(Q1)$ and $(Q2)$. Modulo $(Q1)$, relation (5) becomes $\sigma_3 A_{23}\sigma_3^{-1} = A_{23}^{-1}$. Modulo $(Q2)$, relation (2) says that A_{23} commutes with γ_3, and the case $i = 3$ of relation (4) gives $\sigma_3^2 = 1$. At this point, let us remark that the relations obtained so far say that the subgroup $\langle \gamma_3, A_{23}, \sigma_3\rangle$ is a quotient of $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2)$, where the action of $\mathbb{Z}/2$ is by $-id$.

Continuing our investigation, we remark that relation (3) is a consequence of the previous relation (both sides of it are killed modulo these). Relation (8) becomes $\sigma_1^{-1}A_{23}\sigma_1 = \gamma_3^2 A_{23}^{-1}$ and (remembering that $\sigma_3^2 = 1$) relation (9) becomes $\gamma_1^{-1}\gamma_3\gamma_1 = A_{23}\gamma_3^{-1}$. If we add relations (6) and (7) without change, we get a presentation of G, which will allow us to describe it in an explicit way.

Let us first consider the subgroup $A := \langle \gamma_3, A_{23}\rangle$ (which is abelian by relation (2)). Relations $(Q2)$ and (2) imply that it is stable under conjugation by σ_3 (which equals σ^{-1}, by (4)). Relations (6) and (8) imply that it is stable under conjugation by σ_i^{-1}, and also by σ_i (as one sees by conjugating (8) by σ_i, taking (6) into account). In the same way, relations (7) and (9) imply that it is stable under conjugation by $\gamma_1^{\pm 1}$. Finally, all this implies that it is normal in G. Moreover, the presentation of G/A that we get from the one of G clearly gives $G/A \cong Dic_{16} \times (\mathbb{Z}/2)$ and, in fact, the relations defining G/A were already true in G for the generators σ_1, γ_1 and σ_3, so $G \twoheadrightarrow G/A$ splits, and finally:

$$G \cong A \rtimes (Dic_{16} \times (\mathbb{Z}/2)).$$

We are left with understanding A (which is a quotient of $\mathbb{Z}^2$) and the action of $K := Dic_{16} \times (\mathbb{Z}/2)$ on it. In order to do this, let us consider the relations describing the action of K on A, namely:

$$\begin{cases} (Q2) & \sigma_3\gamma_3\sigma_3^{-1} = \gamma_3^{-1}; \\ (5) & \sigma_3 A_{23}\sigma_3^{-1} = A_{23}^{-1}; \\ (6) & \sigma_1\gamma_3\sigma_1^{-1} = \gamma_3; \\ (8) & \sigma_1 A_{23}\sigma_1^{-1} = A_{23}^{-1}\gamma_3^2; \\ (9) & \gamma_1\gamma_3\gamma_1^{-1} = \gamma_3^{-1}A_{23}; \\ (7) & \gamma_1 A_{23}\gamma_1^{-1} = A_{23}. \end{cases}$$

We remark that these already define an action of K on $\mathbb{Z}^2$, which is exactly the action on $\mathbb{Z}^2 = \Gamma$ considered at the end of §4.5.2. Precisely, with the identifications $\sigma_1 \mapsto \sigma$, $\gamma_1 \mapsto \gamma$ and $\sigma_3 \mapsto \tau$

for generators of K, we get an equivariant map $\Gamma \twoheadrightarrow A$ sending a to A_{23} and c to γ_3 (with the notations of Remark 4.5.15). This induces a surjective morphism from $\Gamma \rtimes K$ onto $A \rtimes K = G$. It is then easy to check that all the relations defining G are in fact already true in $\Gamma \rtimes K$, which allows us to define a converse isomorphism $G \cong \mathbb{Z}^2 \rtimes K$.

Finally, Proposition 4.5.12, together with the equality $\varGamma_*^K(\Lambda) = \varGamma_*^{W_2}(\Lambda)$ from the end of §4.5.2, gives us a complete description of the LCS of G which, in particular, does not stop. □

Remark 4.4.93. The projection onto $Dic_{16} \times (\mathbb{Z}/2)$ in the proof can be seen as coming from the geometry. Precisely, it is the factorisation through G of the projection

$$q\colon \mathbf{B}_{2,2}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2) \times \mathfrak{S}_2 \cong Dic_{16} \times (\mathbb{Z}/2)$$

whose first factor forgets the last two strands and whose second factor forgets the first two strands and then applies the usual projection $\pi\colon \mathbf{B}_2(\mathbb{P}^2) \twoheadrightarrow \mathfrak{S}_2$.

Remark 4.4.94. This quotient looks very much like the one from the proof of Proposition 4.4.89, and the same remark applies (see Remark 4.4.90). Namely, $(Q1)$ is a natural relation to impose (making the extension split), whereas it is much less clear why quotienting by $(Q2)$ (which is the same relation as in the aforementioned proof, up to re-indexing the strands) should work.

We may now complete the proof of Theorem 4.4.78.

Proof of Theorem 4.4.78. The first two statements are part of the general results of Corollary 4.4.52 and Theorem 4.4.53, except for $\mathbf{B}_1(\mathbb{P}^2) \cong \mathbb{Z}/2$ and $\mathbf{B}_2(\mathbb{P}^2)$, which is the dicyclic group of order 16 (Corollary 4.4.83). The third statement combines Propositions 4.4.80 (if λ has blocks of size 1) and 4.4.84 (if λ has blocks of size 2). The fourth statement combines Propositions 4.4.81, 4.4.89 and 4.4.92, together with the fact that $\mathbf{B}_{1,1}(\mathbb{P}^2) = \mathbf{P}_2(\mathbb{P}^2)$ is the quaternion group Q_8 (Corollary 4.4.83). □

The remaining cases to consider are $\mathbf{B}_{2,m}(\mathbb{P}^2)$ for $m \geqslant 3$. These are the only examples of partitioned surface braid groups for which we have not been able to answer the question of whether their LCS stop. We can still say something about these LCS, using the proof of Proposition 4.4.84. Recall that in this proof, the hypothesis on the number of blocks of μ was not used until the end. Moreover, in the case $\mu = (m)$ and $m \geqslant 3$, Proposition 4.4.43 applies, implying that the first quotient G is the quotient by $\varGamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - \{pt\}))$. The latter must be contained in $\varGamma_\infty(\mathbf{B}_{2,m}(\mathbb{P}^2))$ so, in order to understand the LCS of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, we only need to understand the LCS of G. Then, since the central subgroup $A = \langle s_3 \rangle$ (where s_3 is the class of σ_3) is cyclic of order 2, we can apply Lemma 4.1.2 to see that the LCS of G stops if and only if the one of G/A does (with possibly one more step). We have the same presentation of G/A as in the proof of Proposition 4.4.84, from its decomposition as an extension of $\mathbf{B}_2(\mathbb{P}^2) = Dic_{16}$ by $\mathbb{Z}^2$. Note that the action of Dic_{16} on the (abelian) kernel in this extension is exactly the one on Λ from §4.5.2, which is through the quotient $Dic_{16} \twoheadrightarrow W_2$. Precisely, as in the previous proof, $a = a_{23}$ and $c = c_3$ identify with the basis of Λ from Remark 4.5.15. However, this extension is not split, so computing its LCS seems tricky. We can try to make it split, by considering the quotient by the relation $\sigma_1^2 = \gamma_1^2$ (which is equivalent to $a^m = 1$), but then we also kill c^{2m}, getting the finite quotient:

$$(G/A)/\sigma_1^2\gamma_1^{-2} \cong (\Lambda/\langle ma, 2mc\rangle) \rtimes Dic_{16} \cong (\mathbb{Z}/m \times \mathbb{Z}/2m) \rtimes Dic_{16}.$$

In fact, using the notation of Proposition 4.5.12, we have $\langle ma, 2mc\rangle = mV \subset \Lambda$. Thus we can deduce from Proposition 4.5.12 a computation of

$$\varGamma_*^{Dic_{16}}(\mathbb{Z}/m \times \mathbb{Z}/2m) = \varGamma_*^{W_2}(\Lambda/mV).$$

Namely, if $m = 2^\nu m'$, with m' odd, we have that $2^\nu V$ contains mV, and that $2^{\nu+1}V$ equals $2^\nu V$ modulo mV, so $\varGamma_*^{W_2}(\Lambda/mV)$ stops at $\varGamma_{2\nu+2}^{W_2}(\Lambda/mV) = 2^\nu V/mV$. Finally, the LCS of the above quotient stops at $\varGamma_k$, where $k = 2v_2(m) + 2$ if m is even and $k = 4$ is m is odd (in the latter case, note that the relative LCS stops at the second step, but Dic_{16} is 3-nilpotent).

This gives a lower bound for the step at which the LCS of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ stops: it cannot stop before Γ_k for $k = \max\{4, 2v_2(m)+2\}$. However, this lower bound is far from optimal: our experimental calculations [DPS22a] using GAP [GAP] and the package NQ [Nic96] show that, for all $m \leqslant 1024$, the LCS of G/A, and hence those of G and of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, do not stop before Γ_{100} (we also verified this for all $m = 2^\nu$ with $\nu \leqslant 23$). We thus conjecture:

Conjecture 4.4.95. *If $m \geqslant 3$, the LCS of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ does not stop.*

4.5 Appendix: Some calculations of lower central series

This appendix is devoted to computing the LCS of some combinatorially-defined groups. These include notably the Klein group $\mathbb{Z} \rtimes \mathbb{Z}$, the free products $\mathbb{Z}/2 * \mathbb{Z}/2$ and $\mathbb{Z} * \mathbb{Z}/2$, the Artin group of type B_2 and wreath products. Our main tool here is the decomposition of the LCS of a semi-direct product into a semi-direct product of filtrations, which we recall first.

4.5.1 Relative lower central series

In order to obtain actual computations, we need to recall some material from [Dar21, §3] about the LCS of a semi-direct product.

Definition 4.5.1. Let G be a group, of which H is a normal subgroup. We define the *relative lower central series* $\Gamma_*^G(H)$ by:

$$\begin{cases} \Gamma_1^G(H) := H, \\ \Gamma_{k+1}^G(H) := [G, \Gamma_k^G(H)]. \end{cases}$$

If G is the semi-direct product of H with a group K, we write $\Gamma_*^K(H)$ for $\Gamma_*^{H \rtimes K}(H)$ (which does not cause any confusion: if H is a normal subgroup of a group G, then $\Gamma_*^G(H) = \Gamma_*^{H \rtimes G}(H)$, for the semi-direct product associated to the conjugation action of G on H). It was shown in [Dar21] that in this case:

$$\Gamma_*(H \rtimes K) = \Gamma_*^K(H) \rtimes \Gamma_*(K).$$

Moreover, the filtration $\Gamma_*^K(H) = H \cap \Gamma_*(H \rtimes K)$ does have the property that $[\Gamma_i^K(H), \Gamma_j^K(H)] \subseteq \Gamma_{i+j}^K(H)$ (for all $i, j \geqslant 1$), which allows one to define an associated graded Lie ring $\mathcal{L}^K(H)$ (with brackets induced by commutators, as in §4.1.2). Then, the Lie ring of $H \rtimes K$ decomposes into a semi-direct product of Lie rings:

$$\mathcal{L}(H \rtimes K) = \mathcal{L}^K(H) \rtimes \mathcal{L}(K).$$

This is in fact a generalisation of Lemma 4.1.12, which is the degree-one part (one can check that $\mathcal{L}_1^K(H) = (H^{\mathrm{ab}})_K$).

We can devise an analogue of Lemma 4.2.6 in this context, which gives a criterion for the relative LCS to stop:

Lemma 4.5.2. *Let a group K act on a group H. Let the set S_K generate K^{ab} and let the set S_H generate $(H^{\mathrm{ab}})_K$. Suppose that, for each pair $(s,t) \in S_H^2$ (resp. each pair $(s,t) \in S_H \times S_K$), we can find representatives $\tilde{s}, \tilde{t} \in H$ (resp. $\tilde{s} \in H$ and $\tilde{t} \in K$) of s and t such that $\tilde{s}$ and $\tilde{t}$ commute in H (resp. in $H \rtimes K$). Then $\Gamma_2^K(H) = \Gamma_3^K(H)$, which means that $\Gamma_*^K(H)$ stops at $\Gamma_2^K(H)$, and:*

$$\mathcal{L}(H \rtimes K) \cong (H^{\mathrm{ab}})_K \times \mathcal{L}(K),$$

where the first factor is concentrated in degree one.

Proof. On the one hand, by definition of the relative LCS, an element of $\mathcal{L}_2^K(H)$ is a sum of brackets in $\mathcal{L}(G \rtimes K)$, either of two elements of $\mathcal{L}_1^K(H)$, or of an element of $\mathcal{L}_1(K)$ with an element of $\mathcal{L}_1^K(H)$. On the other hand, the relation $[\tilde{s}, \tilde{t}] = 1$ in $H \rtimes K$ readily implies that

$[s,t] = 0$ in $\mathcal{L}(H \rtimes K) \cong \mathcal{L}^K(H) \rtimes \mathcal{L}(K)$. Since S_H linearly generates $\mathcal{L}_1^K(H) = (H^{\mathrm{ab}})_K$ and S_K linearly generates $\mathcal{L}_1(K) = K^{\mathrm{ab}}$, we infer that under our hypothesis, all elements of $\mathcal{L}_2^K(H)$ are trivial, which means that $\Gamma_2^K(H) = \Gamma_3^K(H)$. Moreover, from the definition of $\Gamma_*^K(H)$, this obviously implies that $\Gamma_i^K(H) = \Gamma_{i+1}^K(H)$ for all $i \geqslant 2$. The statement about Lie rings is then just a reformulation of the decomposition $\mathcal{L}(H \rtimes K) \cong \mathcal{L}^K(H) \rtimes \mathcal{L}(K)$ taking into account these conclusions. ◻

4.5.2 Semi-direct products of abelian groups

Let a group G act on an abelian group A. Then the LCS of $A \rtimes G$ can be computed using linear algebra. Indeed, we have:

$$\Gamma_{k+1}^G(A) := \left[A \rtimes G,\ \Gamma_k^G(A)\right] = \left[G,\ \Gamma_k^G(A)\right].$$

These are commutators in $A \rtimes G$, given, for $g \in G$ and $a \in A$, by:

$$[g, a] = g \cdot a - a = (g - id)(a).$$

As a consequence, $\Gamma_{k+1}^G(A)$ is the subgroup of A generated by the $(g - id)(\Gamma_k^G(A))$ (for $g \in G$), which can be computed by studying the endomorphisms $g - id$ of A.

We now study several instances of this situation. We begin by computing the LCS of the Klein group $\mathbb{Z} \rtimes \mathbb{Z}$ (which is the fundamental group of the Klein bottle). We then generalise this calculation to any semi-direct product of an abelian group by $\mathbb{Z}$ acting by $-id$. This can be generalised again, to the case of an action of $\mathbb{Z}$ by an involution. Finally, we compute the LCS of $\Lambda \rtimes W_2$, where $W_2 \cong (\mathbb{Z}/2) \wr \mathfrak{S}_2$ is the Coxeter group of type B_2 (or C_2), acting on the lattice Λ generated by its root system.

The Klein group

There are two distinct automorphisms of $\mathbb{Z}$ (that is, $\pm id$), whence only one non-trivial action of $\mathbb{Z}$ on $\mathbb{Z}$. Thus the following definition makes sense:

Definition 4.5.3. The *Klein group* is the semi-direct product $K = \mathbb{Z} \rtimes \mathbb{Z}$.

Let us denote by x (resp. by t) the element $(1, 0)$ (resp. $(0, 1)$) of $\mathbb{Z} \rtimes \mathbb{Z}$. A presentation of K is given by

$$K = \langle x, t \mid txt^{-1} = x^{-1} \rangle.$$

The LCS of K decomposes as $\Gamma_*(K) = \Gamma_*^{\mathbb{Z}}(\mathbb{Z}) \rtimes \Gamma_*(\mathbb{Z})$. Thus, in order to understand it, we need to understand the filtration $\Gamma_*^{\mathbb{Z}}(\mathbb{Z})$.

Proposition 4.5.4. *The LCS of the Klein group is* $\Gamma_i(\mathbb{Z} \rtimes \mathbb{Z}) = (2^{i-1}\mathbb{Z}) \rtimes \{1\}$ *for* $i \geqslant 2$. *In other words,* $\Gamma_i^{\mathbb{Z}}(\mathbb{Z}) = 2^{i-1}\mathbb{Z}$. *In particular,* $\mathbb{Z} \rtimes \mathbb{Z}$ *is residually nilpotent.*

Proof. This follows from the formula $[x^{2^j}, t] = x^{2^j}(tx^{-2^j}t^{-1}) = x^{2^{j+1}}$, by induction on i. ◻

Corollary 4.5.5. *The (graded) Lie ring of the Klein group identifies with* $(\mathbb{Z}/2)[X] \rtimes \mathbb{Z}$*, where the polynomial ring* $(\mathbb{Z}/2)[X]$ *is seen as a graded abelian Lie ring (where* X^i *is of degree* i*), and the generator* T *of* $\mathbb{Z}$ *(of degree 1) acts via* $[X^i, T] = X^{i+1}$.

Proof. From Proposition 4.5.4, we get a decomposition $\mathcal{L}(K) = \mathcal{L}(2^{*-1}\mathbb{Z}) \rtimes \mathcal{L}(\mathbb{Z})$. Since $\mathbb{Z}$ is abelian, the two factors are abelian Lie rings. The result follows, by calling X^i the class of $x^{2^{i-1}}$ and T the class of t. The formula for brackets comes from $[x^{2^j}, t] = x^{2^{j+1}}$. ◻

Generalised Klein groups

Let A be any abelian group and let $\mathbb{Z}$ act on A via the powers of $-id_A$. The corresponding semi-direct product $K_A = A \rtimes \mathbb{Z}$ is a generalisation of the Klein group $K = K_{\mathbb{Z}}$. We can generalise the above results to this context:

Proposition 4.5.6. *The LCS of K_A is given by $\Gamma_i(A \rtimes \mathbb{Z}) = (2^{i-1}A) \rtimes \{1\}$ for $i \geqslant 2$. In other words, for all $i \geqslant 2$, $\Gamma_i^{\mathbb{Z}}(A) = 2^{i-1}A$. In particular, for any free abelian group A, K_A is residually nilpotent.*

Proof. Let t denote the generator of $\mathbb{Z}$. For all a in A, we have $[a,t] = a - t \cdot a = 2a$ in $A \rtimes \mathbb{Z}$. Hence $[2^j A, t] = 2^{j+1}A$, from which the calculation of the LCS follows. Then K_A is residually nilpotent if and only if the intersection of the $2^j A$ is trivial, which is true for instance when A is finitely generated or when A is free abelian. □

Corollary 4.5.7. *Let us consider the graded abelian Lie ring $\mathcal{L}(2^{*-1}A) = \bigoplus 2^{i-1}A/2^iA$ (where the sum is taken over $i \geqslant 1$). The (graded) Lie ring of K_A identifies with $\mathcal{L}(2^*A) \rtimes \mathbb{Z}$, where the generator T of $\mathbb{Z}$ acts via the degree-one map induced by $a \mapsto 2a$.*

Proof. From Proposition 4.5.6, we get a decomposition $\mathcal{L}(K) = \mathcal{L}(2^{*-1}A) \rtimes \mathcal{L}(\mathbb{Z})$. Since $\mathbb{Z}$ and A are abelian, the two factors are abelian Lie rings. The result follows, since brackets with T come from commutators with t, given by $[a,t] = 2a$ in $A \rtimes \mathbb{Z}$. □

Since t^2 acts trivially on A, it is a central element of K_A (in fact, one easily sees that it generates the centre of K_A if A is not trivial). Thus we can consider $A \rtimes (\mathbb{Z}/2)$ (where $\mathbb{Z}/2$ acts on A via $-id_A$) as a quotient of K_A, which behaves in much the same way:

Corollary 4.5.8. *Consider the group $A \rtimes (\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts on the abelian group A via $-id_A$. We have $\Gamma_i(A \rtimes (\mathbb{Z}/2)) = (2^{i-1}A) \rtimes \{1\}$ for all $i \geqslant 2$. In particular, for any finitely generated abelian group A, $A \rtimes (\mathbb{Z}/2)$ is residually nilpotent. Moreover, the (graded) Lie ring of this group identifies with $\mathcal{L}(2^{*-1}A) \rtimes (\mathbb{Z}/2)$, where the generator T of $\mathbb{Z}/2$ acts via the degree-one map induced by $a \mapsto 2a$.*

Finally, let us spell out the particular case where $A \cong \mathbb{Z}^n$ is free abelian on some basis $x_1, \ldots, x_n$. We then denote K_A by K_n, for short.

Corollary 4.5.9. *The (graded) Lie ring of K_n identifies with $(\mathbb{Z}/2)^n[X] \rtimes \mathbb{Z}$, where the polynomial ring $(\mathbb{Z}/2)^n[X]$ is seen as an abelian Lie ring (where X^i is of degree i), and the generator T of $\mathbb{Z}$ (of degree 1) acts via $[v \cdot X^i, T] = v \cdot X^{i+1}$, for $v \in (\mathbb{Z}/2)^n$.*

Proof. This is just a matter of identifying $\mathcal{L}(2^{*-1}\mathbb{Z}^n)$ with $(\mathbb{Z}/2)^n[X]$, by calling $v \cdot X^i$ the class of the sum of the $2^{i-1}x_k$ such that $v_k = 1$. □

A further generalisation

Let A be an abelian group and let $\mathbb{Z}$ act on A via the powers of some involution τ (for instance, τ could exchange two isomorphic direct factors of A). Let us denote by K_τ the corresponding semi-direct product $A \rtimes \mathbb{Z}$. We have:

$$((\tau - 1) + 2)(\tau - 1) = \tau^2 - 1 = 0,$$

which means that $\tau - 1$ acts via multiplication by -2 on $V := \operatorname{Im}(\tau - 1)$.

Proposition 4.5.10. *The LCS of K_τ is given by $\Gamma_i(A \rtimes \mathbb{Z}) = (2^{i-2}V) \rtimes \{1\}$ for $i \geqslant 2$. In other words, for all $i \geqslant 2$, $\Gamma_i^{\mathbb{Z}}(A) = 2^{i-2}V$. In particular, for any free abelian group A, K_τ is residually nilpotent.*

Proof. For all $a \in A$, we have $[a,t] = a - \tau(a) = (1-\tau)(a)$. This implies that $\Gamma_2^{\mathbb{Z}}(A) = \mathrm{Im}(\tau - 1) = V$. Then for all v in V, we have $[v,t] = v - \tau(v) = 2v$ in $A \rtimes \mathbb{Z}$, and the rest of the proof is similar to that of Proposition 4.5.6. ◻

Corollary 4.5.7 generalises immediately to this context:

Corollary 4.5.11. *The (graded) Lie ring of K_τ identifies with $(A/V \oplus V/2V \oplus 2V/4V \oplus \cdots) \rtimes \mathbb{Z}$, where A/V and $\mathbb{Z}$ are in degree 1, $2^{i-2}V/2^{i-1}V$ is in degree i and the generator T of $\mathbb{Z}$ acts via the degree-one map induced by $1-\tau$ (which coincides with $v \mapsto 2v$ on V).*

The reader can also easily write a generalisation of Corollary 4.5.8 to this context, by factoring the action of $\mathbb{Z}$ through $\mathbb{Z}/2$.

More actions on abelian groups

Let us consider the group defined by the presentation $\langle \sigma, \gamma \mid \sigma^2 = \gamma^2 = (\sigma\gamma)^4 = 1 \rangle$, which is the Coxeter group of type B_2, also denoted by $W_2 = (\mathbb{Z}/2) \wr \mathfrak{S}_2$ in the rest of the chapter. It acts on $\mathbb{R}^2$ in the usual way: γ acts by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and σ by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This action preserves the lattice $\Lambda := \mathbb{Z} \cdot (0,1) \oplus \mathbb{Z} \cdot (\frac{1}{2}, \frac{1}{2})$ (which is generated by roots). It also preserves the lattice $V = \mathbb{Z}^2$, which is of index 2 in Λ.

Proposition 4.5.12. *The filtration $\Gamma_*^{W_2}(\Lambda)$ on Λ is given by:*

$$\Lambda \supset V \supset 2\Lambda \supset 2V \supset 4\Lambda \supset \cdots .$$

In particular, $\Lambda \rtimes W_2$ is residually nilpotent, but not nilpotent.

Proof. One can easily write down explicitly the eight matrices for the actions of elements of W_2 (which are the invertible monomial matrices in $\mathrm{GL}_2(\mathbb{Z})$). Recall that, for every $g \in W_2$, $g - id$ is the commutator by g in $\Lambda \rtimes W_2$. It is then easy to check that the $g - id$ send Λ to V (resp. V to 2Λ) and that the $(g-id)(\Lambda)$ (resp. the $(g-id)(V)$) generate V (resp. 2Λ), whence the result. ◻

Remark 4.5.13. One can look at the proof in a geometric way, by seeing each element $g \cdot v - v$ as the difference between two vertices of a square centred at 0.

Remark 4.5.14. One can compute completely the associated Lie ring, which is a semi-direct product of the abelian Lie ring $(\mathbb{Z}/2)[X]$ by the mod 2 Heisenberg Lie ring $\mathcal{L}(W_2) \cong \mathfrak{n}_3(\mathbb{Z}/2)$.

Remark 4.5.15. Let us use the notations $a := (0,1)$, $b := (1,0)$ and $c := (\frac{1}{2}, \frac{1}{2})$. Then Λ can be described abstractly as the abelian group generated by a, b and c, modulo the relation $a + b = 2c$. Moreover, the action of σ fixes c and exchanges a and b, and the action of γ is via $a \mapsto a$, $b \mapsto -b$ and $c \mapsto c - b$. In particular, since $b = 2c - a$ and $c - b = a - c$, in the basis (a,c) of Λ, σ acts by $\left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and γ by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. Notice that, in this basis, V is just the subgroup of elements whose second coordinate is even.

In the course of the proof of Proposition 4.4.92, we encounter a slight variation on the above action of W_2 on $\Lambda \cong \mathbb{Z}^2$. Namely, we can construct an action of the group $K = Dic_{16} \times (\mathbb{Z}/2)$ (where Dic_{16} is the dicyclic group of order 16, cf. Corollary 4.4.83, which is an index-2 central extension of W_2) on Λ by making $Dic_{16} = \langle \sigma, \gamma \rangle$ act through its quotient W_2 (which is the quotient by σ^2) and making $\mathbb{Z}/2 = \langle \tau \rangle$ act by $-id$. Notice that there is already an element in W_2 acting by $-id$, namely the central element $(\sigma\gamma)^2$ of W_2, so this action is in fact through the quotient $K \twoheadrightarrow W_2$ sending respectively σ, γ and τ to σ, γ and $(\sigma\gamma)^2$. In particular, this implies that we have:

$$\Gamma_*^K(\Lambda) = \Gamma_*^{W_2}(\Lambda).$$

4.5.3 Free products

Two examples

Consider the simplest free product of two groups, which is the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$. We can determine its LCS from its description as a semi-direct product:

Proposition 4.5.16. *There is an isomorphism:*

$$\mathbb{Z}/2 * \mathbb{Z}/2 \;\cong\; \mathbb{Z} \rtimes (\mathbb{Z}/2).$$

As a consequence, this group is residually nilpotent, and its Lie ring is $(\mathbb{Z}/2)[X] \rtimes (\mathbb{Z}/2)$, where both factors are abelian Lie rings, and the generator T of $\mathbb{Z}/2$ acts by $[X^i, T] = X^{i+1}$.

Proof. We know a presentation of each of these groups, namely $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$ and $\mathbb{Z} \rtimes (\mathbb{Z}/2) = \langle a, t \mid tat^{-1} = a^{-1},\ t^2 = 1 \rangle$. It is then easy to check that the assignments $x \mapsto t$, $y \mapsto ta$ and $t \mapsto x$, $a \mapsto xy$ define morphisms inverse to each other. The rest is an application of Corollary 4.5.8 with $A = \mathbb{Z}$. ∎

Consider now the free product $\mathbb{Z} * (\mathbb{Z}/2) = \langle x, y \mid y^2 = 1 \rangle$. We have a similar decomposition into a semi-direct product:

Proposition 4.5.17. *There is an isomorphism:*

$$\mathbb{Z} * (\mathbb{Z}/2) \;\cong\; \mathbf{F}_2 \rtimes (\mathbb{Z}/2),$$

where the action of the generator t of $\mathbb{Z}/2$ is given by exchanging the two elements a and b of a basis of the free group $\mathbf{F}_2$.

Proof. Again, we know a presentation of each these groups, namely $\mathbb{Z} * (\mathbb{Z}/2) = \langle x, y \mid y^2 = 1 \rangle$ and $\mathbf{F}_2 \rtimes (\mathbb{Z}/2) = \langle a, b, t \mid tat^{-1} = b,\ tbt^{-1} = a,\ t^2 = 1 \rangle$. It is then easy to check that the assignments $x \mapsto tb$, $y \mapsto t$ and $t \mapsto y$, $a \mapsto xy$, $b \mapsto yx$ define morphisms inverse to each other. One may alternatively observe that the Tietze transformation removing the generator a turns the second presentation into the first. ∎

Remark 4.5.18. The group $\mathbb{Z} * (\mathbb{Z}/2)$ is isomorphic to $\mathbf{wB}_2$ (or $\mathbf{vB}_2$) and the above isomorphism can be identified with $\mathbf{wB}_2 \cong \mathbf{wP}_2 \rtimes \mathfrak{S}_2$, together with $\mathbf{wP}_2 \cong \mathbf{F}_2$ (with basis (χ_{12}, χ_{21})).

The LCS of $\mathbb{Z} * (\mathbb{Z}/2)$ is much more difficult to compute than the one of $\mathbb{Z}/2 * \mathbb{Z}/2$ above. The reader can find a presentation of the associated Lie ring in [Lab77], where the methods used are somewhat different from ours. Our methods could be adapted to recover this result, together with the residual nilpotence of the group, but we will not do so here. We only give a proof of the following:

Proposition 4.5.19. *The group $\mathbb{Z} * (\mathbb{Z}/2)$ is residually nilpotent, but not nilpotent. Its Lie ring has only 2-torsion elements, except in degree one.*

Proof. Note that the group $\mathbb{Z} * (\mathbb{Z}/2)$ surjects onto $\mathbb{Z}/2 * \mathbb{Z}/2$, whose LCS does not stop, by Proposition 4.5.16. Notice also that the statement about torsion has already been proven in Example 4.2.9. As a consequence, we only need to prove that $\mathbb{Z} * (\mathbb{Z}/2)$ is residually nilpotent, which is the difficult part of the statement. We can prove it using a kind of Magnus expansion. Namely, let us consider the associative algebra of non-commutative formal power series $A := \mathbf{F}_2\langle\langle X, Y \rangle\rangle/(Y^2 = 1)$. We get a morphism Φ from $\mathbb{Z} * (\mathbb{Z}/2)$ to the group $A^\times$ of units of A by sending x to $1 + X$ and y to $1 + Y$. It is injective, by the usual argument: if $x^{a_1} y^{\epsilon_1} \cdots x^{a_l} y^{\epsilon_l} x^{a_{l+1}}$ is a non-trivial reduced expression of some non-trivial element $g \in \mathbb{Z} * (\mathbb{Z}/2)$ (with $\epsilon_i = \pm 1$, $a_i \in \mathbb{Z}$, and a_1 and a_{l+1} possibly trivial), then, by writing $a_i = 2^{b_i}(2c_i + 1)$, using $(1 + X)^{2^k} = 1 + X^{2^k}$ and $(1 + T)^\alpha = 1 + \alpha T + \cdots$, we see that the coefficient of the monomial $X^{2^{b_1}} Y X^{2^{b_2}} Y \cdots Y X^{2^{b_{l+1}}}$ in $\Phi(g)$ is not trivial, hence $\Phi(g) \neq 1$.

Now let us denote by (X,Y) the ideal generated by X and Y in A, and by $A_k^\times$ the subgroup $1+(X,Y)^k$ of $A^\times$ (for $k \geqslant 1$). It is easy to see that $[A_1^\times, A_k^\times] \subset A_{k+1}^\times$ for all $k \geqslant 1$. As a consequence, for all $k \geqslant 1$, we have $\Gamma_k(A_1^\times) \subseteq A_k^\times$. Since the intersection of the $1+(X,Y)^k$ is obviously trivial, $A_1^\times$ is residually nilpotent, whence also $\mathbb{Z} * (\mathbb{Z}/2)$, which is isomorphic to one of its subgroups. ◻

Remark 4.5.20. As mentioned at the beginning of the proof above, if one wishes only to see that the LCS of $\mathbb{Z} * (\mathbb{Z}/2)$ does not stop, one need only apply Lemma 4.1.1 to the obvious projection $\mathbb{Z} * (\mathbb{Z}/2) \twoheadrightarrow (\mathbb{Z}/2) * (\mathbb{Z}/2)$ and one can then deduce the required result from the much simpler explicit computation of the LCS of the infinite dihedral group $(\mathbb{Z}/2) * (\mathbb{Z}/2) \cong \mathbb{Z} \rtimes (\mathbb{Z}/2)$ done in Proposition 4.5.16.

An Artin group

Consider the Artin group of type B_2 (which is also $\mathbf{B}_{1,2}$ – see Lemma 4.3.16), that is:

$$G := \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle.$$

Let $\delta := \sigma x$. Then $(\sigma x)^2 = (x\sigma)^2$ is equivalent to $\delta^2 = \sigma^{-1}\delta^2\sigma$, so that:

$$G = \langle \sigma, \delta \mid \delta^2\sigma = \sigma\delta^2 \rangle.$$

The element δ^2 commutes with the generators δ and σ, hence it is central in G. Since G^{ab} is free on the classes of δ and σ (as is obvious from the presentation), δ^2 is of infinite order. Moreover, the above relation clearly becomes trivial when δ^2 is killed, so that:

$$G/\delta^2 = \langle \sigma, \delta \mid \delta^2 = 1 \rangle \cong \mathbb{Z} * (\mathbb{Z}/2).$$

We thus have a central extension:

$$\mathbb{Z} \hookrightarrow G \twoheadrightarrow \mathbb{Z} * (\mathbb{Z}/2).$$

Proposition 4.5.21. *The group $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$ is residually nilpotent (but not nilpotent).*

Proof. We have observed above that the central subgroup $\langle \delta^2 \rangle$ injects into G^{ab}, i.e., we have $\langle \delta^2 \rangle \cap \Gamma_2(G) = \{1\}$. Thus, we can deduce the (strict) residual nilpotence of G from the fact that $\mathbb{Z}*(\mathbb{Z}/2)$ is (strictly) residually nilpotent (see Proposition 4.5.19) by applying Proposition 4.1.5. ◻

Remark 4.5.22. The weaker fact that the LCS of G does not stop can also be deduced more directly from the reasoning of Remark 4.5.20.

The decomposition of G into a central extension can also be used to describe its Lie ring. Namely, it can be obtained from the Lie ring of $\mathbb{Z} * (\mathbb{Z}/2)$ described in [Lab77]; see §4.5.3.

Proposition 4.5.23. *The Lie ring of $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$ is a central extension of $\mathcal{L}(\mathbb{Z} * (\mathbb{Z}/2))$ by $\mathbb{Z}$, concentrated in degree one. Precisely, $2\overline{\sigma x}$ is central in $\mathcal{L}(G)$, and $\mathcal{L}(G)/(2\overline{\sigma x}) \cong \mathcal{L}(\mathbb{Z} * (\mathbb{Z}/2))$.*

Proof. We have seen that $\langle \delta^2 \rangle$ injects into G^{ab}, which means that $\langle \delta^2 \rangle \cap \Gamma_2(G) = \{1\}$. As a consequence, the projections $\pi\colon \Gamma_k(G) \twoheadrightarrow \Gamma_k(G/\delta^2)$ have trivial kernels for $k \geqslant 2$, which means that they are isomorphisms. Thus, the canonical morphism from $\mathcal{L}(G)$ to $\mathcal{L}(G/\delta^2)$ is an isomorphism in degree at least 2. In degree one, it identifies with the projection of $G^{\mathrm{ab}} \cong \mathbb{Z}^2$ onto $(G/\delta^2)^{\mathrm{ab}} \cong \mathbb{Z} \times (\mathbb{Z}/2)$, whose kernel is generated by $\overline{\delta^2} = 2\bar{\delta}$. Moreover, since δ^2 is central in G, its class in $\mathcal{L}(G)$ must be central, whence our result. ◻

4.5.4 Wreath products

This section is devoted to the study of the LCS of $G \wr \mathfrak{S}_\lambda = G^n \rtimes \mathfrak{S}_\lambda$, where G is any group, $\lambda = (n_1, \dots, n_l)$ is a partition of the integer n and $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_n$ acts on G^n by permuting the factors. By Proposition 4.4.12, wreath products of this form are precisely the (partitioned) braid groups on manifolds of dimensions at least 3. We first compute the abelianisation (Corollary 4.5.25), then we show that the LCS stops under some stability condition (Corollary 4.5.27). Finally, we look at the unstable cases, whose LCS we compute if G is abelian (§4.5.4).

Let us remark that, in order to understand the LCS of $G \wr \mathfrak{S}_\lambda$, we only need to study the LCS of $G \wr \mathfrak{S}_n$. Indeed:

$$G \wr \mathfrak{S}_\lambda \cong \prod_{i=1}^{l} G \wr \mathfrak{S}_{n_i}.$$

Abelianisations

Lemma 4.1.12 allows us to compute abelianisations of wreath products:

Lemma 4.5.24. *For any integer $n \geqslant 2$, we have $(G \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong G^{\mathrm{ab}} \times (\mathbb{Z}/2)$.*

Proof. It is a direct consequence of Lemma 4.1.12, applied to $G \wr \mathfrak{S}_\lambda = G^n \rtimes \mathfrak{S}_\lambda$: $(G \wr \mathfrak{S}_n)^{\mathrm{ab}} = ((G^n)^{\mathrm{ab}})_{\mathfrak{S}_n} \times \mathfrak{S}_n^{\mathrm{ab}} = ((G^{\mathrm{ab}})^n)_{\mathfrak{S}_n} \times (\mathbb{Z}/2) = G^{\mathrm{ab}} \times (\mathbb{Z}/2)$. ◻

Corollary 4.5.25. *For any partition $\lambda = (n_1, \dots, n_l)$ of n, if l' denotes the number of indices $i \leqslant l$ such that $n_i \geqslant 2$, we have $(G \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (G^{\mathrm{ab}})^l \times (\mathbb{Z}/2)^{l'}$.*

Proof. $G \wr \mathfrak{S}_\lambda$ decomposes as the direct product of the $G \wr \mathfrak{S}_{n_i}$, and $(G \wr \mathfrak{S}_{n_i})^{\mathrm{ab}}$ identifies with G^{ab} when $n_i = 1$, and with $G^{\mathrm{ab}} \times (\mathbb{Z}/2)$ if $n_i \geqslant 2$. ◻

The stable case

We now use a disjoint support argument to show that there is a stable behaviour for the LCS of $G \wr \mathfrak{S}_\lambda$, occurring as soon as $n_i \geqslant 3$ for every $i \leqslant l$.

Recall that the usual generators τ_i of $\mathfrak{S}_n$ are conjugate to each other, hence $\mathfrak{S}_n^{\mathrm{ab}} \cong \mathbb{Z}/2$ is generated by their common class τ, and $\Gamma_2 = \Gamma_\infty$ for $\mathfrak{S}_n$.

Proposition 4.5.26. *Let G be a group. If $n \geqslant 3$, the $\tau_i \tau_j^{-1}$ normally generate $\Gamma_2(G \wr \mathfrak{S}_n)$, and $(G \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong G^{\mathrm{ab}} \times \mathbb{Z}/2$. Moreover, the LCS of $G \wr \mathfrak{S}_n$ stops at Γ_2.*

Proof. Let N be the subgroup of $G \wr \mathfrak{S}_n$ normally generated by the $\tau_i \tau_j^{-1}$. These are in $\Gamma_2(\mathfrak{S}_n)$, whence also in $\Gamma_2(G \wr \mathfrak{S}_n)$, hence the latter contains N. In order to show the converse inclusion, we need to show that $(G \wr \mathfrak{S}_n)/N$ is abelian. For any $g \in G$, let us denote by $\overline{g}$ the class of $(g, 1, \dots, 1) \in G^n$ modulo N. We now show that the $\overline{g}$, together with τ, generate $(G \wr \mathfrak{S}_n)/N$, and we use a disjoint support argument to show that they commute with one another.

First, let us remark that $\overline{g}$ commutes with τ. This comes from the fact that τ_2 acts trivially on $(g, 1, \dots, 1)$, thus commutes with it in $G \wr \mathfrak{S}_n$. From this, we deduce that $\overline{g}$ is also the class of $\sigma(g, 1, \dots, 1)\sigma^{-1} = (1, \dots, 1, g, 1, \dots, 1)$ for any $\sigma \in \mathfrak{S}_n$ (whose class modulo N is a power of τ). In particular, for all $g, h \in G$, $(g, 1, \dots, 1)$ commutes with $(1, h, \dots, 1)$, hence $\overline{g}$ commutes with $\overline{h}$.

Now, every $(g_1, \dots, g_n) \in G^n$ is the product of the $(1, \dots, 1, g_i, 1, \dots, 1)$, so that all the elements $(1, \dots, 1, g, 1, \dots, 1)$ (for all $g \in G$ and any choice of position), together with the τ_i, generate $G \wr \mathfrak{S}_n$. This implies that their classes $\overline{g}$, together with τ, generate $(G \wr \mathfrak{S}_n)/N$. Since these generators commute with one another, this ends the proof that $N = \Gamma_2(G \wr \mathfrak{S}_n)$.

The rest of the statement is a direct application of Lemmas 4.5.24 and 4.5.2. ◻

Corollary 4.5.27. *Let G be a group, $n \geqslant 3$ be an integer and $\lambda = (n_1, \dots, n_l)$ be a partition of n with $n_i \geqslant 3$ for all i. Then the $\tau_\alpha \tau_\beta^{-1}$ for α and β in the same block of λ normally generate $\Gamma_2(G \wr \mathfrak{S}_\lambda)$, and $(G \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (G^{\mathrm{ab}} \times \mathbb{Z}/2)^l$. Moreover, the LCS of $G \wr \mathfrak{S}_\lambda$ stops at Γ_2.*

Proof. Apply Proposition 4.5.26 to each factor of $G \wr \mathfrak{S}_\lambda \cong \prod_{i=1}^{l} G \wr \mathfrak{S}_{n_i}$. □

Unstable cases

Since $G \wr \mathfrak{S}_1 = G$, the only case left in our study of $G \wr \mathfrak{S}_\lambda$ is the case of $G \wr \mathfrak{S}_2$, which can be quite complicated. We treat the case where G is an abelian group, which we denote by A.

Proposition 4.5.28. *Let us denote by δA the subgroup $\{(a, -a) \mid a \in A\}$ of A^2. For all $i \geqslant 2$, we have $\Gamma_i(A \wr \mathfrak{S}_2) = 2^{i-2}(\delta A)$. Moreover, the Lie algebra decomposes as:*

$$\mathcal{L}(A \wr \mathfrak{S}_2) \cong (A \oplus A/2A \oplus 2A/4A \oplus \cdots) \rtimes (\mathbb{Z}/2),$$

where A and $\mathbb{Z}/2$ are in degree 1 and each factor of the form $2^{i-2}A/2^{i-1}A$ is in degree i. The Lie ring $A \oplus A/2A \oplus \cdots$ is abelian and the generator T of $\mathbb{Z}/2$ acts on it via the degree-one map:

$$\begin{cases} a \mapsto \overline{a} & \text{in degree } 1, \\ \overline{a} \mapsto \overline{2a} & \text{in degree at least } 2. \end{cases}$$

Proof. This is a straightforward application of Proposition 4.5.10 and Corollary 4.5.11 (adapted to an action of $\mathfrak{S}_2 \cong \mathbb{Z}/2$ instead of $\mathbb{Z}$). Namely, $V = \delta A$ is the subspace of A^2 on which $\mathfrak{S}_2$ acts by $-id$. Moreover, $A^2/V \cong A$ (via $\overline{(a, 0)} \leftarrow\!\shortmid a$) and the map induced by $1 - \tau$ identifies with the one described in our statement. □

Corollary 4.5.29. *Let G be a group and λ be a partition with at least one block of size 2. Suppose that the filtration $2^* G^{\mathrm{ab}}$ of G^{ab} does not stop. Then the LCS of $G \wr \mathfrak{S}_\lambda$ does not stop.*

Proof. The group $G^{\mathrm{ab}} \wr \mathfrak{S}_\lambda$ is a quotient of $G \wr \mathfrak{S}_\lambda$, whose LCS does not stop by Proposition 4.5.28. Thus, our statement follows from Lemma 4.1.1. □

Remark 4.5.30. If G^{ab} is finitely generated, the condition in Corollary 4.5.29 holds if and only if it is infinite. In general, the condition is equivalent to $2^i A \neq \{0\}$ for all $i \geqslant 1$, where A is the quotient of G^{ab} by its maximal 2-divisible subgroup.

4.6 Appendix: Presentation of an extension

Here we recall the classical construction of a presentation of a group extension from a presentation of the quotient and a presentation of the kernel, together with some knowledge of the structure of the extension (see also [HEO05, §2.4.3]). We then apply this construction to show that 2-nilpotent groups whose abelianisation is free are determined by their Lie ring.

Let G be a group, which is an extension of a quotient K by a normal subgroup H. Suppose that presentations of H and K are known, namely $H = \langle X | R \rangle$ and $K = \langle Y | S \rangle$, where R is a subset of the free group $F[X]$ (resp. $S \subset F[Y]$). For each y in Y, let us fix a lift $\tilde{y}$ of the corresponding generator of K to an element of G. Then a presentation of G is given by

$$G = \langle X \sqcup Y | R \cup \widetilde{S} \cup T \rangle,$$

where $\widetilde{S}$ and T are obtained as follows:

- Each $s \in S$ is a word in the elements of Y and their inverses. If we replace each y in s by its chosen lift $\tilde{y}$, we get an element $\tilde{s}$ of G, which is in fact in H, since its projection to K is trivial by construction. Each element of H is represented by a word on the elements of X and their inverses, so we can choose some $w_s \in F[X]$ representing $\tilde{s}$. Then $\widetilde{S}$ is the following set of relations:
$$\widetilde{S} := \{sw_s^{-1} \mid s \in S\} \subset F[X \sqcup Y].$$
- For each $y \in Y$ and each $x \in X$, the element $\tilde{y}x\tilde{y}^{-1}$ is an element of H, which can be represented by a word $w_{x,y} \in F[X]$. Then we define:
$$T := \{yxy^{-1}w_{x,y}^{-1} \mid x \in X,\ y \in Y\} \subset F[X \sqcup Y].$$

Remark 4.6.1. If the presentations of H and K are finite, this construction gives a finite presentation of G.

Remark 4.6.2 (Split extensions)**.** When the extension splits (that is, when G is a semi-direct product of K by H), one usually chooses the lifts of generators of K to be their images under a fixed section. Then the presentation obtained is somewhat simpler, since the relations in S hold in G (that is, $\widetilde{S} = S$).

Proposition 4.6.3. *The above presentation is indeed a presentation of the extension G.*

Proof. Let G_0 be the group defined by the above presentation. By construction, the assignments $x \mapsto x \in H \subset G$ and $y \mapsto \tilde{y}$ induce a well-defined morphism π from G_0 to G. Let H_0 be the subgroup of G_0 generated by X. The morphism π restricts to a morphism $\pi_H \colon H_0 \to H$. Since the relations R are satisfied in H_0, we can construct an inverse to π_H: it is an isomorphism.

The relations T ensure that H_0 is stable under left conjugation by the $y \in G_0$. Moreover, for all $h \in H$, $\pi_H(yhy^{-1}) = \pi(y)\pi(h)\pi(y)^{-1} = \tilde{y}\pi_H(h)\tilde{y}^{-1}$. Since H is normal in G, left conjugation by $\tilde{y}$ is an automorphism of H. Since π_H is an isomorphism, left conjugation by $y \in G_0$ must be an automorphism of H_0, which implies that H_0 is stable under left conjugation by y^{-1}. Finally, H_0 is stable under left conjugation by $y^{\pm 1}$, and also (clearly) by $x^{\pm 1}$, so it is normal in G_0.

As a consequence, π induces a morphism of extensions:

$$\begin{array}{ccccc} H_0 & \hookrightarrow & G_0 & \twoheadrightarrow & G_0/H_0 \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ H & \hookrightarrow & G & \twoheadrightarrow & K. \end{array}$$

The relations R and T become trivial in G_0/H_0, and $\widetilde{S}$ reduces to S there, so this quotient admits the presentation $\langle Y|S\rangle$. This implies that $\overline{\pi}$ is an isomorphism. Since we already know that π_H is an isomorphism, the Five Lemma allows us to conclude that π is an isomorphism. □

Remark 4.6.4. We can replace some of the generators y by their inverses before doing this construction, so we can choose T to encode either left conjugation by y, or right conjugation by y, for each y.

Corollary 4.6.5. *If G is a 2-nilpotent group whose abelianisation is free abelian, then G is determined (up to isomorphism) by its associated Lie ring.*

Proof. We construct a presentation of G which depends only on the structure of $\mathcal{L}(G)$ (and on some choices not involving elements of G). The group G is an extension of $G^{\mathrm{ab}} = \mathcal{L}_1(G)$ by $\Gamma_2(G) = \mathcal{L}_2(G)$, to which we can apply the previous construction. Since G^{ab} is free abelian on some set Y, a presentation of this group is given by generators Y and relations $\{[y,z] \mid y,z \in Y\}$. Let $\langle X|R\rangle$ be a presentation of the group $\mathcal{L}_2(G)$. Then G admits the presentation with generators $X \sqcup Y$ and relations $R \cup \widetilde{S} \cup T$, constructed as above. We need to show that S and T can be recovered from calculations in $\mathcal{L}(G)$ alone.

- Let $s \in S$, that is, $s = [y,z]$ for some $y,z \in Y$. Then, by definition of $\mathcal{L}(G)$, $\tilde{s} = [\tilde{y},\tilde{z}]$ is the element of $\Gamma_2(G) = \mathcal{L}_2(G)$ given by the bracket of $y,z \in \mathcal{L}_1(G)$.

- Since $[G, \Gamma_2(G)] = \{1\}$, the set T consists of the relations $[y, x]$, for $x \in R$ and $y \in S$.

Thus the above presentation of G can be obtained from the data of $\mathcal{L}(G)$, as claimed. □

Remark 4.6.6. Corollary 4.6.5 is not true in general if the abelianisation of G is not free. For example, the dihedral group D_8 of order 8 and the quaternion group Q_8 are not isomorphic, but they are 2-nilpotent groups whose Lie rings are isomorphic. Indeed, in both cases, we have $\mathcal{L}_1 G = (\mathbb{Z}/2)^2$ and $\mathcal{L}_2 G = \mathbb{Z}/2$ and the Lie structure is fully determined by saying that whenever a and b are two distinct non-trivial elements of $\mathcal{L}_1 G$, then $[a, b]$ is non-trivial in $\mathcal{L}_2 G$.

Chapter 5

The Burau representations of loop braid groups

The results of this chapter have been published as [PS22a] in joint work with Arthur Soulié.

Introduction

Loop braid groups appear in many guises in topology and group theory. They may be seen geometrically as fundamental groups of trivial links in $\mathbb{R}^3$, diagrammatically as equivalence classes of *welded braids* (closely related to virtual braids and virtual knot theory), algebraically as subgroups of automorphism groups of free groups or combinatorially via explicit group presentations.

Loop braid groups have been studied, from the topological viewpoint of motions of trivial links in $\mathbb{R}^3$, by Dahm [Dah62], Goldsmith [Gol81], Brownstein and Lee [BL93] and Jensen, McCammond and Meier [JMM06]. In parallel, the *symmetric automorphism groups* and the *braid-permutation groups* (subgroups of $\mathrm{Aut}(F_n)$, which may also be interpreted in terms of welded braids) were studied by McCool [McC86], Collins [Col89] and Fenn, Rimányi and Rourke [FRR97]. In particular, Fenn, Rimányi and Rourke found a finite presentation of the braid-permutation groups. Later, Baez, Wise and Crans [BWC07, Theorem 2.2] showed that their presentation is also a presentation of the group of motions of a trivial link, thus bringing together the two different points of view. Loop braid groups, as well as related groups of "wickets", have also been studied more recently by Brendle and Hatcher [BH13]. For a detailed survey of the many different facets of loop braid groups, see Damiani's survey [Dam17].

The definition that we shall use is the following.

Definition 5.0.1. Let $\mathbb{D}^3$ denote the closed unit ball in $\mathbb{R}^3$ and choose a trivial n-component link U_n in its interior. Let $\mathrm{Emb}(U_n, \mathbb{D}^3)$ denote the set of all smooth embeddings of U_n into the interior of $\mathbb{D}^3$, equipped with the smooth Whitney topology, and write $\mathrm{Emb}^u(U_n, \mathbb{D}^3)$ for the path-component containing the inclusion (the superscript ^{u} stands for "unknotted and unlinked"). There is a natural action of the diffeomorphism group $\mathrm{Diff}(U_n) \cong \mathrm{Diff}(\mathbb{S}^1) \wr \mathfrak{S}_n$ on this space, and we define

$$\mathcal{E}(U_n, \mathbb{D}^3) \coloneqq \mathrm{Emb}^u(U_n, \mathbb{D}^3)/\mathrm{Diff}(U_n).$$

The n-th *extended loop braid group* is the fundamental group $\mathbf{LB}'_n \coloneqq \pi_1(\mathcal{E}(U_n, \mathbb{D}^3))$. Similarly, we define

$$\mathcal{E}^+(U_n, \mathbb{D}^3) \coloneqq \mathrm{Emb}^u(U_n, \mathbb{D}^3)/\mathrm{Diff}^+(U_n),$$

where Diff^+ denotes orientation-preserving diffeomorphisms, and the n-th (non-extended) *loop braid group* is the fundamental group $\mathbf{LB}_n \coloneqq \pi_1(\mathcal{E}^+(U_n, \mathbb{D}^3))$.

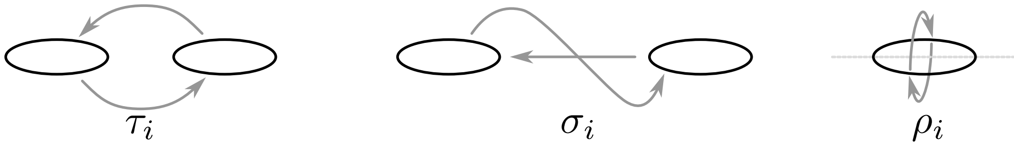


Figure 5.1 The loop braid group $\mathbf{LB}_n$ is generated by the loops of loops $\tau_1, \ldots, \tau_{n-1}$ and $\sigma_1, \ldots, \sigma_{n-1}$. Together with $\rho_1, \ldots, \rho_n$, these generate the extended loop braid group $\mathbf{LB}'_n$.

Thus elements of $\mathbf{LB}'_n$ are thought of as loops of n-component unlinks in $\mathbb{R}^3$, and elements of $\mathbf{LB}_n$ are thought of as loops of *oriented* n-component unlinks in $\mathbb{R}^3$. Since $\mathrm{Diff}^+(U_n)$ is an index-2^n subgroup of $\mathrm{Diff}(U_n)$, the natural quotient map

$$\mathcal{E}^+(U_n, \mathbb{D}^3) \twoheadrightarrow \mathcal{E}(U_n, \mathbb{D}^3)$$

is a 2^n-sheeted covering map, and thus induces an injection

$$\mathbf{LB}_n \hookrightarrow \mathbf{LB}'_n \tag{5.1}$$

of fundamental groups. Thus we view the (non-extended) loop braid group $\mathbf{LB}_n$ as a subgroup (of index 2^n) of the extended loop braid group $\mathbf{LB}'_n$.

Generators. We fix a basepoint for $\mathcal{E}^+(U_n, \mathbb{D}^3)$ where the n circles are arranged on the xy-plane in a row from left to right, as pictured in Figure 5.4. With respect to this basepoint, the loop braid group $\mathbf{LB}_n$ is generated by the elements $\tau_1, \ldots, \tau_{n-1}$ and $\sigma_1, \ldots, \sigma_{n-1}$ illustrated in Figure 5.1. The elements τ_i and σ_i involve only the i-th and $(i+1)$-st loops, which are exchanged; for τ_i, no loop passes through the other; for σ_i, the i-th loop passes through the $(i+1)$-st loop. The extended loop braid group $\mathbf{LB}'_n$ is generated by these elements together with the elements $\rho_1, \ldots, \rho_n$, also illustrated in Figure 5.1. For finite presentations of $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ using these generators, see Fenn, Rimanyi and Rourke [FRR97, §1] and Brendle and Hatcher [BH13, Propositions 3.3 and 3.7]. We note that there are many conflicting conventions for the names of these generators in the literature; in particular, our notation is consistent with [FRR97] but inconsistent with [BH13].

Burau representations of classical braid groups. The classical braid groups $\mathbf{B}_n$ are the fundamental groups of the configuration spaces $C_n(\mathbb{R}^2)$ of points in the plane. One of the oldest interesting representations of $\mathbf{B}_n$ is the *Burau representation* [Bur35]

$$\mathbf{B}_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}]), \tag{5.2}$$

which was defined originally by assigning explicit matrices to the standard generators of $\mathbf{B}_n$, but which is most naturally understood as a homological representation, as follows. The braid group $\mathbf{B}_n$ is naturally isomorphic to the *mapping class group* $\mathrm{MCG}(\mathbb{D}^2_n) = \pi_0(\mathrm{Diff}_\partial(\mathbb{D}^2, Q_n))$, the group of isotopy classes of diffeomorphisms of the 2-disc that act by the identity on its boundary and that preserve a subset Q_n of n points in its interior. In this way, $\mathbf{B}_n$ acts (up to homotopy) on the complement $\mathbb{D}^2_n = \mathbb{D}^2 \smallsetminus Q_n$. There is a projection $\pi_1(\mathbb{D}^2_n) \twoheadrightarrow \mathbb{Z}$ sending a loop to the sum of its winding numbers around each of the points Q_n, and it turns out that the $\mathbf{B}_n$ action on $\mathbb{D}^2_n$ lifts to the corresponding regular covering space $\pi \colon \widetilde{\mathbb{D}}^2_n \twoheadrightarrow \mathbb{D}^2_n$ and commutes with the deck transformations. The induced $\mathbf{B}_n$ action on the first homology $H_1(\widetilde{\mathbb{D}}^2_n)$ therefore respects its structure as a module over the group-ring of the deck transformation group, $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t^{\pm 1}]$, so we obtain a representation

$$\mathbf{B}_n \longrightarrow \mathrm{Aut}_{\mathbb{Z}[t^{\pm 1}]}\big(H_1(\widetilde{\mathbb{D}}^2_n)\big).$$

The homology group $H_1(\widetilde{\mathbb{D}}^2_n)$ is in fact a free $\mathbb{Z}[t^{\pm 1}]$-module of rank $n-1$, so choosing a free basis we may rewrite this as

$$\mathbf{B}_n \longrightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}]). \tag{5.3}$$

This is the *reduced Burau representation.* To obtain the unreduced Burau representation (5.2), we consider instead the induced $\mathbf{B}_n$ action on the *relative* first homology $H_1(\widetilde{\mathbb{D}}^2_n, \pi^{-1}(*))$, where $*$ is

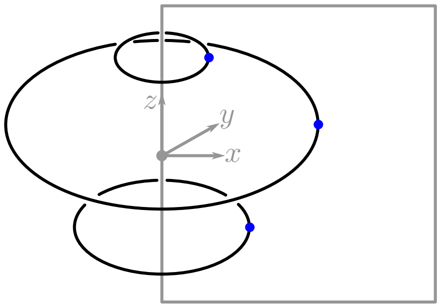


Figure 5.2 Given a configuration of (blue) points in the right-hand xz-plane, we rotate about the z-axis as shown to produce a configuration of unlinked circles in $\mathbb{R}^3$.

a basepoint in the boundary of the disc. This is now a free $\mathbb{Z}[t^{\pm 1}]$-module of rank n, so choosing a free basis we obtain (5.2). The canonical map $H_1(\widetilde{\mathbb{D}}^2_n) \to H_1(\widetilde{\mathbb{D}}^2_n, \pi^{-1}(*))$ is injective, so the reduced Burau representation (5.3) is a subrepresentation of the Burau representation (5.2).

Choosing appropriate ordered free generating sets for $H_1(\widetilde{\mathbb{D}}^2_n)$ and $H_1(\widetilde{\mathbb{D}}^2_n, \pi^{-1}(*))$ over $\mathbb{Z}[t^{\pm 1}]$, the representations (5.2) and (5.3) may be written explicitly as

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2} \tag{5.4}$$

respectively. We note that the Burau representation is sometimes defined using the transposes of these matrices, such as in [KT08], but this is not an essential difference, since the Burau representation is equivalent to its transpose. For more details of these representations, see [KT08].

From classical braids to loop braids. There is an obvious map

$$t\colon C_n(\mathbb{R}^2) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3) \tag{5.5}$$

given by replacing each point in the given configuration with a small circle, oriented positively in $\mathbb{R}^2$, and then including this unlinked configuration of circles into $\mathbb{R}^3$. On fundamental groups, this induces a homomorphism $\mathbf{B}_n \to \mathbf{LB}_n$ sending the standard generators of $\mathbf{B}_n$ to the elements $\tau_1, \ldots, \tau_{n-1}$ of $\mathbf{LB}_n$. In particular, this map factors through the projection $\mathbf{B}_n \to \mathfrak{S}_n$ onto the symmetric group on n letters. There is also a more interesting map

$$s\colon C_n(\mathbb{R}^2) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3) \tag{5.6}$$

defined as follows. Let us identify $\mathbb{R}^2$ with the right-hand xz-plane (the half where the x-coordinate is positive) and the interior of $\mathbb{D}^3$ with $\mathbb{R}^3$. Given a configuration of n points in the right-hand xz-plane, we produce an n-component unlink by rotating the configuration about the z-axis, tracing out n circles while doing so, which all lie in planes parallel to the xy-plane; see Figure 5.2. We orient these circles positively with respect to the parallel copy of the xy-plane in which they lie.

To see that the induced homomorphism on fundamental groups is injective, let us take a basepoint in $C_n(\mathbb{R}^2)$ where the configuration points are arranged in a line along the x-axis; this corresponds to a basepoint of $\mathcal{E}^+(U_n, \mathbb{D}^3)$ with n concentric circles in the xy-plane, centred at the origin. The standard generators of $\mathbf{B}_n$ are sent to loops of the form illustrated on the left-hand side of Figure 5.3. Changing the basepoint of $\mathcal{E}^+(U_n, \mathbb{D}^3)$ to the one chosen earlier, with *non-concentric* circles on the xy-plane, this corresponds to the loop on the right-hand side of Figure 5.3, which is the generator σ_i of $\mathbf{LB}_n$. By [BH13, Proposition 4.3], the group homomorphism $\mathbf{B}_n \to \mathbf{LB}_n$ sending the standard generators of $\mathbf{B}_n$ to the elements $\sigma_1, \ldots, \sigma_{n-1} \in \mathbf{LB}_n$ is injective. As we have just seen, the map (5.6) realises this homomorphism at the space level, and so:

Proposition 5.0.2. *The map* (5.6) *induces an injection on* π_1.

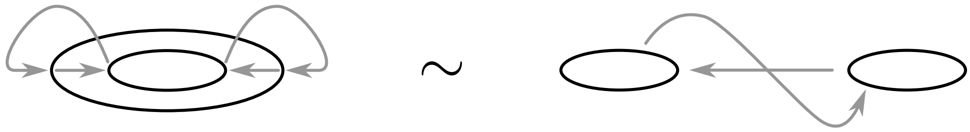


Figure 5.3 The image of the i-th standard generator of $\mathbf{B}_n$ under the map s_* (left) corresponds, by a change of basepoint, to the element σ_i of $\mathbf{LB}_n$ (right).

Remark 5.0.3. The map (5.6) has also been described in §6 of [BB16], where its image is called the *configuration space of linear necklaces*. In particular, [BB16, Theorem 6.1] is equivalent to Proposition 5.0.2 under this interpretation. A small difference is that the map of [BB16] has the space $\mathcal{UR}_n$ as target (see [BH13] for this notation), whereas (5.6) has $\mathcal{E}^+(U_n, \mathbb{D}^3)$ as target. But (5.6) factors as

$$C_n(\mathbb{R}^2) \longrightarrow \mathcal{UR}_n \longrightarrow \mathcal{R}_n^+ \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3),$$

where the left-hand arrow is the map of [BB16]. The middle map is a π_1-isomorphism by [BH13, Proposition 2.3] and the right-hand map is a homotopy equivalence by [BH13, Theorem 1].

Burau representations of loop braid groups. A natural question is whether, and how, one may extend the Burau representations (5.2) and (5.3) along the inclusions

$$\mathbf{B}_n \xhookrightarrow{(5.6)_*} \mathbf{LB}_n \xhookrightarrow{(5.1)} \mathbf{LB}'_n. \tag{5.7}$$

The unreduced Burau representation (5.2) has been extended to $\mathbf{LB}_n$ by Vershinin [Ver01], using a presentation of $\mathbf{LB}_n$ and by assigning explicit matrices to generators. (More precisely, Vershinin's representation of $\mathbf{LB}_n$ restricts to the *transpose* of the unreduced Burau representation of $\mathbf{B}_n$, according to our conventions.)

Our approach will instead be topological, analogous to the description above of the classical Burau representations of $\mathbf{B}_n$, via the homology of covering spaces. In each case, we will find a natural generating set of the relevant homology group, and calculate the matrix that each standard generator of the loop braid group is sent to.

Notation 5.0.4. We write $R = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$ and $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$.

In this notation, the unreduced and reduced Burau representations are R-linear actions $\mathbf{B}_n \curvearrowright R^{\oplus n}$ and $\mathbf{B}_n \curvearrowright R^{\oplus n-1}$ respectively. The case of the non-extended loop braid groups is straightforward:

Theorem 5.A. *These R-linear actions extend to R-linear actions $\mathbf{LB}_n \curvearrowright R^{\oplus n}$ and $\mathbf{LB}_n \curvearrowright R^{\oplus n-1}$. Explicit matrices are given in equations (5.15) and (5.17) respectively.*

To extend further to the extended loop braid groups is a little more subtle. We must first reduce modulo $t^2 - 1$, in other words tensor $- \otimes_R S$ to obtain S-linear actions $\mathbf{LB}_n \curvearrowright S^{\oplus n}$ and $\mathbf{LB}_n \curvearrowright S^{\oplus n-1}$. The unreduced Burau representation then extends directly:

Theorem 5.B. *The S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n}$ extends to an S-linear action $\mathbf{LB}'_n \curvearrowright S^{\oplus n}$. Explicit matrices are given in equations (5.15) and (5.21).*

The reduced Burau representation does not extend directly; instead:

Theorem 5.C. *The S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n-1}$ is a subrepresentation of an S-linear action $\mathbf{LB}_n \curvearrowright S^{\oplus n-1} \oplus S/(t-1)$, and this extends to an S-linear action $\mathbf{LB}'_n \curvearrowright S^{\oplus n-1} \oplus S/(t-1)$. Explicit matrices are given in Table 5.1 on page 158.*

We emphasise that these extensions of the Burau representations to $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ are precisely those that arise naturally via actions on first homology groups of covering spaces, mirroring the topological construction of the classical Burau representations. As a partial summary, we have

$$\begin{array}{ccccccc} \overset{x_1,\ldots,x_{n-1}}{R^{\oplus n-1}} & \twoheadrightarrow & \overset{x_1,\ldots,x_{n-1}}{S^{\oplus n-1}} & \hookrightarrow & \overset{x_1,\ldots,x_{n-1},y}{S^{\oplus n-1}\oplus S/(t-1)} & \hookrightarrow & \overset{a_1,\ldots,a_n}{S^{\oplus n}} \\ \| & & \| & & \| & & \| \\ H_1(\widetilde{\mathbb{D}}^3_n;\mathbb{Z}) & & H_1(\widetilde{\mathbb{D}}^3_n;\mathbb{Z})\otimes_R S & & H_1(\widehat{\mathbb{D}}^3_n;\mathbb{Z}) & & H_1(\widehat{\mathbb{D}}^3_n,\{v,tv\};\mathbb{Z}), \end{array} \tag{5.8}$$

where from left to right we have (1) the reduced Burau representation of $\mathbf{LB}_n$ over $R = \mathbb{Z}[t^{\pm 1}]$ (Theorem 5.A), (2) its reduction modulo t^2-1 over $S = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$, (3) the inclusion of (2) into the reduced Burau representation of $\mathbf{LB}'_n$ over S (Theorem 5.C) and (4) its further inclusion into the unreduced Burau representation of $\mathbf{LB}'_n$ (Theorem 5.B). In each case, an ordered generating set corresponding to the direct sum decomposition is given in blue.

Remark 5.0.5. The matrices of the representations in Theorems 5.A and 5.B are the "obvious" matrices that one may guess by analogy with the matrices for the classical (reduced and unreduced) Burau representations. However, the matrices for the reduced Burau representation of the extended loop braid groups, from Theorem 5.C, do not seem combinatorially or algebraically obvious. However, they arise very naturally *topologically*. Two additional subtleties in this case are the appearance of torsion in the $S/(t-1)$ summand and the *non-locality* of the matrices for the ρ_i generators.

Remark 5.0.6. It is stated in Theorem 5.C that the $\mathbf{LB}_n$-representation $S^{\oplus n-1}$ is a subrepresentation of an $\mathbf{LB}_n$-representation $S^{\oplus n-1} \oplus S/(t-1)$. We remark that it is however *not* a direct summand of the $\mathbf{LB}_n$-representation $S^{\oplus n-1} \oplus S/(t-1)$.

See §5.4 for further remarks on reducibility, kernel and other properties of these representations.

Remark 5.0.7. The Burau representations of the classical braid groups $\mathbf{B}_n$ form the first of an infinite family of *Lawrence-Bigelow* representations [Law90; Big04], and the Burau representations of the loop braid groups $\mathbf{LB}_n$ (or extended loop braid groups $\mathbf{LB}'_n$) may be extended, in more than one way, to an analogous infinite family of representations; see [PS19].

These representations are particularly interesting as the representation theory of the loop braid groups is in the early stages: so far, few other results are known on extensions of representations of the braid groups to loop braid groups and some of their particular subgroups; see Kádár, Martin, Rowell and Wang [Kád+17] and Bellingeri and the second author [BS20]. Furthermore, Damiani, Martin and Rowell [DMR23] have recently studied a finite dimensional quotient $\mathbf{LH}_n$ of the group algebra of $\mathbf{LB}_n$, mimicking the braid group/Iwahori-Hecke algebra paradigm. In particular, the unreduced Burau representation of $\mathbf{LB}_n$ (Theorem 5.A) factors through this quotient algebra $\mathbf{LH}_n$; see [DMR23, §3.2].

5.1 Action on the homology of covering spaces

Let φ be a diffeomorphism of the 3-ball $\mathbb{D}^3$ that restricts to the identity near the boundary. We may restrict φ to our chosen unlink U_n in the interior of $\mathbb{D}^3$ to obtain a new embedding $U_n \hookrightarrow \mathbb{D}^3$. Since φ is isotopic to a diffeomorphism that acts by the identity on U_n (it may be isotoped to act by the identity on a larger and larger collar neighbourhood of the boundary, until this collar neighbourhood contains U_n), this new embedding is isotopic to the inclusion, hence an element of $\mathrm{Emb}^u(U_n, \mathbb{D}^3)$. We therefore have a restriction map

$$\mathrm{Diff}_\partial(\mathbb{D}^3) \longrightarrow \mathrm{Emb}^u(U_n, \mathbb{D}^3), \tag{5.9}$$

where $\mathrm{Diff}_\partial(\mathbb{D}^3)$ denotes the topological group of diffeomorphisms that are the identity on a neighbourhood of the boundary, equipped with the smooth Whitney topology. The map (5.9) is a locally trivial fibration [Pal60; Cer61; Lim63] and the quotient map

$$\mathrm{Emb}^u(U_n, \mathbb{D}^3) \twoheadrightarrow \mathrm{Emb}^u(U_n, \mathbb{D}^3)/\mathrm{Diff}(U_n) = \mathcal{E}(U_n, \mathbb{D}^3) \tag{5.10}$$

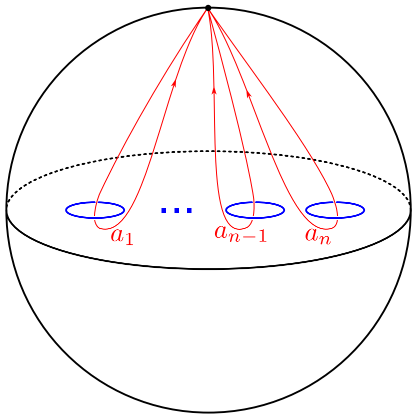


Figure 5.4 The unlink-complement $\mathbb{D}^3_n$ with free generators $a_1, \ldots, a_n$ for $\pi_1(\mathbb{D}^3_n) \cong F_n$.

is also a locally trivial fibration [BF81] (see also [Pal21, §4] for both of these). Putting together (5.9) and (5.10), we have a locally trivial fibration

$$\mathrm{Diff}_\partial(\mathbb{D}^3) \longrightarrow \mathcal{E}(U_n, \mathbb{D}^3). \tag{5.11}$$

If we modify (5.10) to quotient only by $\mathrm{Diff}^+(U_n)$, it remains a locally trivial fibration, and together with (5.9) we obtain a locally trivial fibration

$$\mathrm{Diff}_\partial(\mathbb{D}^3) \longrightarrow \mathcal{E}^+(U_n, \mathbb{D}^3). \tag{5.12}$$

Together with Hatcher's proof [Hat83] of the Smale conjecture, this implies the following, where $\mathrm{Diff}_\partial(\mathbb{D}^3, U_n) \leqslant \mathrm{Diff}_\partial(\mathbb{D}^3)$ is the subgroup of diffeomorphisms that preserve U_n (setwise) and $\mathrm{Diff}_\partial(\mathbb{D}^3, U_n^+) \leqslant \mathrm{Diff}_\partial(\mathbb{D}^3)$ is the subgroup of diffeomorphisms that preserve U_n and its orientation.

Lemma 5.1.1. *There are isomorphisms*

$$\mathbf{LB}'_n = \pi_1(\mathcal{E}(U_n, \mathbb{D}^3)) \cong \pi_0(\mathrm{Diff}_\partial(\mathbb{D}^3, U_n))$$
$$\mathbf{LB}_n = \pi_1(\mathcal{E}^+(U_n, \mathbb{D}^3)) \cong \pi_0(\mathrm{Diff}_\partial(\mathbb{D}^3, U_n^+)).$$

Proof. The topological group $\mathrm{Diff}_\partial(\mathbb{D}^3)$ is contractible, in particular simply-connected, by [Hat83], and these isomorphisms then follow from the long exact sequences associated to (5.11) and (5.12). □

Notation 5.1.2. We will abbreviate $\mathbb{D}^3_n = \mathbb{D}^3 \smallsetminus U_n$, where U_n is the n-component unlink in the interior of $\mathbb{D}^3$ chosen previously. See Figure 5.4 for an illustration of a particular choice.

By the mapping class group interpretation of loop braid groups (Lemma 5.1.1), the group $\mathbf{LB}'_n$ (and hence also its subgroup $\mathbf{LB}_n$) acts, up to homotopy, on the unlink-complement $\mathbb{D}^3_n$ by diffeomorphisms (in particular, homeomorphisms).

Remark 5.1.3. We will speak of actions of mapping class groups up to homotopy, which induce (strict) actions on homology. An alternative, equivalent viewpoint would be that the corresponding *diffeomorphism group* acts (strictly) at the level of spaces, and then observing that its induced action on homology factors through the mapping class group, since homology groups are discrete.

Note that the fundamental group of $\mathbb{D}^3_n$ is the free group F_n on n generators. This is easy to see: the unlink-complement $\mathbb{D}^3_n \subseteq \mathbb{D}^3$ deformation retracts onto a wedge of n circles and n copies of the 2-sphere. The n circles $a_1, \ldots, a_n$ are shown in Figure 5.4. Now let

$$\phi \colon \pi_1(\mathbb{D}^3_n) \longrightarrow \mathbb{Z}$$

be the surjective homomorphism defined by $\phi(a_i) = 1$ for all $i = 1, \ldots, n$ and let

$$\phi' \colon \pi_1(\mathbb{D}^3_n) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

be the composition of ϕ with the unique surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

Definition 5.1.4. We denote by $\widetilde{\mathbb{D}}^3_n$ the regular covering space corresponding to $\ker(\phi)$ and by $\widehat{\mathbb{D}}^3_n$ the regular covering space corresponding to $\ker(\phi')$. We therefore have regular coverings

$$\eta\colon \widetilde{\mathbb{D}}^3_n \longrightarrow \mathbb{D}^3_n \qquad \text{and} \qquad \eta'\colon \widehat{\mathbb{D}}^3_n \longrightarrow \mathbb{D}^3_n$$

whose deck transformations groups are $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively.

In general, if a group G acts (up to homotopy) on a based space X and we choose a surjection $\psi\colon \pi_1(X) \twoheadrightarrow Q$ that is invariant under the induced action of G on $\pi_1(X)$, then the G-action on X lifts uniquely to the regular covering space corresponding to ψ and commutes with the action of Q by deck transformations.

Let us first take $G = \mathbf{LB}_n$ and $X = \mathbb{D}^3_n$ with basepoint $* \in \partial\mathbb{D}^3 = \partial\mathbb{D}^3_n$. We note that the quotient ϕ is invariant under the action of $\mathbf{LB}_n$: for this it suffices to check that each generator τ_i, σ_i of $\mathbf{LB}_n$ sends each generator a_j of $\pi_1(\mathbb{D}^3_n)$ to an element in $\phi^{-1}(1)$, and this follows since, up to conjugation, τ_i and σ_i simply permute the generators a_j. We therefore have an induced action (up to homotopy) of $\mathbf{LB}_n$ on $\widetilde{\mathbb{D}}^3_n$ commuting with the deck transformation action of $\mathbb{Z}$. Thus the first integral homology groups

$$H_1\big(\widetilde{\mathbb{D}}^3_n;\mathbb{Z}\big) \qquad \text{and} \qquad H_1\big(\widetilde{\mathbb{D}}^3_n, \eta^{-1}(*);\mathbb{Z}\big) \tag{5.13}$$

are $\mathbb{Z}[\mathbb{Z}]$-modules via the deck transformation action, and are $\mathbf{LB}_n$-representations over $\mathbb{Z}[\mathbb{Z}]$ via the lifted $\mathbf{LB}_n$-action on $\widetilde{\mathbb{D}}^3_n$.

Definition 5.1.5. The $\mathbf{LB}_n$-representations (5.13) are the *reduced* and the *unreduced Burau representations* of loop braid groups over $\mathbb{Z}[\mathbb{Z}] = R$.

Let us now take $G = \mathbf{LB}'_n$ and again $X = \mathbb{D}^3_n$ with basepoint $* \in \partial\mathbb{D}^3 = \partial\mathbb{D}^3_n$. This time ϕ is not invariant under the induced action of $\mathbf{LB}'_n$, since, for example, the generator ρ_i sends $a_i \in \phi^{-1}(1)$ to $a_i^{-1} \in \phi^{-1}(-1)$. However, the deeper quotient ϕ' (namely ϕ reduced mod 2) is clearly invariant under the action of $\mathbf{LB}'_n$. We therefore have an induced action (up to homotopy) of $\mathbf{LB}'_n$ on $\widehat{\mathbb{D}}^3_n$ commuting with the deck transformation action of $\mathbb{Z}/2\mathbb{Z}$. Thus the first integral homology groups

$$H_1\big(\widehat{\mathbb{D}}^3_n;\mathbb{Z}\big) \qquad \text{and} \qquad H_1\big(\widehat{\mathbb{D}}^3_n, (\eta')^{-1}(*);\mathbb{Z}\big) \tag{5.14}$$

are $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-modules via the deck transformation action, and are $\mathbf{LB}'_n$-representations over $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ via the lifted $\mathbf{LB}'_n$-action on $\widehat{\mathbb{D}}^3_n$.

Definition 5.1.6. The $\mathbf{LB}'_n$-representations (5.14) are the *reduced* and the *unreduced Burau representations* of extended loop braid groups over $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] = S$.

Let us make these covering spaces more concrete by building explicit models for each of them. We embed n pairwise disjoint closed 3-discs into the interior of the unit 3-disc $\mathbb{D}^3$ as pictured in Figure 5.5, so that each little 3-disc looks like a "lens shape" and the union of their equators is precisely the n-component unlink that we fixed earlier. Let $\mathring{\mathbb{D}}^3_n$ denote $\mathbb{D}^3_n$ minus the interiors of these n little 3-discs, equivalently, $\mathbb{D}^3$ minus the interiors and equators of the n little 3-discs. Also, write N_i for the open northern hemisphere of the boundary of the i-th little 3-disc, and write S_i for the open southern hemisphere of the boundary of the i-th little 3-disc. Now consider

$$\mathbb{Z} \times \mathring{\mathbb{D}}^3_n$$

and glue $\{j\} \times N_i$ to $\{j-1\} \times S_i$ via the homeomorphism $N_i \cong S_i$ given by reflection in the plane passing through the equator. This is an explicit model for $\widetilde{\mathbb{D}}^3_n$. Similarly, we may consider

$$\mathbb{Z}/2\mathbb{Z} \times \mathring{\mathbb{D}}^3_n$$

and glue as before, where j is now considered mod 2. This is an explicit model for $\widehat{\mathbb{D}}^3_n$.

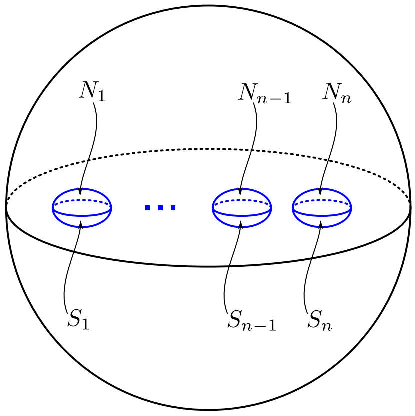


Figure 5.5 The complement $\mathring{\mathbb{D}}^3_n$ of the interiors and equators of n closed little 3-discs ("lens shapes") in the interior of the closed unit 3-disc $\mathbb{D}^3$. The boundary of $\mathring{\mathbb{D}}^3_n$ decomposes as the disjoint union of $2n+1$ components: $\partial\mathring{\mathbb{D}}^3_n = \partial\mathbb{D}^3 \sqcup N_1 \sqcup \ldots \sqcup N_n \sqcup S_1 \sqcup \ldots \sqcup S_n$.

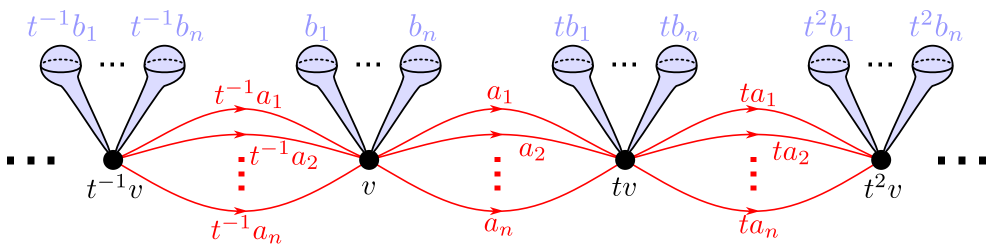


Figure 5.6 The deformation retract X of the $\mathbb{Z}$-covering $\widetilde{\mathbb{D}}^3_n$.

5.2 Matrices for non-extended loop braid groups

We first consider the $\mathbf{LB}_n$-representations (5.13). The calculations of these representations are unsurprising, but they are a useful warm-up to the slightly more subtle ones in the next section, for the $\mathbf{LB}'_n$-representations (5.14).

The modules. As noted above, the unlink-complement $\mathbb{D}^3_n$ deformation retracts onto a wedge of n circles and n copies of the 2-sphere. This deformation retraction lifts to a deformation retraction of the covering space $\widetilde{\mathbb{D}}^3_n$ onto the space pictured in Figure 5.6. This is an infinite 2-dimensional cell complex X with vertices indexed by $\mathbb{Z}$, with exactly n edges between consecutive vertices (and none between non-consecutive vertices) and with exactly n copies of the 2-sphere wedged onto each vertex. Its fundamental group is freely generated by $t^k.(a_2\bar{a}_1), \ldots\ldots, t^k.(a_n\bar{a}_{n-1})$ for all $k \in \mathbb{Z}$, where $\bar{a}$ denotes the reverse of a path a. Abelianising and writing $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$, we see that its first homology is freely generated, as a $\mathbb{Z}[t^{\pm 1}]$-module, by $x_1 \coloneqq a_2\bar{a}_1, \ldots\ldots, x_{n-1} \coloneqq a_n\bar{a}_{n-1}$.

The relative homology group $H_1(\widetilde{\mathbb{D}}^3_n, \eta^{-1}(*);\mathbb{Z})$ is isomorphic to the first homology of X relative to its set of vertices (since these vertices are fixed by the deformation retraction described above), which is freely generated, as a $\mathbb{Z}[t^{\pm 1}]$-module, by $a_1, \ldots, a_n$. Summarising, we have natural isomorphisms

$$H_1\big(\widetilde{\mathbb{D}}^3_n, \eta^{-1}(*);\mathbb{Z}\big) \cong \mathbb{Z}[t^{\pm 1}]\{a_1, \ldots, a_n\}$$
$$H_1\big(\widetilde{\mathbb{D}}^3_n;\mathbb{Z}\big) \cong \mathbb{Z}[t^{\pm 1}]\{x_1, \ldots, x_{n-1}\},$$

and the canonical homomorphism $H_1(\widetilde{\mathbb{D}}^3_n;\mathbb{Z}) \to H_1(\widetilde{\mathbb{D}}^3_n, \eta^{-1}(*);\mathbb{Z})$ is given under these identifications by $x_i \mapsto a_{i+1} - a_i$.

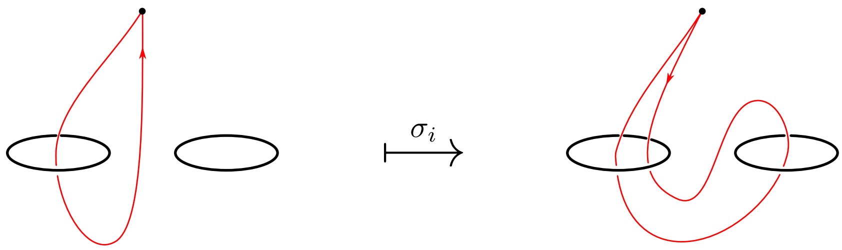


Figure 5.7 The action of $\sigma_i \in \mathbf{LB}_n$ on the homological generator a_i. The right-hand loop may be decomposed (writing from the left to right) as $a_i \cdot ta_{i+1} \cdot t\bar{a}_i$.

The unreduced representation. It is easy to calculate visually how the $\mathbf{LB}_n$ generators τ_i and σ_i act on the homological generators a_j. Clearly τ_i simply interchanges a_i and a_{i+1}. On the other hand, σ_i acts by

$$\sigma_i(a_i) = (1-t).a_i + t.a_{i+1} \qquad \sigma_i(a_{i+1}) = a_i \qquad \sigma_i(a_j) = a_j \text{ (for } j \notin \{i, i+1\}),$$

where the first formula comes from the fact that, at the fundamental group level, σ_i sends the loop a_i to the loop $a_i \cdot ta_{i+1} \cdot t\bar{a}_i$, which may be read off from Figure 5.7. Thus we see that the matrices for the unreduced Burau representation $\mathbf{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ are given by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \quad \text{and} \quad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix} \oplus I_{n-i-1}. \tag{5.15}$$

Remark 5.2.1. These are precisely the transposes of the matrices used in [Ver01]. Related to this, we note that, as observed in [Ibr21, Theorem 3.2], one may extend the (transpose of the) unreduced Burau representation to the virtual braid group $\mathbf{VB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}, u^{\pm 1}])$ by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & u^{-1} \\ u & 0 \end{bmatrix} \oplus I_{n-i-1} \quad \text{and} \quad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}. \tag{5.16}$$

This factors through the projection $\mathbf{VB}_n \twoheadrightarrow \mathbf{LB}_n$ if one sets $u = 1$, but in general it does not. It would be interesting to find a topological construction of this representation, in the sense of the present chapter, although it is unclear how this could be done, as we are unaware of a topological interpretation of the virtual braid group as a motion group, analogous to the realisation of the loop braid group as the group of motions of an oriented trivial link in $\mathbb{R}^3$.

The reduced representation. Using this computation of the unreduced representation, and the explicit formula ($x_i \mapsto a_{i+1} - a_i$) for the inclusion of the reduced representation into the unreduced one, it is an easy exercise to read off the following explicit formulas for the reduced Burau representation $\mathbf{LB}_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}])$:

$$\tau_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2} \quad \text{and} \quad \sigma_i \longmapsto I_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-i-2}. \tag{5.17}$$

If $i = 1$ or $i = n-1$, one should ignore the "I_{-1}" on the left or right, and instead remove the left column and top row, respectively the right column and bottom row, from the displayed matrix.

Observe that, when restricted to the σ_i generators, the formulas (5.15) and (5.17) are precisely the matrices (5.4) defining the unreduced and reduced Burau representations of the classical braid groups. This concludes the proof of Theorem 5.A.

Action on the second homology. The other non-trivial homology group of the covering space $\widetilde{\mathbb{D}}^3_n \simeq X$ is in degree two, where we have $H_2(\widetilde{\mathbb{D}}^3_n; \mathbb{Z}) \cong \mathbb{Z}[t^{\pm 1}]\{b_1, \ldots, b_n\}$, where b_i are illustrated in blue in Figure 5.6. The generator $\tau_i \in \mathbf{LB}_n$ clearly acts by swapping the homological generators b_i and b_{i+1}. The generator $\sigma_i \in \mathbf{LB}_n$ acts as illustrated in Figures 5.8 and 5.9. It sends b_i, considered

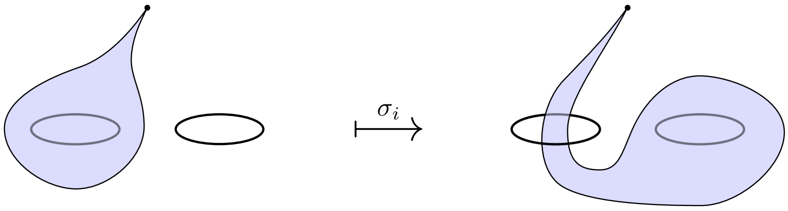


Figure 5.8 The action of $\sigma_i \in \mathbf{LB}_n$ on the element $b_i \in \pi_2(\widetilde{\mathbb{D}}^3_n)$. The right-hand side is homotopic to $a_i \cdot b_{i+1}$, where $\cdot$ denotes the action of π_1 on π_2.

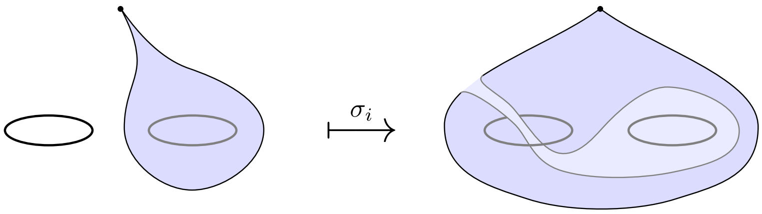


Figure 5.9 The action of $\sigma_i \in \mathbf{LB}_n$ on the element $b_{i+1} \in \pi_2(\widetilde{\mathbb{D}}^3_n)$. The right-hand side is homotopic to $b_i + b_{i+1} - a_i \cdot b_{i+1}$, where $\cdot$ denotes the action of π_1 on π_2.

as an element of $\pi_2(\widetilde{\mathbb{D}}^3_n)$, to $a_i \cdot b_{i+1}$, where $\cdot$ denotes the canonical action of $\pi_1(\widetilde{\mathbb{D}}^3_n)$ on $\pi_2(\widetilde{\mathbb{D}}^3_n)$. Inspecting Figure 5.6, we see that, viewed as an element of $H_2(\widetilde{\mathbb{D}}^3_n;\mathbb{Z})$, this is tb_{i+1}. Similarly, σ_i sends b_{i+1}, considered as an element of $\pi_2(\widetilde{\mathbb{D}}^3_n)$, to $b_i + b_{i+1} - a_i \cdot b_{i+1}$, which is $b_i + (1-t).b_{i+1}$ as an element of $H_2(\widetilde{\mathbb{D}}^3_n;\mathbb{Z})$. Thus, with respect to the ordered basis $(b_1,\ldots,b_n)$ of $H_2(\widetilde{\mathbb{D}}^3_n;\mathbb{Z})$, this representation $\mathbf{LB}_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$ is given by

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ t & 1-t \end{bmatrix} \oplus I_{n-i-1}. \tag{5.18}$$

Reversing the ordering, i.e. using instead the ordered basis $(b_n,\ldots,b_1)$, we obtain

$$\tau_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1} \qquad \text{and} \qquad \sigma_i \longmapsto I_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}, \tag{5.19}$$

which is the transpose of the unreduced Burau representation (5.15) of $\mathbf{LB}_n$, and agrees with the matrices used in [Ver01].

5.3 Matrices for extended loop braid groups

The modules. As in §5.2, the deformation retraction of the unlink-complement $\mathbb{D}^3_n$ onto a wedge of circles and 2-spheres lifts to a deformation retraction of its covering $\widehat{\mathbb{D}}^3_n$ onto the space pictured in Figure 5.10. This is a finite 2-dimensional cell complex with two vertices $\{v, tv\}$, $2n$ edges between them and with n copies of the 2-sphere wedged onto each vertex. Its fundamental group is freely generated by the $2n-1$ loops

$$\begin{gathered} x_1 = a_2\bar{a}_1, \ldots\ldots, x_{n-1} = a_n\bar{a}_{n-1} \\ tx_1 = ta_2\, t\bar{a}_1, \ldots\ldots, tx_{n-1} = ta_n\, t\bar{a}_{n-1} \\ y \coloneqq a_n\, ta_n. \end{gathered}$$

Hence its first homology $H_1(\widehat{\mathbb{D}}^3_n;\mathbb{Z})$ is generated, as an abelian group, by the same $2n-1$ loops, viewed as homology classes. The first $2n-2$ of these classes generate a free module of rank $n-1$ over $S = \mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$, whereas the last element y generates a summand isomorphic to $\mathbb{Z}$ viewed as a *trivial* $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-module, in other words $S/(t-1)$.

The relative homology group $H_1(\widehat{\mathbb{D}}^3_n, (\eta')^{-1}(*);\mathbb{Z})$ is isomorphic to the first homology of this complex relative to its vertices $\{t, tv\}$. This is much simpler: it is a free module over

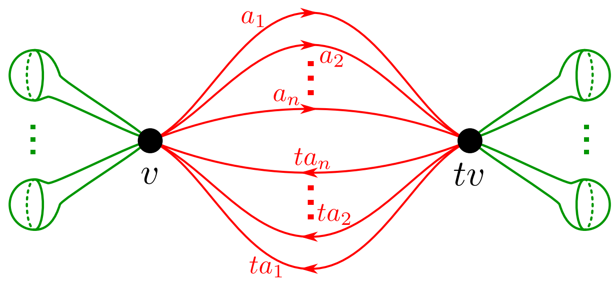


Figure 5.10 The deformation retract of the double covering $\widehat{\mathbb{D}}^3_n$ corresponding to quotienting Figure 5.6 by the action of t^2.

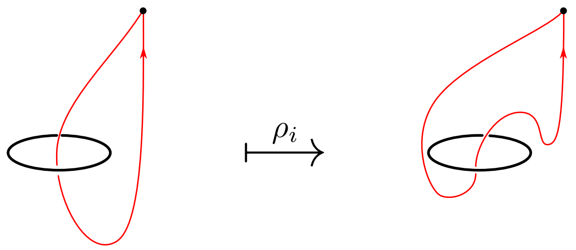


Figure 5.11 The action of $\rho_i \in \mathbf{LB}'_n$ on the homological generator a_i. The right-hand loop is $t\bar{a}_i$. Note that it is not simply $\bar{a}_i$, since this is a path from tv to v, whereas $\rho_i(a_i)$ is a path from $v = t^2 v$ to tv.

$S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$ of rank n, generated by $a_1, \ldots, a_n$. Summarising, we have natural isomorphisms

$$H_1\big(\widehat{\mathbb{D}}^3_n, (\eta')^{-1}(*); \mathbb{Z}\big) \cong S\{a_1, \ldots, a_n\}$$
$$H_1\big(\widetilde{\mathbb{D}}^3_n; \mathbb{Z}\big) \cong S\{x_1, \ldots, x_{n-1}\} \oplus S/(t-1)\{y\}.$$

The canonical homomorphism $H_1(\widehat{\mathbb{D}}^3_n; \mathbb{Z}) \to H_1(\widehat{\mathbb{D}}^3_n, (\eta')^{-1}(*); \mathbb{Z})$ is given under these identifications by $x_i \mapsto a_{i+1} - a_i$ and $y \mapsto (1+t)a_n$. As a matrix, this is:

$$\begin{bmatrix} -1 & 0 & 0 & & & \\ 1 & -1 & 0 & & & \\ 0 & 1 & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & 1 & 1+t \end{bmatrix}. \tag{5.20}$$

Note that this matrix describes an injective homomorphism. (Viewed as an endomorphism of $S^{\oplus n}$, it is of course *not* injective, since $1+t$ is a zero-divisor. But its kernel is precisely the submodule $0^{\oplus n-1} \oplus (t-1)$ of $S^{\oplus n}$ so once we replace the domain with $S^{\oplus n-1} \oplus S/(t-1)$ it becomes injective.)

The unreduced representation. The action of the $\mathbf{LB}'_n$ generators τ_i and σ_i on the homological generators a_j is exactly as in §5.2, and given by the matrices (5.15), considered now over the ring $S = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$ instead of $R = \mathbb{Z}[t^{\pm 1}]$. The $\mathbf{LB}'_n$ generator ρ_i acts trivially on a_j for $j \neq i$ and sends a_i to $-ta_i$. This last formula comes from the fact that, at the fundamental group level, ρ_i sends the loop a_i to the loop $t\bar{a}_i$, which may be read off from Figure 5.11. Thus we see that the matrices for the unreduced Burau representation $\mathbf{LB}'_n \to \mathrm{GL}_n(S)$ are given by

$$(5.15) \qquad \text{and} \qquad \rho_i \longmapsto I_{i-1} \oplus \begin{bmatrix} -t \end{bmatrix} \oplus I_{n-i}. \tag{5.21}$$

In particular, the restriction of this representation of $\mathbf{LB}'_n$ to the generators τ_i and σ_i (i.e. to $\mathbf{LB}_n$) is the reduction modulo t^2 of the representation (5.15) of Theorem 5.A. This concludes the proof of Theorem 5.B.

The reduced representation. It is now a purely algebraic exercise, using the formulas (5.15) and (5.21) for the unreduced Burau representation, together with the explicit description (5.20) of the inclusion, to deduce explicit formulas for the reduced Burau representation

$$\mathbf{LB}'_n \longrightarrow \mathrm{Aut}_S(S^{\oplus n-1} \oplus S/(t-1)).$$

These are given in Table 5.1, where we abbreviate $\delta \coloneqq 1 + t$. Note that the matrices for the extended generators ρ_i are, in a sense, "non-local".

Remark 5.3.1. Since these matrices describe automorphisms of $S^{\oplus n-1} \oplus S/(t-1)$, each entry above the bottom row should be considered as an element of S, whereas each element of the bottom row should be considered as an element of $S/(t-1) \cong \mathbb{Z}$. In other words, we set $t = 1$ on the bottom row.

More precisely, the entries in the bottom row lie in $\mathrm{Hom}_S(S, S/(t-1)) \cong S/(t-1)$, except the bottom-right entry, which lies in $\mathrm{Hom}_S(S/(t-1), S/(t-1)) \cong S/(t-1)$. The entries in the right-hand column (except the bottom-right entry) lie in $\mathrm{Hom}_S(S/(t-1), S) \cong (1+t)S = \delta S \subset S$. (The S-modules $S/(t-1)$ and δS are abstractly isomorphic, but they are related differently to S.)

In particular, the restriction of this representation of $\mathbf{LB}'_n$ to the group generators τ_i and σ_i and to the homological generators $x_1, \ldots, x_{n-1}$ is equal to the reduction modulo t^2 of the representation (5.17) of Theorem 5.A. This concludes the proof of Theorem 5.C.

Remark 5.3.2. It is an amusing exercise to verify explicitly that the matrices in Table 5.1 indeed satisfy all of the relations of the extended loop braid group $\mathbf{LB}'_n$, as described for example in [BH13, §3] or [Dam17, §3]. (Warning: the papers [BH13], [Dam17] and the present chapter pairwise disagree on notation for the three families of generators of $\mathbf{LB}'_n$.) One should bear in mind that braid words are written from left to right in [BH13] and [Dam17] (as is usual for composition of loops), whereas matrix multiplication goes from right to left (as is usual for function composition), so in fact the matrices in Table 5.1 satisfy the *opposite* of the relations of $\mathbf{LB}'_n$ described in [BH13; Dam17]. (Technically, this means that we have constructed a representation of the opposite group $(\mathbf{LB}'_n)^{\mathrm{op}}$, but, except for computations, we ignore this subtlety, since this is abstractly isomorphic to $\mathbf{LB}'_n$.) For an explicit verification, using Sage, that the matrices in Table 5.1 satisfy the 40 relations of the extended loop braid group $\mathbf{LB}'_n$ in the case $n = 4$, see the supplementary materials [PS22c].

	$i = 1$	$2 \leqslant i \leqslant n-2$	$i = n-1$
τ_i	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$	$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -\delta \\ 0 & 0 & 1 \end{bmatrix}$
σ_i	$\begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-2}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{I}_{n-i-1}$	$\mathrm{I}_{n-3} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & -\delta \\ 0 & 0 & 1 \end{bmatrix}$

	$i = 1$	$2 \leqslant i \leqslant n-1$	$i = n$
ρ_i	$\begin{bmatrix} -t & 0 & \cdots & 0 \\ -\delta & & & \\ \vdots & & \mathrm{I}_{n-1} & \\ -\delta & & & \\ 1 & & & \end{bmatrix}$	$\mathrm{I}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \delta & -t & 0 & \cdots & 0 \\ \delta & -\delta & & & \\ \vdots & \vdots & & \mathrm{I}_{n-i} & \\ \delta & -\delta & & & \\ -1 & 1 & & & \end{bmatrix}$	$\mathrm{I}_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

Table 5.1 Explicit matrices for the reduced Burau representation of the extended loop braid group $\mathbf{LB}'_n$. Notation: $\delta = 1 + t$. All entries lie in $S = \mathbb{Z}[t^{\pm 1}]/(t^2 - 1)$, except for the bottom row, where they lie in $S/(t-1) \cong \mathbb{Z}$, in other words we set $t = 1$ on the bottom row.

5.4 Properties

Irreducibility. The unreduced Burau representations (5.15) of $\mathbf{LB}_n$ and (5.21) of $\mathbf{LB}'_n$ are clearly reducible, since they contain the reduced Burau representations (5.17) and (Table 5.1) respectively. The reduced Burau representation (5.17) of $\mathbf{LB}_n$ becomes irreducible when we pass to the field of fractions $\mathbb{Q}(t)$ of $\mathbb{Z}[t^{\pm 1}]$. This follows because its restriction to the symmetric group $\mathfrak{S}_n \subset \mathbf{LB}_n$ is the standard $(n-1)$-dimensional representation of $\mathfrak{S}_n$, which is irreducible over any field.

On the other hand, for the reduced Burau representation (Table 5.1) of $\mathbf{LB}'_n$, we cannot directly pass to a field of fractions, since its ground ring $S = \mathbb{Z}[t^{\pm 1}]/(t^2-1)$ is not an integral domain. Instead, we may tensor over S with $\mathbb{Q}$, setting either $t=-1$ or $t=1$. In the first case, the additional $S/(t-1)$ summand is killed and we obtain an $(n-1)$-dimensional representation of $\mathbf{LB}'_n$ over $\mathbb{Q}$, which is irreducible, again because its restriction to $\mathfrak{S}_n \subset \mathbf{LB}'_n$ is the standard representation of $\mathfrak{S}_n$. In the second case, we obtain an n-dimensional representation:

$$\mathbf{LB}'_n \longrightarrow \mathrm{GL}_n(\mathbb{Q}). \tag{5.22}$$

Lemma 5.4.1. *The representation* (5.22) *of* $\mathbf{LB}'_n$ *is irreducible.*

Proof. Suppose that $V \subseteq \mathbb{Q}^n$ is a non-trivial subrepresentation; we will show that $V = \mathbb{Q}^n$. Write $v = \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1} + \beta y$.

Step 1. It suffices to find $v \in V$ with $v \neq 0$ and $\beta = 0$.
First note that $\mathrm{span}_{\mathbb{Q}}\{x_1, \dots, x_{n-1}\}$ is an irreducible subrepresentation (it is irreducible since its restriction to $\mathfrak{S}_n$ is the standard representation of $\mathfrak{S}_n$). Thus V must contain $\mathrm{span}_{\mathbb{Q}}\{x_1, \dots, x_{n-1}\}$. But then it must also contain $x_{n-1} - \rho_n(x_{n-1}) = y$, and so $V = \mathbb{Q}^n$.

Step 2. It suffices to find $v \in V$ with $v \neq 0$ and $\alpha_{n-2} - 2\alpha_{n-1} - 2\beta \neq 0$.
For such a v, we have $\tau_{n-1}(v) - v = (\alpha_{n-2} - 2\alpha_{n-1} - 2\beta)x_{n-1}$, and we are done by Step 1.

Step 3. It suffices to find $v \in V$ with $v \neq 0$ and $\alpha_{n-1} + 2\beta \neq 0$.
For such a v, we have $\rho_n(v) - v = -(\alpha_{n-1} + 2\beta)y$, and we are done by Step 2.

Step 4. Let $v \in V$ be a non-zero vector. By the previous steps, we may assume that its coefficients satisfy $\alpha_{n-2} = \alpha_{n-1} = -2\beta$ and $\beta \neq 0$. We then have $\tau_{n-2}(v) - v = (\alpha_{n-3} + 2\beta)x_{n-2}$, so we are done by Step 1 unless $\alpha_{n-3} = -2\beta$. On the other hand, if $\alpha_{n-3} = -2\beta$, we have $\tau_{n-3}(v) - v = (\alpha_{n-4} + 2\beta)x_{n-3}$, so again we are done by Step 1 unless $\alpha_{n-4} = -2\beta$. Repeating this a further $n-5$ times, we see that we are done unless $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = -2\beta$. But in this case we have $\tau_1(v) - v = 2\beta x_1$, and we are done by Step 1. ∎

Remark 5.4.2. The restriction of the representation (5.22) to $\mathfrak{S}_n \subset \mathbf{LB}'_n$ is isomorphic to the regular representation of $\mathfrak{S}_n$. To see this, note that $\mathrm{span}_{\mathbb{Q}}\{x_1, \dots, x_{n-1}\}$ is a subrepresentation isomorphic to the standard representation of $\mathfrak{S}_n$, and the quotient is a trivial 1-dimensional representation. Thus, by Maschke's theorem, $(5.22)|_{\mathfrak{S}_n}$ is isomorphic to the sum of the standard representation and a trivial 1-dimensional representation, which is isomorphic to the regular representation of $\mathfrak{S}_n$.

Kernel. The classical (unreduced) Burau representation $\mathbf{B}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is known to be faithful for $n \leqslant 3$ and unfaithful for $n \geqslant 5$ [Moo91; LP93; Big99]. By contrast, the unreduced Burau representation (5.15): $\mathbf{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is unfaithful for all $n \geqslant 2$, by [Bar05, Lemmas 4 and 5]. The unreduced Burau representations for $\mathbf{LB}_n$ and $\mathbf{LB}'_n$ fit together in the commutative square

$$\begin{array}{ccc} \mathbf{LB}_n & \xrightarrow{(5.15)} & \mathrm{GL}_n(R) \\ {\scriptstyle \text{incl.}}\downarrow & & \downarrow{\scriptstyle -\otimes_R S} \\ \mathbf{LB}'_n & \xrightarrow[(5.21)]{} & \mathrm{GL}_n(S), \end{array} \tag{5.23}$$

(recall that $R = \mathbb{Z}[t^{\pm 1}]$ and $S = R/(t^2-1)$) so the unreduced Burau representation (5.21): $\mathbf{LB}'_n \to \mathrm{GL}_n(S)$ is also unfaithful for all $n \geqslant 2$. In fact, the kernel of the composition $\mathbf{LB}_n \to \mathrm{GL}_n(S)$ across the diagonal of (5.23) is larger than the kernel of (5.15): $\mathbf{LB}_n \to \mathrm{GL}_n(R)$ since, for example, $(\tau_1\sigma_1)^2$ is sent to $\left[\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right] \oplus \mathrm{I}_{n-2}$.

The transpose of the Burau representation. As mentioned in the introduction, the (unreduced) Burau representation of the classical braid group $\mathbf{B}_n$ is equivalent to its transpose. Explicitly, conjugation by the diagonal matrix $\mathrm{Diag}(1, t, \ldots, t^{n-1})$ passes between the Burau representation and its transpose. On the other hand:

Lemma 5.4.3. *The unreduced Burau representation (5.15) of the loop braid group $\mathbf{LB}_n$ is not equivalent to its transpose.*

Proof. Let (5.15) = Bur. The aim is to show that there is no invertible matrix $M \in \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ such that $\mathrm{Bur}(g)M = M\mathrm{Bur}(g)^{\mathrm{t}}$ for all $g \in \mathbf{LB}_n$, where $(-)^{\mathrm{t}}$ denotes the transpose of a matrix. For $n = 2$, the equations $\mathrm{Bur}(\sigma_1)M = M\mathrm{Bur}(\sigma_1)^{\mathrm{t}}$ and $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ imply that M is of the form $\left[\begin{smallmatrix} a & -a \\ -a & a \end{smallmatrix}\right]$, which is not invertible. For $n \geqslant 3$, we will prove by induction on n the statement that the only matrix M satisfying $\mathrm{Bur}(g)M = M\mathrm{Bur}(g)^{\mathrm{t}}$ for all $g \in \mathbf{LB}_n$ is the zero matrix. We begin with the base case $n = 3$. Applying the argument for $n = 2$ to the top-left and bottom-right 2×2 blocks of M, we deduce that M must be of the form $\left[\begin{smallmatrix} a & -a & b \\ -a & a & -a \\ c & -a & a \end{smallmatrix}\right]$. Given this, the equation $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ then implies that $b = c = -a$, and the equation $\mathrm{Bur}(\sigma_1)M = M\mathrm{Bur}(\sigma_1)^{\mathrm{t}}$ implies that $a = at$, thus $a = 0$ and M is the zero matrix. For the inductive step, we may apply the inductive hypothesis to the top-left and bottom right $(n-1) \times (n-1)$ blocks of M to see that the entries of M are all zero except possibly the top-right and bottom-left entries. But then the equation $\mathrm{Bur}(\tau_1)M = M\mathrm{Bur}(\tau_1)^{\mathrm{t}}$ implies that these are zero too. □

Thus the representations (5.15) and $(5.15)^{\mathrm{tr}}$ are non-equivalent representations of $\mathbf{LB}_n$. However, we have constructed both of these topologically: (5.15) is the action of $\mathbf{LB}_n$ on the first homology group $H_1(\widetilde{\mathbb{D}}^3_n, \eta^{-1}(*); \mathbb{Z})$ and $(5.15)^{\mathrm{tr}}$ is the action of $\mathbf{LB}_n$ on the second homology group $H_2(\widetilde{\mathbb{D}}^3_n; \mathbb{Z})$, as shown at the end of §5.2.

Chapter 6

Actions of subgroups of MCGs on Heisenberg homology

The results of this chapter are accepted for publication as [BPS23] in joint work with Christian Blanchet and Awais Shaukat.

Introduction

In recent work [BPS21], we constructed a twisted action of the mapping class group of any compact, connected, oriented surface Σ with one boundary component on the homology of configuration spaces with local coefficients determined by a representation V of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$. The details of this construction are recalled briefly in §6.1. For specific representations V of $\mathcal{H}$ we were able to untwist and obtain genuine, untwisted linear representations of the mapping class group (for the linearisation $\mathcal{H} \oplus \mathbb{Z}$ of the affine translation action of $\mathcal{H}$ on itself) or linear representations of central extensions of the mapping class group (for the Schrödinger representation of $\mathcal{H}$).

Our goal here is to complete the study of this action on Heisenberg homology. In §6.2 we identify the kernel of the action of the mapping class group on the Heisenberg group as the Chillingworth subgroup (Proposition 6.2.6); hence we obtain a linear representation of this subgroup for any representation V of $\mathcal{H}$ (Theorem 6.2.12). We also identify the *projective kernel* of this action (i.e. the subgroup of elements that act by inner automorphisms) with the Torelli group (Proposition 6.2.7) and use this fact in §6.3 to obtain untwisted linear representations of the Torelli group for any V (Theorem 6.3.3). In the special case where V is the Schrödinger representation – where in [BPS21] we obtained an action of the stably universal central extension of the mapping class group – we show in §6.4 that, restricting to a so-called *Earle-Morita subgroup* defined in Definition 6.2.1, we obtain linear representations without passing to any extension (Theorem 6.4.14).

Using the local system given by the Schrödinger representation at an odd root of 1, De Renzi and Martel [DM22] produced a homological model for TQFT representations derived from quantum $sl(2)$. Our results shed light on a few points in their paper. First, Proposition 6.2.6 identifies a certain subgroup of the mapping class group denoted by $\mathcal{M}_g^{\mathbb{H}}$ in [DM22, Proposition 2.21] with the Chillingworth subgroup. Second, the construction of [DM22, §6.2] depends on the identification (Proposition 6.2.7) of the projective kernel of the action of the mapping class group on $\mathcal{H}$ with the Torelli group.

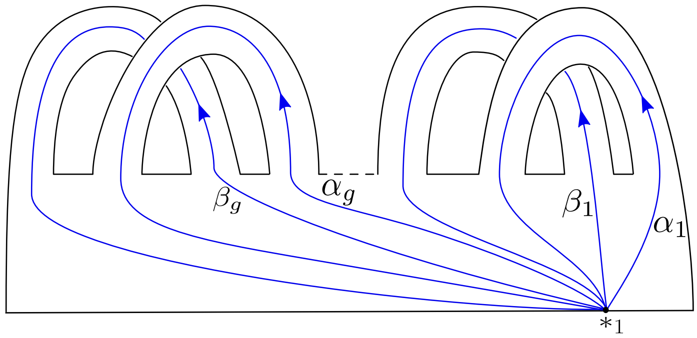


Figure 6.1 Model for Σ.

6.1 Twisted representations of the mapping class group

6.1.1 A review of Heisenberg homology

Let $\Sigma = \Sigma_{g,1}$, for $g \geqslant 1$, be a compact, connected, oriented surface of genus g with one boundary component. For $n \geqslant 2$, the unordered configuration space of n points in $\Sigma_{g,1}$ is

$$\mathcal{C}_n(\Sigma_{g,1}) = \{\{c_1, \ldots, c_n\} \subset \Sigma_{g,1} \mid c_i \neq c_j \text{ for } i \neq j\}.$$

The surface braid group is then defined as $\mathbb{B}_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma), *)$. A presentation for this group was first obtained by G. P. Scott [Sco70] and subsequently revisited by González-Meneses [Gon01] and Bellingeri [Bel04]. We fix a collection of based loops, $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$, as depicted in Figure 6.1. The base point $*_1$ belongs to the base configuration $*$. We will use the same notation α_r, β_s for the corresponding braids where only the first point is moving.

The braid group $\mathbb{B}_n(\Sigma)$ has generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$, together with the classical generators $\sigma_1, \ldots, \sigma_{n-1}$, and relations:

$$\begin{cases} \textbf{(BR1)}\ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geqslant 2, \\ \textbf{(BR2)}\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \textbf{(CR1)}\ [\zeta, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } \zeta \text{ among the } \alpha_r, \beta_s, \\ \textbf{(CR2)}\ [\zeta, \sigma_1\zeta\sigma_1] = 1 & \text{for all } \zeta \text{ among the } \alpha_r, \beta_s, \\ \textbf{(CR3)}\ [\zeta, \sigma_1^{-1}\eta\sigma_1] = 1 & \text{for all } \zeta \neq \eta \text{ among the } \alpha_r, \beta_s, \text{ with} \\ & \{\zeta, \eta\} \neq \{\alpha_r, \beta_r\}, \\ \textbf{(SCR)}\ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases} \tag{6.1}$$

Composition of loops is written from right to left.

We will use the notation $x.y$ for the standard intersection form on $H_1(\Sigma; \mathbb{Z})$. The *Heisenberg group* $\mathcal{H}(\Sigma)$ is the central extension of the homology group $H_1(\Sigma; \mathbb{Z})$ defined using the 2-cocycle $(x, y) \mapsto x.y$. As a set, it is equal to $\mathbb{Z} \times H_1(\Sigma; \mathbb{Z})$, and the operation is given by

$$(k, x)(l, y) = (k + l + x.y, x + y). \tag{6.2}$$

We will often denote the Heisenberg group simply by $\mathcal{H} = \mathcal{H}(\Sigma)$ when the surface Σ under consideration is clear. We will use the notation a_r, b_s, for the homology classes of α_r, β_s. From the presentation (6.1) we deduced the following in [BPS21, §1].

Proposition 6.1.1. *For each $g \geqslant 1$ and $n \geqslant 2$, the quotient of the braid group $\mathbb{B}_n(\Sigma)$ by the subgroup $[\sigma_1, \mathbb{B}_n(\Sigma)]^N$ normally generated by the commutators $[\sigma_1, x]$, $x \in \mathbb{B}_n(\Sigma)$, is isomorphic to the Heisenberg group $\mathcal{H}(\Sigma)$. An isomorphism*

$$\mathbb{B}_n(\Sigma)/[\sigma_1, \mathbb{B}_n(\Sigma)]^N \cong \mathcal{H}(\Sigma) \tag{6.3}$$

is represented by the surjective homomorphism

$$\phi\colon \mathbb{B}_n(\Sigma) \longrightarrow \mathcal{H}(\Sigma)$$

sending each σ_i *to* $u = (1,0)$*,* α_r *to* $\tilde{a}_r = (0, a_r)$ *and* β_s *to* $\tilde{b}_s = (0, b_s)$*.*

From the homomorphism ϕ we obtain a regular covering $\widetilde{\mathcal{C}}_n(\Sigma)$ of the configuration space $\mathcal{C}_n(\Sigma)$. The homology of this covering space is the homology of $\mathcal{C}_n(\Sigma)$ with local coefficients defined by ϕ, which we call *Heisenberg homology* and denote by $H_*(\mathcal{C}_n(\Sigma), \mathbb{Z}[\mathcal{H}])$. It is equipped with a right $\mathbb{Z}[\mathcal{H}]$-module structure defined by deck transformations.

Let us denote by $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ the singular chain complex of the Heisenberg covering $\widetilde{\mathcal{C}}_n(\Sigma)$; this is a complex of right $\mathbb{Z}[\mathcal{H}]$-modules. Given a (left) representation $\rho\colon \mathcal{H} \to GL(V)$, the corresponding twisted homology is that of the complex

$$\mathcal{S}_*(\mathcal{C}_n(\Sigma); V) := \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V \tag{6.4}$$

This will be called the *Heisenberg homology* of surface configurations with coefficients in V.

We also consider the Borel-Moore homology

$$H_*^{BM}(\mathcal{C}_n(\Sigma); V) = \varprojlim_T H_*(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma) \setminus T; V), \tag{6.5}$$

where the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$. We denote by $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ the closed subspace of configurations containing at least one point in a fixed closed interval $\partial^-(\Sigma) \subset \partial\Sigma$. The relative Borel-Moore homology is defined similarly as

$$\begin{aligned} &H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) \\ &\qquad = \varprojlim_T H_*\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \setminus T); V\big). \end{aligned} \tag{6.6}$$

The following theorem computes the relative Borel-Moore homology as a module.

Theorem 6.1.2 ([BPS21, §2])**.** *Let* V *be any representation of the discrete Heisenberg group* $\mathcal{H}(\Sigma)$*. Then, for* $n \geqslant 2$*, there is an isomorphism of modules*

$$H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) \;\cong\; \bigoplus_{k \in \mathcal{K}} V.$$

Furthermore, this is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$*.*

There is also an explicit geometric description of the indexing set $\mathcal{K}$ of the direct sum decomposition of Theorem 6.1.2, which is explained in [BPS21, §2]; however, it will not be essential for the present chapter.

6.1.2 An action of the mapping class group on the Heisenberg group

The *mapping class group* of Σ, denoted by $\mathfrak{M}(\Sigma)$, is the group of orientation-preserving diffeomorphisms of Σ fixing the boundary pointwise, modulo isotopies relative to the boundary. The isotopy class of a diffeomorphism f is denoted by $[f]$. An orientation-preserving self-diffeomorphism $f\colon \Sigma \to \Sigma$ fixing the boundary pointwise gives a homeomorphism $\mathcal{C}_n(f)\colon \mathcal{C}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$, defined by $\{x_1, x_2, \ldots, x_n\} \mapsto \{f(x_1), f(x_2), \ldots, f(x_n)\}$. If we ensure that the basepoint configuration of $\mathcal{C}_n(\Sigma)$ is contained in $\partial\Sigma$, then it is fixed by $\mathcal{C}_n(f)$ and this in turn induces an automorphism $f_{\mathbb{B}_n(\Sigma)} = \pi_1(\mathcal{C}_n(f))\colon \mathbb{B}_n(\Sigma) \to \mathbb{B}_n(\Sigma)$, which depends only on the isotopy class $[f]$ of f. In [BPS21, §3] we proved:

Proposition 6.1.3. *There exists a unique automorphism* $f_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}$ *such that the following square commutes:*

$$\begin{array}{ccc} \mathbb{B}_n(\Sigma) & \xrightarrow{f_{\mathbb{B}_n(\Sigma)}} & \mathbb{B}_n(\Sigma) \\ \phi\downarrow & & \downarrow\phi \\ \mathcal{H} & \xrightarrow{f_{\mathcal{H}}} & \mathcal{H} \end{array} \tag{6.7}$$

Thus, there is an action of $\mathfrak{M}(\Sigma)$ on the Heisenberg group $\mathcal{H}$ given by

$$\Psi \colon f \mapsto f_{\mathcal{H}} \colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H}). \tag{6.8}$$

6.1.3 Twisted representations of the mapping class group

In [BPS21] we explained how to use the action (6.8) to obtain twisted representations of $\mathfrak{M}(\Sigma)$ for each representation V of $\mathcal{H}$ over a ring R and each integer $n \geqslant 2$. We recall this briefly here.

For a (left) representation $\rho \colon \mathcal{H} \to \mathrm{Aut}_R(V)$ and an automorphism $\tau \in \mathrm{Aut}(\mathcal{H})$, the τ-twisted representation $\rho \circ \tau$ is denoted by ${}_{\tau}V$. Also, for any representation V of $\mathcal{H}$, we denote the induced local system

$$\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} V$$

on the configuration space $\mathcal{C}_n(\Sigma)$ simply by V, by abuse of notation. We then write

$$\mathcal{V}_n(V) = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) \tag{6.9}$$

for the relative Borel-Moore homology with coefficients in this local system. In this notation, [BPS21, §4.1] explains that each mapping class $f \in \mathfrak{M}(\Sigma)$ induces an automorphism of $\mathcal{C}_n(\Sigma)$ covered by an isomorphism

$${}_{\tau \circ f_{\mathcal{H}}}V \longrightarrow {}_{\tau}V$$

of local systems for each $\tau \in \mathrm{Aut}(\mathcal{H})$. Taking relative Borel-Moore homology, we therefore obtain isomorphisms

$$\mathcal{V}_n\big({}_{\tau \circ f_{\mathcal{H}}}V\big) \longrightarrow \mathcal{V}_n\big({}_{\tau}V\big) \tag{6.10}$$

of R-modules. This may be described succinctly as a representation of the *action groupoid* associated to the action (6.8) of $\mathfrak{M}(\Sigma)$ on $\mathcal{H}$.

Definition 6.1.4. For a group G with a left action $a \colon G \to \mathrm{Sym}(X)$ on a set X, the *action groupoid* $\mathrm{Ac}(G \curvearrowright X)$ is the groupoid whose set of objects is $a(G)$, whose set of morphisms $\sigma \to \tau$ is the subset $a^{-1}(\tau^{-1}\sigma) \subseteq G$ and whose composition is given by multiplication in G.

Theorem 6.1.5 ([BPS21, Theorem A(b)])**.** *Associated to any representation V of $\mathcal{H}$ over R and any integer $n \geqslant 2$, there is a functor*

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Mod}_R \tag{6.11}$$

sending each object $\tau \colon \mathcal{H} \to \mathcal{H}$ to the R-module $\mathcal{V}_n({}_{\tau}V)$ and sending each morphism $f \colon \tau \circ f_{\mathcal{H}} \to \tau$ to the R-linear isomorphism (6.10)*.*

Remark 6.1.6. The basic strategy to upgrade this to an *untwisted* representation is to try to construct coefficient isomorphisms $V \cong {}_{f_{\mathcal{H}}}V$ for each f. Given this, one may then pre-compose (6.10) with the induced isomorphism of twisted homology groups to obtain *automorphisms* of $\mathcal{V}_n(V)$. We explained how to do this in [BPS21, §4.2] when V is the linearisation $L = \mathcal{H} \oplus \mathbb{Z}$ of the (affine) translation action of $\mathcal{H}$ on itself. We also explained in [BPS21, §5] how to do this – after passing to a certain central extension of $\mathfrak{M}(\Sigma)$ – when V is the Schrödinger representation of $\mathcal{H}$ (this is recalled briefly in §6.4.1). The goal of the present chapter is to explain how to untwist on the Torelli group $\mathfrak{T}(\Sigma) \subset \mathfrak{M}(\Sigma)$ for *any* representation V of $\mathcal{H}$, as well as how to untwist on Earle-Morita subgroups of $\mathfrak{M}(\Sigma)$ when V is the Schrödinger representation of $\mathcal{H}$ (without passing to any central extension).

6.2 Action on the Heisenberg group

The goal of this section is to study the action (6.8) of the mapping class group $\mathfrak{M}(\Sigma)$ on the Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, as well as the crossed homomorphism naturally associated to this action.

6.2.1 Automorphisms of the Heisenberg group

As a first step, we recall a semi-direct product decomposition of an index-2 subgroup of the automorphism group $\mathrm{Aut}(\mathcal{H})$.

Since the element $u = (1, 0)$ of $\mathcal{H}$ generates its centre, which is infinite cyclic, any automorphism of $\mathcal{H}$ must send it either to itself or its inverse. We denote the group of automorphisms of $\mathcal{H}$ that fix u by $\mathrm{Aut}^+(\mathcal{H})$. The structure of this group was studied in [BPS21, §3]. There is a natural homomorphism $\mathcal{L}\colon \mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$, where we write $H = H_1(\Sigma;\mathbb{Z})$, and a split short exact sequence

$$1 \longrightarrow H^1(\Sigma;\mathbb{Z}) \xrightarrow{\ j\ } \mathrm{Aut}^+(\mathcal{H}) \xrightarrow{\ \mathcal{L}\ } Sp(H) \longrightarrow 1 \tag{6.12}$$

where $j(c) = [(k, x) \mapsto (k + c(x), x)]$. The splitting gives a decomposition $\mathrm{Aut}^+(\mathcal{H}) \cong Sp(H) \ltimes H^1(\Sigma;\mathbb{Z})$, where the semi-direct product structure on the right-hand side is induced by the natural action of $Sp(H)$ on $\mathrm{Hom}(H,\mathbb{Z}) \cong H^1(\Sigma;\mathbb{Z})$. The projection onto the right-hand factor of this decomposition is a function $(-)^\diamond\colon \mathrm{Aut}^+(\mathcal{H}) \to H^1(\Sigma;\mathbb{Z}) \cong \mathrm{Hom}(H,\mathbb{Z})$ (which is not a group homomorphism) given by the assignment $\varphi \mapsto \varphi^\diamond = \mathrm{pr}_1(\varphi(0,-))$.

For a mapping class $f \in \mathfrak{M}(\Sigma)$, the map $f_{\mathcal{H}}$ of (6.7) is represented as follows:

$$f_{\mathcal{H}}\colon (k, x) \mapsto (k + \delta_f(x), f_*(x)), \tag{6.13}$$

where $\delta_f = (f_{\mathcal{H}})^\diamond \in H^1(\Sigma;\mathbb{Z}) \cong \mathrm{Hom}(H,\mathbb{Z})$. We proved in [BPS21, §3] that the map $\delta\colon \mathfrak{M}(\Sigma) \to H^1(\Sigma;\mathbb{Z})$ given by $f \mapsto \delta_f$ is a crossed homomorphism, meaning that

$$\delta_{g\circ f}(x) = \delta_f(x) + f^*(\delta_g)(x)\ .$$

We also identified this crossed homomorphism explicitly, by showing that it coincides with the combinatorially-defined crossed homomorphism $\mathfrak{d}$ constructed by Morita [Mor89a], who proved that it represents a generator of $H^1(\mathfrak{M}(\Sigma); H^1(\Sigma;\mathbb{Z}))$, which is infinite cyclic. In fact, a different crossed homomorphism ψ had been constructed somewhat earlier by Earle [Ear78], and turned out, in light of [Mor89a], also to represent a generator of $H^1(\mathfrak{M}(\Sigma); H^1(\Sigma;\mathbb{Z}))$. The precise relationship between the crossed homomorphisms $\mathfrak{d}$ and ψ was elucidated in [Kun09]. In §6.2.3 below, we discuss the relationship of $\mathfrak{d} = \delta$ to the *Trapp representation* [Tra92] (see Proposition 6.2.10).

Definition 6.2.1. The *Earle-Morita subgroup* $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ is defined to be the kernel of $\mathfrak{d}$.

Remark 6.2.2. The Earle-Morita subgroup is not a normal subgroup of $\mathfrak{M}(\Sigma)$, despite being a kernel; this is because it is a kernel of a *crossed* homomorphism. We remark also that there are many Earle-Morita subgroups, since the definition of $\mathfrak{d}$ (or, equivalently, the definition of δ) depends on a choice. The definition of δ above uses the splitting of the short exact sequence (6.12). Note that it depends on the choice of isomorphism (6.3) in Proposition 6.1.1, identifying the relevant quotient of the surface braid group with (an explicit model for) the Heisenberg group.

6.2.2 The Chillingworth subgroup

Recall that the *Torelli subgroup* $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ consists of those elements of the mapping class group whose natural action on $H_1(\Sigma;\mathbb{Z})$ is trivial. The restriction of the crossed homomorphism $\delta\colon f \mapsto \delta_f$ to the Torelli group is a homomorphism. We will first describe this homomorphism in relation with the action of the Torelli group on homotopy classes of vector fields. Recall that the set $\Xi(\Sigma)$ of homotopy classes of non-vanishing vector fields on Σ supports a natural simply transitive action of $H^1(\Sigma;\mathbb{Z})$ (in other words, an affine structure over $\mathbb{Z}$ with associated $\mathbb{Z}$-module $H^1(\Sigma;\mathbb{Z})$), and the action of $\mathfrak{M}(\Sigma)$ is compatible with this action. It follows that the Torelli group acts by translation on $\Xi(\Sigma)$, which defines a homomorphism $e\colon \mathfrak{T}(\Sigma) \to H^1(\Sigma;\mathbb{Z})$. A formula for $e(f)([\gamma])$, where γ is a regular curve, is given by the variation of the winding number. For convenience we recall some details about the winding number below.

Fix a Riemannian metric on Σ. A non-vanishing vector field X gives a trivialisation of the unit tangent bundle $T_1(\Sigma) \cong \Sigma \times S^1$. The winding number $\omega_X(\gamma)$ of a regular oriented curve γ

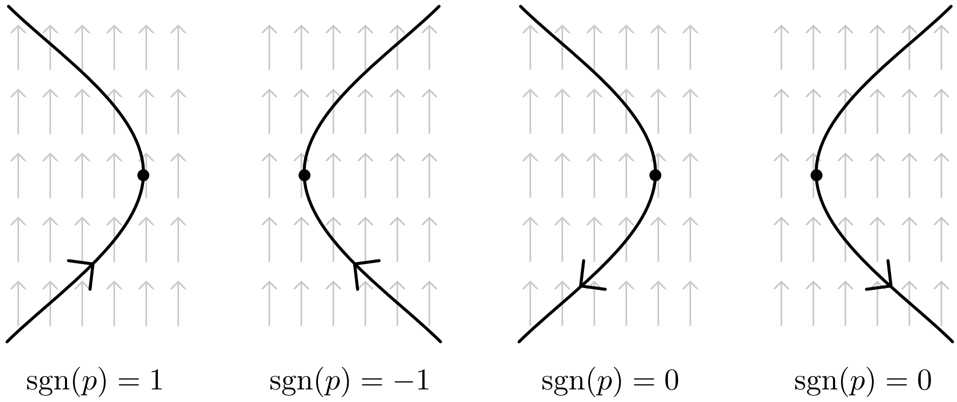


Figure 6.2 The sign of a point on γ that is tangent to X.

is the degree of the second component of the unit tangent vector. It can be computed as follows. Assuming that γ is transverse to X except at a finite set $\gamma \pitchfork X$ of points, where it looks locally as in Figure 6.2, then

$$\omega_X(\gamma) = \sum_{p \in \gamma \pitchfork X} \mathrm{sgn}(p),$$

where $\mathrm{sgn}(p)$ is defined in Figure 6.2. Notice that only the points p where the tangent vector of γ is *parallel* to X count towards this sum; those that point in the opposite direction to X do not.

Definition 6.2.3. The *Chillingworth homomorphism* $e\colon \mathfrak{T}(\Sigma) \to H^1(\Sigma;\mathbb{Z})$, studied in [Chi72; Joh80], is defined by

$$e(f)([\gamma]) = \omega_X(f \circ \gamma) - \omega_X(\gamma)\ . \tag{6.14}$$

Its kernel is the *Chillingworth subgroup* $\mathrm{Chill}(\Sigma)$. We note that e does not depend on the choice of non-vanishing vector field X, but extends to a crossed homomorphism $e_X\colon \mathfrak{M}(\Sigma) \to H^1(\Sigma;\mathbb{Z})$ that does, as we shall discuss in §6.2.3 (see Definition 6.2.9).

Remark 6.2.4. Since we have $\delta = \mathfrak{d}$ (as recalled in §6.2.1 above) and $e = \delta$ on the Torelli group (Lemma 6.2.5 below), we have:

$$\mathrm{Chill}(\Sigma) = \ker(e) = \mathfrak{T}(\Sigma) \cap \ker(\mathfrak{d}) = \mathfrak{T}(\Sigma) \cap \mathrm{Mor}(\Sigma).$$

Equivalently, we may say that $\mathrm{Chill}(\Sigma)$ is the intersection of the kernels of $\mathfrak{s}$ and $\mathfrak{d}$, in other words it is the kernel of $(\mathfrak{s},\mathfrak{d})\colon \mathfrak{M}(\Sigma) \to Sp(H) \ltimes H$. Notice in particular that, although the Earle-Morita subgroup $\mathrm{Mor}(\Sigma)$ depends on a non-canonical choice (see Remark 6.2.2), its intersection with the Torelli group does not.

The following lemma is Proposition 3.7 of [Bre02]. The proof there uses a result of [Mor93]. We give an independent proof below.

Lemma 6.2.5. *The homomorphisms δ and e coincide on the Torelli group and have image given by $\delta(\mathfrak{T}(\Sigma)) = 2.H^1(\Sigma;\mathbb{Z})$.*

From formula (6.13) it follows that the kernel of the action $\Psi\colon \mathfrak{M}(\Sigma) \to \mathrm{Aut}^+(\mathcal{H})$ is contained in the Torelli group; we may therefore identify this kernel as a corollary of Lemma 6.2.5.

Proposition 6.2.6. *For any genus $g \geqslant 1$, we have $\ker(\Psi) = \mathrm{Chill}(\Sigma)$.*

Proof. From formula (6.13) we see that $\ker(\Psi) = \mathfrak{T}(\Sigma) \cap \ker(\delta)$; by Lemma 6.2.5 this is equal to $\ker(e) = \mathrm{Chill}(\Sigma)$. □

Denote by $\mathrm{Inn}(\mathcal{H})$ the group of *inner* automorphisms of the Heisenberg group $\mathcal{H}$. From Lemma 6.2.5, we may also identify the *projective kernel* of the action Ψ, namely the subgroup $\Psi^{-1}(\mathrm{Inn}(\mathcal{H}))$ that acts by inner automorphisms.

Proposition 6.2.7. *For any genus $g \geqslant 1$, we have $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$.*

Proof. Conjugation in the Heisenberg group $\mathcal{H}$ is given by the formula

$$(l,x)(k,y)(-l,-x) = (l,y)(k,x)(-l,-x) = (k+2x.y,y) \tag{6.15}$$

First, if $\Psi(f) = f_{\mathcal{H}}$ is an inner automorphism, then its induced action on H must be trivial. This means that f lies in the Torelli group. Conversely, if $f \in \mathfrak{T}(\Sigma)$, we have from Lemma 6.2.5 that δ_f is in $2.H^1(\Sigma;\mathbb{Z})$. Using Poincaré duality, we obtain $x \in H$ such that $\delta_f(y) = 2x.y$ for every $y \in H$. Comparing formulas (6.13) and (6.15), we deduce that $f_{\mathcal{H}}$ is inner. □

We will use the following basic lemma about crossed homomorphisms in the proof of Lemma 6.2.5.

Lemma 6.2.8. *Let G be a group acting on an abelian group K and denote by $N \subseteq G$ the kernel of this action. Suppose that $S \subseteq N$ normally generates N in G. If two crossed homomorphisms $\theta_1, \theta_2 \colon G \to K$ agree on S, then they agree on N.*

Proof. The assumption that S normally generates N in G means that

$$T = \{gsg^{-1} \mid s \in S, g \in G\} \subseteq N$$

generates N. It will therefore suffice to show that θ_1 and θ_2 agree on T. Let $s \in S$ and $g \in G$. We know by hypothesis that $\theta_1(s) = \theta_2(s)$, and we need to show that $\theta_1(g^{-1}sg) = \theta_2(g^{-1}sg)$. First, observe that, for $i = 1, 2$, we have

$$\theta_i(g) + g.\theta_i(g^{-1}) = \theta_i(g^{-1}g) = \theta_i(1) = 0.$$

Using this, and the fact that $s \in N$, so it acts trivially on K, we deduce that

$$\begin{aligned}\theta_i(g^{-1}sg) &= \theta_i(g) + g.\theta_i(s) + gs.\theta_i(g^{-1}) \\ &= \theta_i(g) + g.\theta_i(s) + g.\theta_i(g^{-1}) \\ &= g.\theta_i(s).\end{aligned}$$

Thus $\theta_1(g^{-1}sg) = g.\theta_1(s) = g.\theta_2(s) = \theta_2(g^{-1}sg)$, as required. □

Proof of Lemma 6.2.5. The Torelli group is generated by genus-one bounding pair diffeomorphisms [Joh79, Theorem 2], and this generating set is a single conjugacy class in the full mapping class group. It follows that the Torelli group is normally generated by a(ny) single genus-one bounding pair diffeomorphism f. We may therefore apply Lemma 6.2.8 to the setting where $G = \mathfrak{M}(\Sigma)$, $K = H^1(\Sigma;\mathbb{Z})$, $N = \mathfrak{T}(\Sigma)$ and $S = \{f\}$. Both δ and e extend to crossed homomorphisms defined on the full mapping class group (for e such an extension is given by the Trapp representation, as recalled in §6.2.3 below). To prove that δ and e coincide on $\mathfrak{T}(\Sigma)$, it is therefore sufficient to show that δ and e agree on the single element f. Moreover, to show that $\delta(\mathfrak{T}(\Sigma)) \subseteq 2.H^1(\Sigma;\mathbb{Z})$ it is enough to show that this common value $\delta_f = e(f)$ lies in $2.H^1(\Sigma;\mathbb{Z})$. Specifically, we will take this element to be

$$f = BP(\gamma,\delta) = T_\gamma . T_\delta^{-1},$$

the genus one bounding pair diffeomorphism depicted in Figure 6.3, and we will show that the elements $e(f)$ and δ_f of $H^1(\Sigma;\mathbb{Z}) \cong \mathrm{Hom}(H,\mathbb{Z})$ are both equal to the homomorphism $H = H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$ given by

$$a_1 \mapsto 2 \quad , \quad a_i \mapsto 0 \text{ for } i \geqslant 2 \quad \text{and} \quad b_i \mapsto 0 \text{ for } i \geqslant 1. \tag{6.16}$$

As explained just above, this calculation will show that $\delta = e$ on $\mathfrak{T}(\Sigma)$ and that $\delta(\mathfrak{T}(\Sigma)) \subseteq 2.H^1(\Sigma;\mathbb{Z})$. The opposite inclusion will also follow, since $\delta_f = (6.16)$ is a free generator of $2.H^1(\Sigma;\mathbb{Z})$ and the other $2g-1$ elements of the evident free generating set may be realised similarly as $\delta_{f'}$ for analogous elements f'.

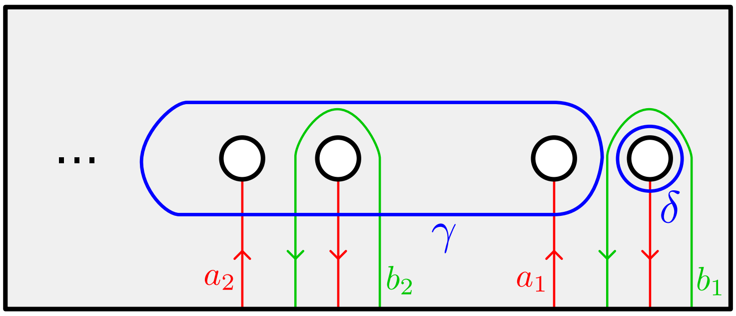


Figure 6.3 The surface Σ is obtained by identifying the $2g$ interior boundary components (four of which are depicted above) in g pairs by reflections. The bounding pair map from the proof of Lemma 6.2.5 is $BP(\gamma, \delta) = T_\gamma . T_\delta^{-1}$, for the blue curves γ and δ. The red and green arcs form a symplectic basis for the first homology of Σ relative to the bottom edge $\partial^-(\Sigma)$.

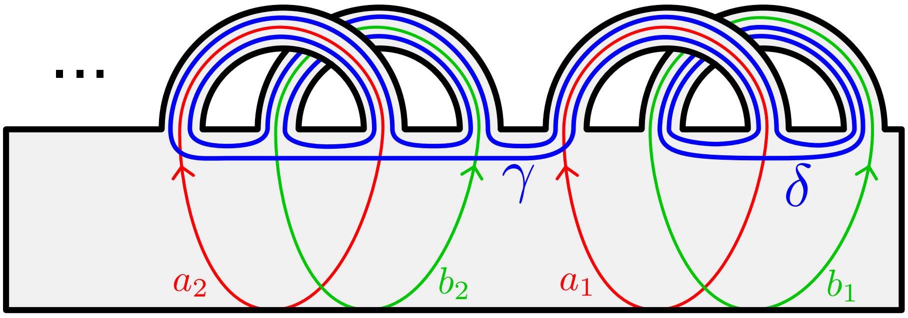


Figure 6.4 An alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis for the first homology of Σ relative to the bottom edge $\partial^-(\Sigma)$.

It therefore remains to calculate that $\delta_f = e(f) = (6.16)$.

We first calculate δ_f from the automorphism $f_{\mathcal{H}}$. We may directly read off from Figure 6.3 the effect of $f_{\mathcal{H}}$ on the elements $\tilde{a}_i$ and $\tilde{b}_i$ of $\mathcal{H}$. It clearly acts trivially except possibly on the three elements $\tilde{a}_2 = (0, a_2)$, $\tilde{b}_2 = (0, b_2)$ and $\tilde{a}_1 = (0, a_1)$, since the others may be realised disjointly from $\gamma \cup \delta$, and:

$$\begin{aligned} \tilde{a}_1 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 = u^2 \tilde{a}_1 = (2, a_1) \\ \tilde{a}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{a}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{a}_2 \\ \tilde{b}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{b}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{b}_2. \end{aligned}$$

This gives the calculation $\delta_f = (6.16)$.

To calculate $e(f)$, we use the alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis a_i, b_i for H depicted in Figure 6.4. This model for Σ has the advantage of having an obvious non-vanishing vector field X, which simply points *upwards* according to the standard framing of the page.

Using this vector field X and comparing to Figure 6.2, we observe that the winding numbers of the symplectic generators a_i and b_i (more precisely, their smooth, closed representatives pictured in Figure 6.4) are given by

$$\omega_X(a_i) = -1 \qquad \text{and} \qquad \omega_X(b_i) = +1.$$

We recall that, by definition, $e(f)(c) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) \in \mathbb{Z}$ for any $c = [\bar{c}] \in H$. We clearly have $f \circ \bar{c} = \bar{c}$ for $\bar{c} = a_i$ or b_i with $i \geqslant 3$ or for $\bar{c} = b_1$, since these curves may be represented disjointly from $\gamma \cup \delta$. Hence $e(f)([\bar{c}]) = 0$ for these $\bar{c}$.

The curve $f \circ a_1$ is depicted in Figure 6.5.

There are precisely three points on this curve where its tangent vector is equal to the vector field X, i.e., where its tangent vector is pointing vertically upwards: two are positive and one is

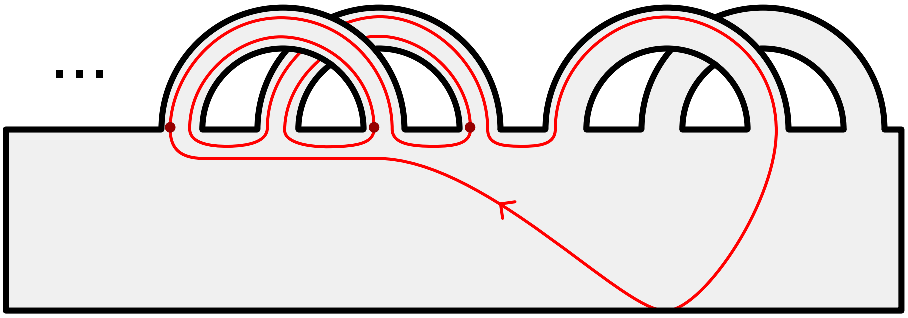


Figure 6.5 The curve $f \circ a_1$ for $f = T_\gamma . T_\delta^{-1}$. The three points where its tangent vector points vertically upwards are marked with dark red points: the left-most one is negative according to Figure 6.2, and the other two are positive. [At first sight it may look like there are two more, but these are not allowed since they do not fit either of the local models of Figure 6.2. We therefore perturb the curve slightly to get rid of these two tangencies with the vector field X. Alternatively, we may perturb it differently, to turn each of these disallowed tangencies into a pair of two allowed tangencies with opposite signs, which will therefore cancel in the expression for $\omega_X(f \circ a_1)$.]

negative (compare the local models in Figure 6.2), hence

$$\begin{aligned} e(f)(a_1) &= \omega_X(f \circ a_1) - \omega_X(a_1) \\ &= (2-1) - (-1) \\ &= 2. \end{aligned}$$

Now let $\bar{c}$ be either a_2 or b_2. In this case the effect of f is simply to conjugate $\bar{c}$ by γ, so we have that

$$\begin{aligned} \omega_X(f \circ \bar{c}) &= \omega_X(\gamma) + \omega_X(\bar{c}) - \omega_X(\gamma) \\ &= \omega_X(\bar{c}), \end{aligned}$$

since positive/negative tangencies with X for γ are negative/positive tangencies with X for γ^{-1} respectively, and so $e(f)([\bar{c}]) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) = 0$. Thus we have shown that $e(f) \colon H \to \mathbb{Z}$ is also given by (6.16). □

6.2.3 The Trapp representation

We next recall the *Trapp representation* [Tra92], and show that our representation (6.8) of $\mathfrak{M}(\Sigma)$ on $\mathcal{H}$ may be identified with it, up to "coboundaries", when the genus g of Σ is at least 2. This provides an alternative proof of Proposition 6.2.6 (except when $g = 1$), since the kernel of the Trapp representation is equal to the Chillingworth subgroup $\mathrm{Chill}(\Sigma)$ when $g \geqslant 2$ [Tra92, Corollary 2.7].

Definition 6.2.9. Write $H = H_1(\Sigma;\mathbb{Z})$ and $H^* = \mathrm{Hom}(H,\mathbb{Z}) \cong H^1(\Sigma;\mathbb{Z})$. The representation of Trapp [Tra92] is defined as a homomorphism

$$\Phi_X : \mathfrak{M}(\Sigma) \longrightarrow Sp(H) \ltimes H^* \subset GL_{2g+1}(\mathbb{Z}) \tag{6.17}$$

lifting the standard symplectic action $\mathfrak{s} \colon \mathfrak{M}(\Sigma) \to Sp(H)$. Having fixed this choice of symplectic action, the homomorphism (6.17) corresponds to a choice of crossed homomorphism

$$e_X : \mathfrak{M}(\Sigma) \longrightarrow H^*. \tag{6.18}$$

This crossed homomorphism is given by the variation of the winding number with respect to a fixed non-vanishing vector field X on Σ, as already discussed in §6.2.2; see the formula (6.14).

We therefore have two homomorphisms

$$\Psi = (6.8) \text{ and } \Phi_X = (6.17) \colon \mathfrak{M}(\Sigma) \longrightarrow Sp(H) \ltimes H^*$$

corresponding to crossed homomorphisms δ and $e_X \colon \mathfrak{M}(\Sigma) \to H^*$. We have proven above (Lemma 6.2.5) that these crossed homomorphisms are equal when restricted to the Torelli group. We now strengthen this to show that δ and e_X agree, modulo coboundaries, on the whole mapping class group.

Proposition 6.2.10. *For $g \geqslant 2$, the crossed homomorphisms δ and e_X represent the same cohomology class in $H^1(\mathfrak{M}(\Sigma); H^*) \cong \mathbb{Z}$. In other words, they are equal modulo principal crossed homomorphisms, i.e. coboundaries.*

Proof. We will use the homomorphism

$$H^1(\mathfrak{M}(\Sigma); H^*) \longrightarrow \mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \tag{6.19}$$

given by restricting a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^*$ to the Torelli group. This is well-defined since principal crossed homomorphisms (coboundaries) are trivial on the Torelli group. The right-hand side of (6.19) is rather large: by a theorem of Johnson [Joh85], the abelianisation of $\mathfrak{T}(\Sigma)$ is isomorphic to $\wedge^3 H \oplus (\text{torsion})$, so $\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \cong \mathrm{Hom}(\wedge^3 H, H^*)$, which is free abelian of rank $2g\binom{2g}{3}$. However, it has the advantage that it is easy to detect when its elements are equal, since it is just a group of homomorphisms (rather than crossed homomorphisms modulo principal ones). On the other hand, the left-hand side of (6.19) is much smaller. Indeed, Morita proved in [Mor89a, Proposition 6.4] that the group $H^1(\mathfrak{M}(\Sigma); H^*)$ is infinite cyclic. (In fact, it is generated by $[\mathfrak{d}]$, which we showed in [BPS21] is equal to $[\delta]$, but we will not need this.) In Lemma 6.2.5 we have proven that δ and e_X coincide, and are non-trivial, on the Torelli subgroup. Since $\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*)$ is torsion-free, the homomorphism (6.19) is injective and the result follows. ◻

Remark 6.2.11. In summary, we have considered three crossed homomorphisms

$$\delta, \mathfrak{d}, e_X \colon \mathfrak{M}(\Sigma) \longrightarrow H^* \cong H^1(\Sigma; \mathbb{Z}),$$

where δ is the crossed homomorphism corresponding to the action (6.8) of the mapping class group on the Heisenberg group, $\mathfrak{d}$ is Morita's crossed homomorphism (whose precise relationship to Earle's crossed homomorphism ψ is described in [Kun09]) and e_X is Chillingworth's crossed homomorphism, depending on a choice of non-vanishing vector field X on Σ. We showed in [BPS21] that $\delta = \mathfrak{d}$ on $\mathfrak{M}(\Sigma)$. In this section, we have shown (Lemma 6.2.5) that $\delta = e_X$ when restricted to $\mathfrak{T}(\Sigma)$ and, moreover, that $\delta = e_X$ on $\mathfrak{M}(\Sigma)$ modulo coboundaries (Proposition 6.2.10). We note, however, that only the weaker statement of Lemma 6.2.5 was needed to deduce (Propositions 6.2.6 and 6.2.7) that the kernel of Ψ is $\mathrm{Chill}(\Sigma)$ and the projective kernel of Ψ is $\mathfrak{T}(\Sigma)$.

6.2.4 Restricting to the Chillingworth subgroup

Using Proposition 6.2.6, we deduce that the twisted representations of $\mathfrak{M}(\Sigma)$ constructed in Theorem 6.1.5 are in fact untwisted when restricted to $\mathrm{Chill}(\Sigma) \subset \mathfrak{M}(\Sigma)$.

Theorem 6.2.12. *Associated to any representation V of $\mathcal{H}$ over R and any integer $n \geqslant 2$, there is a representation*

$$\mathrm{Chill}(\Sigma) \longrightarrow \mathrm{Aut}_R\bigl(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)\bigr), \tag{6.20}$$

which is a restriction of (6.11) to a single object of the action groupoid.

Proof. This follows from the construction described in §6.1.3, with each element $f \in \mathrm{Chill}(\Sigma)$ acting by the automorphism (6.10) (setting $\tau = \mathrm{id}$), since we know from Proposition 6.2.6 that $f_{\mathcal{H}} = \Psi(f) = \mathrm{id}$ for each $f \in \mathrm{Chill}(\Sigma)$. ◻

6.3 Representations of the Torelli group

We now restrict to the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$. By Proposition 6.2.7, the Torelli group acts by inner automorphisms under Ψ, so we have a homomorphism

$$\Psi \colon \mathfrak{T}(\Sigma) \longrightarrow \mathrm{Inn}(\mathcal{H}) \subset \mathrm{Aut}(\mathcal{H}). \tag{6.21}$$

We may therefore pull back the $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \cong \mathcal{Z}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow \mathrm{Inn}(\mathcal{H}) \to 1 \tag{6.22}$$

along (6.21) to obtain a $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \longrightarrow \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathfrak{T}(\Sigma) \to 1 \tag{6.23}$$

and a homomorphism

$$\widetilde{\Psi} \colon \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathcal{H} \tag{6.24}$$

lifting (6.21).

Remark 6.3.1. The inner automorphism group $\mathrm{Inn}(\mathcal{H})$ naturally identifies with the first homology group $H = H_1(\Sigma;\mathbb{Z})$. Under this identification, (6.21) becomes the crossed homomorphism $\delta \colon \mathfrak{T}(\Sigma) \to H^*$ (which is a *homomorphism* on the Torelli group) composed with the Poincaré duality isomorphism $H^* \cong H$. A 2-cocycle representing the extension (6.22) is the intersection form . on H. A 2-cocycle representing (6.23) is therefore given by $(f,f') \mapsto \delta(f)^\sharp.\delta(f')^\sharp$, where $(\ \ )^\sharp$ denotes Poincaré duality. As mentioned above, we showed in [BPS21] that $\delta = \mathfrak{d}$, so this 2-cocycle may also be written as $(f,f') \mapsto \mathfrak{d}(f)^\sharp.\mathfrak{d}(f')^\sharp$.

Recall from §6.1.1 that, for any choice of representation V of $\mathcal{H}$, the Heisenberg homology module $H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma,\partial^-(\Sigma)); V\big)$ is obtained from the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ of the Heisenberg covering $\widetilde{\mathcal{C}}_n(\Sigma)$ by:

- tensoring over $\mathbb{Z}[\mathcal{H}]$ with V (6.4);
- for each compact $T \subset \mathcal{C}_n(\Sigma)$, taking the quotient of this complex by the subcomplex corresponding to the subspace $\mathcal{C}_n(\Sigma,\partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \setminus T)$;
- passing to homology and then taking the inverse limit over T (6.6).

The isomorphism (6.10) (for $\tau = (f_{\mathcal{H}})^{-1}$) is induced by the natural twisted action of $f \in \mathfrak{M}(\Sigma)$ on $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$, which is of the form

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}}. \tag{6.25}$$

Now, for an element $h \in \mathcal{H}$, let us denote by $c_h = h{-}h^{-1}$ the corresponding inner automorphism $c_h \in \mathrm{Inn}(\mathcal{H})$. One may verify that the isomorphism

$$- \cdot h \colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{c_h} \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \tag{6.26}$$

of singular chain complexes given by the right-action of h is $\mathbb{Z}[\mathcal{H}]$-linear. For each $\widetilde{f} \in \widetilde{\mathfrak{T}}(\Sigma)$, we may then take the composition

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \xrightarrow{(6.25)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}} \xrightarrow{(6.26)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \ , \tag{6.27}$$

where f denotes the projection of $\widetilde{f}$ to $\mathfrak{T}(\Sigma)$ and we set $h = \widetilde{\Psi}(\widetilde{f}) \in \mathcal{H}$. The fact that $f_{\mathcal{H}} = c_h$ follows from the fact that (6.24) is a lift of (6.21). This defines an untwisted, $\mathbb{Z}[\mathcal{H}]$-linear action of $\widetilde{\mathfrak{T}}(\Sigma)$ on the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$. By the construction recalled above, this in turn induces an untwisted, R-linear action of $\widetilde{\mathfrak{T}}(\Sigma)$ on Heisenberg homology:

$$\widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathrm{Aut}_R\big(H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma,\partial^-(\Sigma)); V\big)\big). \tag{6.28}$$

To complete the construction, we show that:

Lemma 6.3.2. *The central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of $\mathfrak{T}(\Sigma)$ is trivial, i.e. it is isomorphic to the product $\mathfrak{T}(\Sigma) \times \mathbb{Z}$.*

Theorem 6.3.3. *Associated to any representation V of $\mathcal{H}$ over R and any integer $n \geqslant 2$, there is a well-defined representation of the Torelli group*

$$\mathfrak{T}(\Sigma) \longrightarrow \mathrm{Aut}_R\big(H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)\big) \tag{6.29}$$

that lifts a projective action of $\mathfrak{T}(\Sigma)$ on this homology module.

Proof. Let us abbreviate $\mathcal{V}_n(V) = H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$. The group homomorphism (6.28) must send the subgroup $\mathbb{Z} \subset \widetilde{\mathfrak{T}}(\Sigma)$ (the kernel of the central extension of $\mathfrak{T}(\Sigma)$) to the centre of $\mathrm{Aut}_R(\mathcal{V}_n(V))$, so it descends to

$$\mathfrak{T}(\Sigma) \longrightarrow \mathrm{PAut}_R(\mathcal{V}_n(V)), \tag{6.30}$$

where the projective automorphism group $\mathrm{PAut}_R(A)$ of an R-module A is the quotient of $\mathrm{Aut}_R(A)$ by its centre. Note that the centre of $\mathrm{Aut}_R(A)$ is equal to $\{- \cdot \lambda \mid \lambda \in \mathcal{Z}(R^\times)\}$ when A is a free R-module, but may be larger when A is not free. This is a projective action of the Torelli group. To lift it to a linear action, we compose (6.28) with any section of the central extension $\widetilde{\mathfrak{T}}(\Sigma) \to \mathfrak{T}(\Sigma)$, which exists by Lemma 6.3.2. ◻

To prove Lemma 6.3.2, we will need a lemma describing the behaviour of the crossed homomorphism $\mathfrak{d}$ with respect to increasing genus. Consider the inclusion of surfaces $\Sigma_{g,1} \subseteq \Sigma_{h,1}$ given by boundary connected sum with $\Sigma_{h-g,1}$. This induces an inclusion of mapping class groups

$$\mathfrak{M}(\Sigma_{g,1}) \longhookrightarrow \mathfrak{M}(\Sigma_{h,1}) \tag{6.31}$$

by extending diffeomorphisms by the identity on $\Sigma_{h-g,1}$.

Lemma 6.3.4 ([BPS21, §5.2])**.** *The diagram*

$$\begin{array}{ccc} \mathfrak{M}(\Sigma_{g,1}) & \xhookrightarrow{(6.31)} & \mathfrak{M}(\Sigma_{h,1}) \\ \mathfrak{d}\downarrow & & \downarrow\mathfrak{d} \\ H^1(\Sigma_{g,1};\mathbb{Z}) & \longrightarrow & H^1(\Sigma_{h,1};\mathbb{Z}) \end{array} \tag{6.32}$$

commutes, where the bottom arrow is the map induced by the inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_{h,1}$ on $H_1(-;\mathbb{Z})$, conjugated by Poincaré duality.

Proof of Lemma 6.3.2. We begin by showing that it suffices to prove the statement for all sufficiently large g; we will then be able to assume $g \geqslant 3$ in the rest of the proof. For $g < h$, consider the inclusion of Torelli groups

$$\iota\colon \mathfrak{T}(\Sigma_{g,1}) \longhookrightarrow \mathfrak{T}(\Sigma_{h,1}). \tag{6.33}$$

We claim that the pullback of the central extension $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ along (6.33) is $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$. To see this, recall from Remark 6.3.1 that the central extension $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$ of $\mathfrak{T}(\Sigma_{g,1})$ is represented by the 2-cocycle $(f, f') \mapsto \mathfrak{d}(f)^\sharp.\mathfrak{d}(f')^\sharp$. Similarly, the pullback of the central extension $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ of $\mathfrak{T}(\Sigma_{h,1})$ along the inclusion (6.33) is represented by the 2-cocycle $(f, f') \mapsto \mathfrak{d}(\iota(f))^\sharp.\mathfrak{d}(\iota(f'))^\sharp$. Lemma 6.3.4, together with the fact that the map $H_1(\Sigma_{g,1};\mathbb{Z}) \to H_1(\Sigma_{h,1};\mathbb{Z})$ preserves the intersection form, implies that these 2-cocycles are equal. Thus triviality of $\widetilde{\mathfrak{T}}(\Sigma_{h,1})$ will imply triviality of $\widetilde{\mathfrak{T}}(\Sigma_{g,1})$ for any $g < h$. For the remainder of this proof, we assume that $g \geqslant 3$ and abbreviate $\Sigma_{g,1}$ to Σ, as usual.

By [Ben+20, Lemma A.1(xiii)] and homological stability [Wah13, Theorem 1.2], the canonical surjection $\mathfrak{M}(\Sigma) \twoheadrightarrow Sp(H)$ induces an isomorphism on $H^2(-;\mathbb{Z})$ when $g \geqslant 3$. It follows that the inclusion $\mathfrak{T}(\Sigma) \hookrightarrow \mathfrak{M}(\Sigma)$ induces the trivial map on $H^2(-;\mathbb{Z})$. This means that every $\mathbb{Z}$-central extension of $\mathfrak{M}(\Sigma)$ becomes trivial when restricted to $\mathfrak{T}(\Sigma)$. To prove the lemma, it will therefore suffice to show that $\widetilde{\mathfrak{T}}(\Sigma)$ is the restriction of a $\mathbb{Z}$-central extension defined on the whole mapping class group $\mathfrak{M}(\Sigma)$. Recall from Remark 6.3.1 that the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of $\mathfrak{T}(\Sigma)$ is represented by the 2-cocycle c' given by $c'(f,f') = \mathfrak{d}(f)^\sharp.\mathfrak{d}(f')^\sharp$. We therefore just have to show that the 2-cocycle c' extends to $\mathfrak{M}(\Sigma)$.

Now, there is a 2-cocycle c on $\mathfrak{M}(\Sigma)$, defined by Morita [Mor89b], given by the formula $c(f,f') = \mathfrak{d}(f^{-1})^\sharp.\mathfrak{d}(f')^\sharp$. By general properties of crossed homomorphisms, we have $\mathfrak{d}(f^{-1})^\sharp = -f_*^{-1}(\mathfrak{d}(f)^\sharp)$, and so we may rewrite this as

$$c(f,f') = -f_*^{-1}(\mathfrak{d}(f)^\sharp).\mathfrak{d}(f')^\sharp = -\mathfrak{d}(f)^\sharp.f_*(\mathfrak{d}(f')^\sharp), \tag{6.34}$$

where for the second equality we have used the fact that the automorphism f_* of H preserves the intersection form. Restricted to the Torelli group, we have $f_* = \mathrm{id}$, so $c(f,f') = -\mathfrak{d}(f)^\sharp.\mathfrak{d}(f')^\sharp = -c'(f,f')$ for $f,f' \in \mathfrak{T}(\Sigma)$. Thus the 2-cocycle c' extends to the 2-cocycle $-c$ on $\mathfrak{M}(\Sigma)$. □

6.4 Representations of Earle-Morita subgroups

We now identify another large subgroup of the mapping class group $\mathfrak{M}(\Sigma)$ on which we may construct linear representations without passing to a central extension. We will do this in the setting where we take coefficients in the *Schrödinger representation* of $\mathcal{H}$ and the subgroup under consideration is the *Earle-Morita subgroup* of $\mathfrak{M}(\Sigma)$.

Recall from §6.2.1 that the *Earle-Morita subgroup* $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ is the kernel of the crossed homomorphism $\mathfrak{d}\colon \mathfrak{M}(\Sigma) \to H^1(\Sigma;\mathbb{Z})$ defined by Morita [Mor89a], and that this crossed homomorphism coincides with the one associated to the action $\mathfrak{M}(\Sigma) \to \mathrm{Aut}^+(\mathcal{H}) \cong Sp(H) \ltimes H^1(\Sigma;\mathbb{Z})$ from Proposition 6.1.3. We also recall, from Remark 6.2.2, that this crossed homomorphism, and its kernel $\mathrm{Mor}(\Sigma)$, depend on the parametrisation of the surface Σ.

An important representation of the Heisenberg group is the *Schrödinger representation*, which is parametrised by a non-zero real number $\hbar$ (called the Planck constant). It is given by the right action $\Pi_\hbar$ of $\mathcal{H}$ on the Hilbert space $W := L^2(\mathbb{R}^g)$ determined by the following formula:

$$\left[\Pi_\hbar\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](s) = e^{i\hbar\frac{k-p\cdot q}{2}} e^{i\hbar p\cdot s}\psi(s-q). \tag{6.35}$$

In fact, this is an action of the *continuous* Heisenberg group $\mathcal{H}_\mathbb{R}$, which is the central extension of $H_\mathbb{R} := H_1(\Sigma;\mathbb{R})$ by $\mathbb{R}$ corresponding to the intersection form. There is a natural inclusion $\mathcal{H} \subset \mathcal{H}_\mathbb{R}$. The Schrödinger representation is a unitary action on $W = L^2(\mathbb{R}^g)$, so it may be written as

$$\Pi_\hbar\colon \mathcal{H}_\mathbb{R} \longrightarrow U(W). \tag{6.36}$$

We recall also that the group $\mathrm{Aut}^+(\mathcal{H}_\mathbb{R})$ of automorphisms acting trivially on the centre of $\mathcal{H}_\mathbb{R}$ decomposes as $\mathrm{Aut}^+(\mathcal{H}_\mathbb{R}) \cong Sp(H_\mathbb{R}) \ltimes H^1(\Sigma;\mathbb{R})$, similarly to the decomposition of $\mathrm{Aut}^+(\mathcal{H})$ described in §6.2.1. From these decompositions we see that there is a natural inclusion $\mathrm{Aut}^+(\mathcal{H}) \subset \mathrm{Aut}^+(\mathcal{H}_\mathbb{R})$.

6.4.1 Untwisting on a central extension of the mapping class group

We first recall from [BPS21, §5] how to untwist the twisted representation (6.11) on the *stably universal central extension* of $\mathfrak{M}(\Sigma)$ when the $\mathcal{H}$-representation V is the Schrödinger representation. In §6.4.2–§6.4.4 we then explain how to untwist on the Earle-Morita subgroup *without* passing to a central extension.

As recalled in [BPS21, §5.1], an immediate corollary of the *Stone-von Neumann theorem* (see for example [LV80, p. 19]) is the following.

Corollary 6.4.1 (of the Stone-von Neumann theorem)**.** *Fix a positive real number $\hbar$ and let $\rho\colon \mathcal{H}_{\mathbb{R}} \to U(W)$ be an irreducible unitary representation whose restriction to the centre $\mathbb{R} \subset \mathcal{H}_{\mathbb{R}}$ is given by $\rho(t,0) = e^{\hbar i t/2}.\mathrm{id}_W$. Then there is an element $u \in U(W)$, unique up to rescaling by an element of S^1, such that $\rho = u.\Pi_{\hbar}.u^{-1}$.*

In particular, we may apply this result to the representation $\rho := \Pi_{\hbar} \circ \varphi$, for any automorphism $\varphi \in \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$. Sending φ to the element $u \in U(W)/S^1 = PU(W)$ provided by Corollary 6.4.1 defines a homomorphism

$$T\colon \mathrm{Aut}(\mathcal{H}_{\mathbb{R}}) \longrightarrow PU(W), \tag{6.37}$$

which is the *Segal-Shale-Weil projective representation*. Restricting this to the subgroup $\mathrm{Aut}^+(\mathcal{H}) \subset \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}) \subset \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$, we may pre-compose it with the action $\Psi\colon \mathfrak{M}(\Sigma) \to \mathrm{Aut}^+(\mathcal{H})$ from Proposition 6.1.3 to obtain a projective representation

$$\mathfrak{M}(\Sigma) \longrightarrow PU(W) \tag{6.38}$$

of the mapping class group. This is the key ingredient for untwisting the twisted representations of the mapping class group constructed in §6.1.3.

Definition 6.4.2. Let $\overline{\mathfrak{M}}(\Sigma)$ denote the central extension of $\mathfrak{M}(\Sigma)$ by S^1 pulled back from the central extension $U(W)$ of $PU(W)$ along (6.38). By construction, the projective representation (6.38) lifts to a linear representation

$$\overline{\mathfrak{M}}(\Sigma) \longrightarrow U(W) \tag{6.39}$$

on this central extension.

Definition 6.4.3. For $g \geqslant 4$, the mapping class group $\mathfrak{M}(\Sigma)$ is perfect and we have $H_2(\mathfrak{M}(\Sigma);\mathbb{Z}) \cong \mathbb{Z}$, so it has a universal central extension with kernel $\mathbb{Z}$. Let us denote this extension by $\widetilde{\mathfrak{M}}(\Sigma)$. For $h \geqslant g \geqslant 4$, the pullback of $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ along the inclusion (6.31) is $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$. Thus we may define, for $g \geqslant 1$, the *stably universal central extension* $\widetilde{\mathfrak{M}}(\Sigma_{g,1})$ of $\mathfrak{M}(\Sigma_{g,1})$ to be the pullback of the universal central extension $\widetilde{\mathfrak{M}}(\Sigma_{h,1})$ of $\mathfrak{M}(\Sigma_{h,1})$ along the inclusion (6.31), for any $h \geqslant 4$.

We note that there is a canonical morphism of central extensions

$$\widetilde{\mathfrak{M}}(\Sigma) \longrightarrow \overline{\mathfrak{M}}(\Sigma). \tag{6.40}$$

When $g \geqslant 4$ this morphism exists and is unique by universality of $\widetilde{\mathfrak{M}}(\Sigma)$. For $g \leqslant 3$ it may be pulled back from the $g \geqslant 4$ case via the inclusion (6.31), as explained in [BPS21, §5.3].

Using these ingredients, we showed in [BPS21, §5.3] how to untwist the twisted representation (6.11) of $\mathfrak{M}(\Sigma)$ when taking coefficients in the Schrödinger representation W of $\mathcal{H}$, after passing to the stably universal central extension $\widetilde{\mathfrak{M}}(\Sigma)$. Let us recall briefly how this works, following the philosophy of Remark 6.1.6. In the notation of §6.1.3, the construction of the twisted representation (6.11) provides isomorphisms

$$\mathcal{V}_n\big({}_{f_{\mathcal{H}}}V\big) \longrightarrow \mathcal{V}_n(V) \tag{6.41}$$

for each $f \in \mathfrak{M}(\Sigma)$. (This is simply (6.10) with $\tau = \mathrm{id}_{\mathcal{H}}$.) For each $\bar{f} \in \overline{\mathfrak{M}}(\Sigma)$ lifting $f \in \mathfrak{M}(\Sigma)$, its image under (6.39) is a unitary automorphism of W that intertwines the left Schrödinger action of $\mathcal{H}$, as long as we twist the action on the codomain by $f_{\mathcal{H}}$. In other words, it is a (unitary) isomorphism of left $\mathcal{H}$-representations of the form $W \cong {}_{f_{\mathcal{H}}}W$. This is a direct consequence of the defining property of the Segal-Shale-Weil projective representation from Corollary 6.4.1. This isomorphism of coefficients induces an isomorphism $\mathcal{V}_n(W) \cong \mathcal{V}_n({}_{f_{\mathcal{H}}}W)$, which we may compose with (6.41) (for $V = W$) to obtain:

Theorem 6.4.4 ([BPS21, §5])**.** *For any $n \geqslant 2$, there is a representation*

$$\overline{\mathfrak{M}}(\Sigma) \longrightarrow GL(\mathcal{V}_n(W)) \tag{6.42}$$

induced by the natural action of the mapping class group on (6.9) with coefficients in the Schrödinger representation $V = W$. Via the morphism (6.40), we may view this as a representation of the stably universal central extension of $\mathfrak{M}(\Sigma)$.

The purpose of this section is to show that, when we restrict to the Earle-Morita subgroup $\mathrm{Mor}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ and take coefficients in the Schrödinger representation $V = W$, we may obtain a representation of $\mathrm{Mor}(\Sigma)$ itself, *without* passing to any central extension.

We shall do this as follows. We first recall the *metaplectic extension* $\widehat{\mathfrak{M}}(\Sigma)$ of the mapping class group, which is an extension by $\mathbb{Z}/2$. Denoting by $\widehat{\mathrm{Mor}}(\Sigma)$ and $\overline{\mathrm{Mor}}(\Sigma)$ the restrictions of $\widehat{\mathfrak{M}}(\Sigma)$ and $\overline{\mathfrak{M}}(\Sigma)$ to the Earle-Morita subgroup, we show that $\overline{\mathrm{Mor}}(\Sigma)$ contains $\widehat{\mathrm{Mor}}(\Sigma)$ and that $\widehat{\mathrm{Mor}}(\Sigma)$ is a trivial extension. It will then follow that we may restrict (6.42) to $\widehat{\mathrm{Mor}}(\Sigma) \subset \overline{\mathrm{Mor}}(\Sigma) \subset \overline{\mathfrak{M}}(\Sigma)$ and pre-compose with a section of the trivial extension $\widehat{\mathrm{Mor}}(\Sigma)$ to obtain a representation of the Earle-Morita subgroup $\mathrm{Mor}(\Sigma)$.

6.4.2 Metaplectic extensions

We first consider two extensions of the symplectic group $Sp(H_{\mathbb{R}})$.

Definition 6.4.5. Recall that the fundamental group of $Sp(H_{\mathbb{R}}) \cong Sp_{2g}(\mathbb{R})$ is infinite cyclic. It therefore has a unique connected double covering group, which is called the *metaplectic group*, denoted by $Mp(H_{\mathbb{R}})$.

Definition 6.4.6. Consider the restriction of the projective representation (6.37) to the subgroup $Sp(H_{\mathbb{R}}) \subset Sp(H_{\mathbb{R}}) \ltimes H^1(\Sigma;\mathbb{Z}) \cong \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}) \subset \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$, which is a projective representation

$$Sp(H_{\mathbb{R}}) \longrightarrow PU(W) \tag{6.43}$$

of the symplectic group. It is in fact this restriction that is more usually referred to by the name *Segal-Shale-Weil projective representation.* Denote by $\overline{Sp}(H_{\mathbb{R}})$ the pullback of the central extension $U(W)$ of $PU(W)$. This is a central extension of $Sp(H_{\mathbb{R}})$ by S^1.

The main technical result of this subsection is the following:

Proposition 6.4.7. *There is an inclusion $Mp(H_{\mathbb{R}}) \subset \overline{Sp}(H_{\mathbb{R}})$ of central extensions, restricting to the inclusion $\mathbb{Z}/2 \cong \{\pm 1\} \subset S^1$ on fibres.*

Before proving this, we record its implications under pulling back to the mapping class group. We first define the relevant extensions of the mapping class group and its subgroups.

Definition 6.4.8. Denote by $\mathfrak{s}\colon \mathfrak{M}(\Sigma) \to Sp(H)$ the standard symplectic action of the mapping class group on $H = H_1(\Sigma;\mathbb{Z})$. The *metaplectic extension* $\widehat{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$ is defined to be its central extension by $\mathbb{Z}/2$ given by pulling back the $\mathbb{Z}/2$-central extension $Mp(H_{\mathbb{R}}) \to Sp(H_{\mathbb{R}})$ along $\mathfrak{s}$ and the inclusion $Sp(H) \subset Sp(H_{\mathbb{R}})$.

Definition 6.4.9. For a subgroup $G \subseteq \mathfrak{M}(\Sigma)$, we denote the restrictions of the central extensions $\overline{\mathfrak{M}}(\Sigma)$ and $\widehat{\mathfrak{M}}(\Sigma)$ to G by $\overline{G}$ and $\widehat{G}$ respectively.

Corollary 6.4.10. *There is an inclusion $\widehat{\mathrm{Mor}}(\Sigma) \subset \overline{\mathrm{Mor}}(\Sigma)$ of central extensions, restricting to the inclusion $\mathbb{Z}/2 \cong \{\pm 1\} \subset S^1$ on fibres.*

Proof. By definition, the central extension $\overline{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$ is pulled back from the central extension $U(W)$ of $PU(W)$ along the top row of the following diagram.

$$\begin{array}{ccccccc}
 & \overset{\Psi}{\nearrow} & \mathrm{Aut}^+(\mathcal{H}) & \hookrightarrow & \mathrm{Aut}^+(\mathcal{H}_{\mathbb{R}}) & \xrightarrow{T} & PU(W) \\
 & & \wr\Vert & & \wr\Vert & & \\
\mathfrak{M}(\Sigma) & \xrightarrow[(\mathfrak{s},\mathfrak{d})]{} & Sp(H) \ltimes H^1(\Sigma;\mathbb{Z}) & \hookrightarrow & Sp(H_{\mathbb{R}}) \ltimes H^1(\Sigma;\mathbb{R}) & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Mor}(\Sigma) & \xrightarrow{\mathfrak{s}} & Sp(H) & \hookrightarrow & Sp(H_{\mathbb{R}}) & &
\end{array}$$

Since this diagram commutes (for the bottom-left square this is because $\mathfrak{d} \equiv 0$ on $\mathrm{Mor}(\Sigma)$), it follows that its restriction $\overline{\mathrm{Mor}}(\Sigma)$ to the Earle-Morita subgroup is the pullback of the central extension $\overline{Sp}(H_{\mathbb{R}})$ of $Sp(H_{\mathbb{R}})$ along the bottom row of the diagram. On the other hand, the metaplectic extension $\widehat{\mathrm{Mor}}(\Sigma)$ is by definition the pullback of $Mp(H_{\mathbb{R}})$ along the bottom row of the diagram. Thus the inclusion of central extensions $Mp(H_{\mathbb{R}}) \subset \overline{Sp}(H_{\mathbb{R}})$ of $Sp(H_{\mathbb{R}})$ from Proposition 6.4.7 pulls back to the desired inclusion $\widehat{\mathrm{Mor}}(\Sigma) \subset \overline{\mathrm{Mor}}(\Sigma)$ of central extensions of $\mathrm{Mor}(\Sigma)$. ◻

Remark 6.4.11. The argument above does *not* show that the metaplectic extension includes into the S^1-extension pulled back via the Segal-Shale-Weil representation on the *full* mapping class group. This is because the inclusion of central extensions essentially arises at the level of the symplectic group (Proposition 6.4.7), so we have to restrict to the kernel of $\mathfrak{d}$ to ensure that the S^1-extension pulls back via the symplectic group.

Proof of Proposition 6.4.7. Let us first slightly rewrite the statement in notation that makes the dependence on g explicit: our goal is to prove that, over the group $Sp_{2g}(\mathbb{R})$, there is an embedding of central extensions $Mp_{2g}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$ (which must necessarily restrict to the inclusion $\mathbb{Z}/2 \cong \{\pm 1\} \subset S^1$ on fibres).

We first show that it suffices to prove this statement for all g sufficiently large; we will then be able to assume for the rest of the proof that $g \geqslant 4$, which is the stable range for (co)homology of degree at most 2 for $Sp_{2g}(\mathbb{R})$ and $\mathfrak{M}(\Sigma_{g,1})$. For any $g < h$ there is an inclusion map $Sp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2h}(\mathbb{R})$ given by extending symplectic automorphisms of $\mathbb{R}^{2g}$ by the identity on $\mathbb{R}^{2h-2g}$. We claim that the pullbacks of $Mp_{2h}(\mathbb{R})$ and of $\overline{Sp}_{2h}(\mathbb{R})$ under this inclusion are $Mp_{2g}(\mathbb{R})$ and $\overline{Sp}_{2g}(\mathbb{R})$ respectively. For the metaplectic central extensions this follows from the fact that the induced map $\pi_1(Sp_{2g}(\mathbb{R})) \cong \mathbb{Z} \to \mathbb{Z} \cong \pi_1(Sp_{2h}(\mathbb{R}))$ is an isomorphism and the metaplectic double covering corresponds to the unique index-2 subgroup of π_1. For $\overline{Sp}$, note that the Segal-Shale-Weil projective representations in genus g and h fit into a commutative square as follows:

$$\begin{array}{ccccc} Sp_{2g}(\mathbb{R}) & \xrightarrow{(6.43)} & PU(L^2(\mathbb{R}^g)) & \twoheadleftarrow & U(L^2(\mathbb{R}^g)) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Sp_{2h}(\mathbb{R}) & \xrightarrow{(6.43)} & PU(L^2(\mathbb{R}^h)) & \twoheadleftarrow & U(L^2(\mathbb{R}^h)) \end{array} \tag{6.44}$$

The right-hand side square of this diagram arises as follows. We consider $L^2(\mathbb{R}^g)$ as the closed subspace of $L^2(\mathbb{R}^h)$ consisting of those L^2-functions that factor through $\mathbb{R}^h = \mathbb{R}^g \times \mathbb{R}^{h-g} \twoheadrightarrow \mathbb{R}^g$. Any closed subspace of a Hilbert space has an orthogonal complement, so we may extend unitary automorphisms of $L^2(\mathbb{R}^g)$ by the identity on this complement to obtain a homomorphism $U(L^2(\mathbb{R}^g)) \to U(L^2(\mathbb{R}^h))$, which descends to the projective unitary groups, forming a pullback square. By definition, $\overline{Sp}_{2g}(\mathbb{R})$ is the pullback along (6.43) of the extension $U(L^2(\mathbb{R}^g))$ of $PU(L^2(\mathbb{R}^g))$. Commutativity of (6.44) then implies that the pullback of $\overline{Sp}_{2h}(\mathbb{R})$ along the inclusion is $\overline{Sp}_{2g}(\mathbb{R})$. Thus the existence of an embedding $Mp_{2h}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2h}(\mathbb{R})$ will imply the existence of an embedding $Mp_{2g}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$ for $g < h$. We henceforth assume that $g \geqslant 4$ in this proof (this is only needed in the last paragraph).

First, we recall from [LV80, §1.7] that a particular choice of cocycle

$$\omega_{Sp} \colon Sp_{2g}(\mathbb{R}) \times Sp_{2g}(\mathbb{R}) \longrightarrow S^1,$$

representing the central extension $\overline{Sp}_{2g}(\mathbb{R})$, takes values in the finite cyclic subgroup $\mathbb{Z}/8 \subseteq S^1$, so there is an embedding of central extensions $Sp_{2g}(\mathbb{R})^{(8)} \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$, for a certain $\mathbb{Z}/8$-central extension $Sp_{2g}(\mathbb{R})^{(8)}$ of $Sp_{2g}(\mathbb{R})$. Moreover, this central extension is classified by the element $-[\tau].8\mathbb{Z} \in H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z}/8)$, the reduction modulo 8 of the element $-[\tau] \in H^2(Sp_{2g}(\mathbb{R}); \mathbb{Z})$ represented by the negative of the *Maslov cocycle* τ (see formula 1.7.7 on page 70 of [LV80]).

Second, we also recall from [LV80, §1.7] that there is a function

$$s \colon Sp_{2g}(\mathbb{R}) \longrightarrow \mathbb{Z}/4 \subseteq S^1$$

such that $\omega_{Sp}(g,h)^2 = s(g)^{-1}s(h)^{-1}s(gh)$ (formula 1.7.8 on page 70 of [LV80]). It follows that the subset of $Sp_{2g}(\mathbb{R})^{(8)}$ of those pairs (t,g) for which $t^2 = s(g)$ is a subgroup. The projection onto $Sp_{2g}(\mathbb{R})$ restricted to this subgroup is a double covering, and so this subgroup must either be the trivial covering $Sp_{2g}(\mathbb{R}) \times \mathbb{Z}/2$ or the metaplectic covering $Mp_{2g}(\mathbb{R})$.

To finish the proof, we just have to show that it cannot be the trivial covering. Suppose for a contradiction that it is. Then $Sp_{2g}(\mathbb{R})^{(8)}$ admits a section, so it is a trivial extension and we must have $[\tau].8\mathbb{Z} = 0 \in H^2(Sp_{2g}(\mathbb{R});\mathbb{Z}/8)$. However, the pullback of $[\tau]$ along the projection $\mathfrak{M}(\Sigma) \to Sp_{2g}(\mathbb{R})$, also denoted by $[\tau]$, is precisely 4 times a generator of $H^2(\mathfrak{M}(\Sigma);\mathbb{Z}) \cong \mathbb{Z}$ (here we are using the assumption that $g \geqslant 4$). Thus $[\tau].8\mathbb{Z} \in H^2(\mathfrak{M}(\Sigma);\mathbb{Z}/8) \cong \mathbb{Z}/8$ is non-zero. Hence we must have $[\tau].8\mathbb{Z} \neq 0$ already in $H^2(Sp_{2g}(\mathbb{R});\mathbb{Z}/8)$. This completes the proof. ∎

6.4.3 Triviality of an extension

The last ingredient that we will need is the following.

Proposition 6.4.12. *The $\mathbb{Z}/2$-central extension $\widehat{\mathrm{Mor}}(\Sigma)$ of $\mathrm{Mor}(\Sigma)$ is trivial.*

Proof. We first note that it suffices to prove this statement for all sufficiently large g. This is because the inclusion of mapping class groups $\mathfrak{M}(\Sigma_{g,1}) \hookrightarrow \mathfrak{M}(\Sigma_{h,1})$ restricts to an inclusion of Earle-Morita subgroups $\mathrm{Mor}(\Sigma_{g,1}) \hookrightarrow \mathrm{Mor}(\Sigma_{h,1})$ (as an immediate consequence of Lemma 6.3.4), and the pullback of $\widehat{\mathrm{Mor}}(\Sigma_{h,1})$ along this inclusion is $\widehat{\mathrm{Mor}}(\Sigma_{g,1})$, for any $g < h$. This last statement follows from the fact that the pullback of $Mp_{2h}(\mathbb{R})$ along $Sp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2h}(\mathbb{R})$ is $Mp_{2g}(\mathbb{R})$, which was explained during the proof of Proposition 6.4.7. We now assume that $g \geqslant 4$ for the rest of the proof.

Recall from the proof of Proposition 6.4.7 that there is an embedding of central extensions $Mp_{2g}(\mathbb{R}) \hookrightarrow Sp_{2g}(\mathbb{R})^{(8)}$, where $Sp_{2g}(\mathbb{R})^{(8)}$ is a certain central extension of $Sp_{2g}(\mathbb{R})$ by $\mathbb{Z}/8$. Pulling back along the symplectic action $\mathfrak{M}(\Sigma) \to Sp_{2g}(\mathbb{R})$, we obtain an embedding of central extensions $\widehat{\mathfrak{M}}(\Sigma) \hookrightarrow \mathfrak{M}(\Sigma)^{(8)}$, where $\mathfrak{M}(\Sigma)^{(8)}$ is classified by $-[\tau].8\mathbb{Z} \in H^2(\mathfrak{M}(\Sigma);\mathbb{Z}/8) \cong \mathbb{Z}/8$. Now, $H^2(\mathfrak{M}(\Sigma);\mathbb{Z})$ is infinite cyclic, generated by the first Chern class c_1, and we have $[\tau] = 4c_1$. There is also a cocycle $c\colon \mathfrak{M}(\Sigma) \times \mathfrak{M}(\Sigma) \to \mathbb{Z}$ defined by Morita [Mor89b] given by the formula $c(f,f') = \mathfrak{d}(f^{-1})^\sharp.\mathfrak{d}(f')^\sharp$ (see also the proof of Lemma 6.3.2) and we have $[c] = 12c_1$ in $H^2(\mathfrak{M}(\Sigma);\mathbb{Z})$. Thus, in particular, we have $3[\tau] = [c]$. Since $\mathrm{Mor}(\Sigma) = \ker(\mathfrak{d})$, Morita's cocycle c vanishes on $\mathrm{Mor}(\Sigma)$, and so after restricting to the Earle-Morita subgroup we have $3[\tau] = [c] = 0 \in H^2(\mathrm{Mor}(\Sigma);\mathbb{Z})$. Reducing modulo 8 we therefore have $3[\tau].8\mathbb{Z} = 0 \in H^2(\mathrm{Mor}(\Sigma);\mathbb{Z}/8)$. But this cohomology group is a $\mathbb{Z}/8$-module, and 3 is invertible modulo 8, so we may divide by 3 and deduce that $[\tau].8\mathbb{Z} = 0$ in $H^2(\mathrm{Mor}(\Sigma);\mathbb{Z}/8)$. Hence the restriction $\mathrm{Mor}(\Sigma)^{(8)}$ of $\mathfrak{M}(\Sigma)^{(8)}$ to the Earle-Morita subgroup $\mathrm{Mor}(\Sigma)$ is a trivial extension. It therefore follows from the embedding $\widehat{\mathrm{Mor}}(\Sigma) \hookrightarrow \mathrm{Mor}(\Sigma)^{(8)}$ that $\widehat{\mathrm{Mor}}(\Sigma)$ is also a trivial extension. ∎

Remark 6.4.13. In summary, we have considered, in this subsection and the previous one, three nested central extensions $Mp(H_\mathbb{R}) \subset Sp(H_\mathbb{R})^{(8)} \subset \overline{Sp}(H_\mathbb{R})$ of the symplectic group $Sp(H_\mathbb{R})$ with fibres $\mathbb{Z}/2 \subset \mathbb{Z}/8 \subset S^1$. Clearly they are either all trivial or all non-trivial. We have seen that their pullbacks along the symplectic action $\mathfrak{M}(\Sigma) \to Sp(H_\mathbb{R})$ are non-trivial (and hence they must also be non-trivial to begin with), but their further pullbacks (restrictions) to the Earle-Morita subgroup $\mathrm{Mor}(\Sigma) \subset \mathfrak{M}(\Sigma)$ are trivial.

6.4.4 Untwisted representations of Earle-Morita subgroups

We may now conclude with the main result of this section:

Theorem 6.4.14. *For any $n \geqslant 2$, there is a representation*

$$\mathrm{Mor}(\Sigma) \longrightarrow GL(\mathcal{V}_n(W)) \qquad (6.45)$$

induced by the natural action of the mapping class group on the twisted Borel-Moore homology group (6.9) with coefficients in the Schrödinger representation $V = W$.

Proof. By Theorem 6.4.4, such a representation is defined on the central extension $\overline{\mathfrak{M}}(\Sigma)$ of the mapping class group by S^1. By Corollary 6.4.10, the restriction of this extension to $\mathrm{Mor}(\Sigma) \subset \mathfrak{M}(\Sigma)$ contains the metaplectic extension $\widehat{\mathrm{Mor}}(\Sigma)$, so we may further restrict to this subgroup. By Proposition 6.4.12, the central extension $\widehat{\mathrm{Mor}}(\Sigma)$ of $\mathrm{Mor}(\Sigma)$ is trivial, i.e., it admits a section. Hence, composing with any such section, we obtain the desired representation (6.45). ∎

Part III

Future directions

F.1 Stable homology of moduli spaces

There are several families of moduli spaces for which homological stability is known to hold, but the stable homology is not at all understood.

Configurations of higher-dimensional submanifolds. For example, as recalled in §O.1.2, homological stability for unordered configuration spaces $C_n(M)$ in an ambient manifold M that is connected and non-compact generalises to moduli spaces $C_{nL}(M)$ of higher-dimensional closed submanifolds of M that are isotopic to the disjoint union of finitely many parallel copies of a given $L \subset M$ (the case $L = \{*\}$ recovering the classical setting of point-configurations).

In contrast, the *stable homology*

$$\underset{n\to\infty}{\mathrm{colim}} \big(H_*(C_{nL}(M))\big) \tag{F.1}$$

is known when $L = \{*\}$ (by Segal [Seg73] and McDuff [McD75]) but is almost completely unknown whenever $\dim(L) > 0$. The first stable homology group of $C_{nS^1}(\mathbb{R}^3)$ (for the component of the unlink) is known to be isomorphic to $(\mathbb{Z}/2)^3$ by Brendle and Hatcher [BH13], who in fact calculated $\pi_1(C_{nS^1}(\mathbb{R}^3))$ for all n, but otherwise the stable homology groups (F.1) are very mysterious.

In particular, the techniques of [Seg73; McD75] involving *scanning maps* cannot apply (at least, not directly) for higher-dimensional L. For example, the naive generalisation of the scanning argument gives the wrong answer in the setting of $C_{nS^1}(\mathbb{R}^3)$, as one may check using the calculations of [BH13] mentioned above. More conceptually, scanning maps cannot capture all of the information of a positive-dimensional configuration, since they only "see" infinitesimally-extended objects and thus the global topology of the configuration is invisible to them as soon as it consists of more than just points.

The following question is therefore still wide open:

Question F.1.1. What are the stable homology groups (F.1) when $\dim(L) > 0$?

Moduli spaces of manifolds with conical singularities. As proven in Chapter 2 (see also Theorem J in §O.1.2), homological stability for configuration spaces of higher-dimensional submanifolds $C_{nL}(M)$ may be used to deduce homological stability for the moduli spaces $B\mathrm{Diff}^{\partial\mathcal{T}(L)}(M_{n\cdot L})$ of manifolds equipped with conical $\partial\mathcal{T}(L)$-singularities, as the number of singularities goes to infinity. One may ask the same question about their stable homology:

Question F.1.2. What are the stable homology groups $\underset{n\to\infty}{\mathrm{colim}} \big(H_*(B\mathrm{Diff}^{\partial\mathcal{T}(L)}(M_{n\cdot L}))\big)$?

Recall that the manifold-with-singularities $M_{n\cdot L}$ is constructed by collapsing a tubular neighbourhood of each copy of L to a point. This means that the global information about the diffeomorphism type (and isotopy class) of the embedded copies of L has been concentrated at the isolated singularities of $M_{n\cdot L}$. It may therefore be possible, in this setting, to successfully adapt the classical scanning techniques to answer Question F.1.2. Using the relationship between $B\mathrm{Diff}^{\partial\mathcal{T}(L)}(M_{n\cdot L})$ and $C_{nL}(M)$ (in the opposite direction to the logic of the proof of homological stability), this may in turn shed light on Question F.1.1.

F.2 Big mapping class groups

A number of open questions concerning big mapping class groups were asked in Chapter 3, some of which we recall here.

Question F.2.1 (Question 3.7.1)**.** Amongst infinite type surfaces S, is there a dichotomy between those for which $H_i(\mathrm{Map}(S))$ is *finitely generated* for all i and those for which $H_i(\mathrm{Map}(S))$ is *uncountable* for all i?

Remark. This is compatible with all currently known calculations, in particular with those of Theorems K and M of §O.1.3, although these only cover a few corners of the gamut of all possible infinite type surfaces.

Question F.2.2 (cf. Questions 3.0.11 and 3.0.12)**.** Is there an infinite type surface S for which $H_*(\mathrm{Map}(S))$ or $H_*(\mathrm{PMap}(S))$ contains an uncountable torsion subgroup?

Remark. All currently known methods for constructing uncountable subgroups of $H_*(\mathrm{Map}(S))$ (including [Dom22], [MT23], Chapter 3) produce subgroups isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Z}$ or $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ (where $\mathfrak{c}$ denotes the cardinality of the continuum). However, I expect that the answer to Question F.2.2 must surely be *yes*; its purpose is to highlight that such a basic question is still unknown.

The co̲homology of big (pure) mapping class groups is expected to have even larger cardinality:

Question F.2.3 (Question 3.7.6)**.** If S is an infinite type surface, does it follow that $H^i(\mathrm{PMap}(S))$ has cardinality $\geqslant 2^{\mathfrak{c}}$ (where $\mathfrak{c} =$ cardinality of the continuum) for each $i \geqslant 2$?

Remark. This is true if S has infinitely many non-planar ends, by Proposition 3.7.7. Note that Proposition 3.7.7 also implies the existence of an uncountable torsion subgroup of $H^*(\mathrm{PMap}(S))$ under this assumption. This does not help with Question F.2.2, however, since this uncountable torsion subgroup arises, via the universal coefficient theorem, from the Ext-group $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ of the two torsion-free groups $\mathbb{Z}^{\mathbb{N}}$ and $\mathbb{Z}$.

Question F.2.4 (Question 3.7.2)**.** Denote by $S_{g,1}$ the connected, compact, orientable surface of genus g with one boundary component and consider a Cantor subspace $\mathcal{C} \subset S_{g,1}$ of its interior. Is the natural map $\mathrm{Map}(S_{g,1} \smallsetminus \mathcal{C}) \to \mathrm{Map}(S_{g,1})$ a homology isomorphism?

Remark. As pointed out in Remark 3.7.3, this is known to be true in homological degree one (by [CC22, Theorem 2.3]) and in all degrees for $g = 0$ (by Theorem K in §O.1.3.1).

Compactly-supported homology classes. For any infinite-type surface S, one may consider the inclusion

$$\operatorname*{colim}_{\Sigma \subset S}(\mathrm{Map}(\Sigma)) = \mathrm{Map}_c(S) \longhookrightarrow \mathrm{Map}(S) \tag{F.2}$$

of the *compactly-supported mapping class group* (the colimit is taken over all compact subsurfaces $\Sigma \subset S$) and its induced map on homology

$$\operatorname*{colim}_{\Sigma \subset S}(H_*(\mathrm{Map}(\Sigma))) = H_*(\mathrm{Map}_c(S)) \longrightarrow H_*(\mathrm{Map}(S)). \tag{F.3}$$

For example, when S is the *Loch Ness monster surface* $L = \mathrm{colim}_{g \to \infty}(S_{g,1})$, the left-hand side of (F.3) is the *stable homology* of (compact, oriented) mapping class groups, which is understood by the Madsen-Weiss theorem [MW07], whereas the right-hand side is uncountable in every positive degree by Proposition 3.4.3.

Question F.2.5. For which infinite type surfaces S is the map (F.3) non-trivial, i.e., when does $H_*(\mathrm{Map}(S))$ contain non-trivial compactly-supported classes?

Remark. Ongoing joint work with Xiaolei Wu gives a partial answer to this question, which seems to be particularly subtle when S has genus zero.

F.3 Homological representations

The most famous — and extremely difficult — open question concerning (homological) representations is whether the mapping class groups $\mathrm{Map}(\Sigma_{g,1})$ are *linear*, i.e. admit faithful representations on finite-dimensional vector spaces (see [Mar19, §1]). This question is also open in general for the *surface braid groups* $\mathbf{B}_n(S)$ (except when S has very low genus [Big01; Kra02; GG12]) and for the *loop braid groups* $\mathbf{wB}_n$.

These wide-open questions are one motivation for the construction of new, homological representations of mapping class groups and motion groups. (*Homological*, since those representations that have proven themselves useful in answering the linearity question in the positive have, so far, all been homological in nature.) Other motivations are given by the following potential applications (phrased as speculative questions):

Question F.3.1. Can one construct *Chern-Simons theory* for $G = SU(2)$ purely in terms of the twisted homology of configuration spaces on surfaces?

Question F.3.2. Can one use new families of representations of braid groups to construct novel *polynomial link invariants* via Markov functions?

Question F.3.3. Can one use homological representations of the ribbon Thompson group (viewed as an asymptotically rigid mapping class group) to produce new link invariants via Jones' construction [Jon17; Jon19] of links from Thompson groups?

Bibliography

[Ang20] C. A.-M. Anghel. *Coloured Jones and Alexander polynomials as topological intersections of cycles in configuration spaces.* ArXiv:2002.09390. 2020 (↑ 10).

[AP20] C. A.-M. Anghel and M. Palmer. *Lawrence-Bigelow representations, bases and duality.* ArXiv: 2011.02388. 2020 (↑ 10).

[APV20] J. Aramayona, P. Patel and N. G. Vlamis. *The first integral cohomology of pure mapping class groups*. *Int. Math. Res. Not. IMRN* 2020.22 (2020), pp. 8973–8996 (↑ 59, 70, 71, 77).

[AV20] J. Aramayona and N. G. Vlamis. *Big mapping class groups: an overview. In the Tradition of Thurston: Geometry and Topology.* Ed. by K. Ohshika and A. Papadopoulos. Springer, 2020, pp. 459–496 (↑ 7, 59).

[Are46] R. Arens. *Topologies for homeomorphism groups*. *Amer. J. Math.* 68 (1946), pp. 593–610 (↑ 80).

[Arn70] V. I. Arnol'd. *Certain topological invariants of algebraic functions. Trudy Moskov. Mat. Obšč.* 21 (1970), pp. 27–46. (In Russian; English translation in Trans Moscow Math Soc, vol 21, 30–52, 1970) (↑ 31).

[AH88] M. Atiyah and N. Hitchin. *The geometry and dynamics of magnetic monopoles*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988, pp. viii+134 (↑ 18, 19).

[Baa73] N. A. Baas. *On bordism theory of manifolds with singularities*. *Math. Scand.* 33 (1973), 279–302 (1974) (↑ 51, 52).

[BWC07] J. C. Baez, D. K. Wise and A. S. Crans. *Exotic statistics for strings in 4D BF theory*. *Adv. Theor. Math. Phys.* 11.5 (2007), pp. 707–749 (↑ 147).

[Bar05] V. G. Bardakov. *Extending representations of braid groups to the automorphism groups of free groups*. *J. Knot Theory Ramifications* 14.8 (2005), pp. 1087–1098 (↑ 159).

[BB09a] V. G. Bardakov and P. Bellingeri. *Combinatorial properties of virtual braids*. *Topology Appl.* 156.6 (2009), pp. 1071–1082 (↑ 93).

[BB09b] V. G. Bardakov and P. Bellingeri. *On residual properties of pure braid groups of closed surfaces*. *Comm. Algebra* 37.5 (2009), pp. 1481–1490 (↑ 83, 114).

[BNS19] V. G. Bardakov, T. R. Nasybullov and M. Singh. *Automorphism groups of quandles and related groups*. *Monatsh. Math.* 189.1 (2019), pp. 1–21 (↑ 91).

[Bar+16] V. G. Bardakov, R. V. Mikhailov, V. V. Vershinin and J. Wu. *On the pure virtual braid group PV_3*. *Comm. Algebra* 44.3 (2016), pp. 1350–1378 (↑ 83).

[Bau10] G. Baumslag. *Some reflections on proving groups residually torsion-free nilpotent. I*. *Illinois J. Math.* 54.1 (2010), pp. 315–325 (↑ 113).

[Bel04] P. Bellingeri. *On presentations of surface braid groups*. *J. Algebra* 274.2 (2004), pp. 543–563 (↑ 106–108, 162).

[BB16] P. Bellingeri and A. Bodin. *The braid group of a necklace*. *Math. Z.* 283.3-4 (2016), pp. 995–1010 (↑ 150).

[BG16] P. Bellingeri and S. Gervais. *On p-almost direct products and residual properties of pure braid groups of nonorientable surfaces*. *Algebr. Geom. Topol.* 16.1 (2016), pp. 547–568 (↑ 83, 85).

[BGG08] P. Bellingeri, S. Gervais and J. Guaschi. *Lower central series of Artin-Tits and surface braid groups*. *J. Algebra* 319.4 (2008), pp. 1409–1427 (↑ 85, 93, 112, 114, 115, 119, 123).

[BGG17] P. Bellingeri, E. Godelle and J. Guaschi. *Abelian and metabelian quotient groups of surface braid groups*. *Glasg. Math. J.* 59.1 (2017), pp. 119–142 (↑ 85, 94, 119).

[BS20] P. Bellingeri and A. Soulié. *A note on representations of welded braid groups*. *J. Knot Theory Ramifications* 29.12, 2050082 (2020) (↑ 151).

[BM14] M. Bendersky and J. Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815 (↑ 5).

[Ben+20] D. Benson, C. Campagnolo, A. Ranicki and C. Rovi. *Signature cocycles on the mapping class group and symplectic groups*. *J. Pure Appl. Algebra* 224.11 (2020), pp. 106400, 49 (↑ 172).

[BP09] B. Berceanu and Ş. Papadima. *Universal representations of braid and braid-permutation groups*. *J. Knot Theory Ramifications* 18.7 (2009), pp. 999–1019 (↑ 83).

[BBF15] M. Bestvina, K. Bromberg and K. Fujiwara. *Constructing group actions on quasi-trees and applications to mapping class groups*. *Publ. Math. Inst. Hautes Études Sci.* 122 (2015), pp. 1–64 (↑ 65).

[Big99] S. Bigelow. *The Burau representation is not faithful for $n = 5$*. *Geom. Topol.* 3 (1999), pp. 397–404 (↑ 159).

[Big01] S. Bigelow. *Braid groups are linear*. *J. Amer. Math. Soc.* 14.2 (2001), pp. 471–486 (↑ 9, 10, 182).

[Big04] S. Bigelow. *Homological representations of the Iwahori-Hecke algebra*. *Proceedings of the Casson Fest*. Vol. 7. Geom. Topol. Monogr. Geom. Topol. Publ., Coventry, 2004, pp. 493–507 (↑ 151).

[BF81] E. Binz and H. R. Fischer. *The manifold of embeddings of a closed manifold. Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978)*. Vol. 139. Lecture Notes in Phys. With an appendix by P. Michor. Springer, Berlin-New York, 1981, pp. 310–329 (↑ 152).

[Bir69] J. S. Birman. *On braid groups*. *Comm. Pure Appl. Math.* 22 (1969), pp. 41–72 (↑ 103).

[Bir70] J. S. Birman. *Abelian quotients of the mapping class group of a 2-manifold*. *Bull. Amer. Math. Soc.* 76 (1970), pp. 147–150 (↑ 67).

[Bir74] J. S. Birman. *Braids, links, and mapping class groups.* Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974, pp. ix+228 (↑ 103, 104, 109).

[BPS21] C. Blanchet, M. Palmer and A. Shaukat. *Heisenberg homology on surface configurations.* ArXiv:2109.00515. 2021 (↑ iv, vi, 12, 13, 161–165, 170–174).

[BPS23] C. Blanchet, M. Palmer and A. Shaukat. *Action of subgroups of the mapping class group on Heisenberg homologies.* To appear in Contemporary Mathematics; ArXiv: 2306.08614. 2023 (↑ iv, vi, 13, 161).

[BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123 (↑ 5).

[Bot92] B. I. Botvinnik. *Manifolds with singularities and the Adams-Novikov spectral sequence*. Vol. 170. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1992, pp. xvi+181 (↑ 52).

[Bre02] T. Brendle. *The Torelli group and representations of mapping class groups*. Thesis (Ph.D.)–Columbia University. ProQuest LLC, Ann Arbor, MI, 2002, p. 112 (↑ 166).

[BH13] T. Brendle and A. Hatcher. *Configuration spaces of rings and wickets*. *Comment. Math. Helv.* 88.1 (2013), pp. 131–162 (↑ 36, 147–150, 158, 181).

[BM79] E. M. Brown and R. Messer. *The classification of two-dimensional manifolds*. *Trans. Amer. Math. Soc.* 255 (1979), pp. 377–402 (↑ 100).

[Bro82] K. S. Brown. *Cohomology of groups.* Vol. 87. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. x+306 (↑ 65, 78).

[Bro62] M. Brown. *Locally flat imbeddings of topological manifolds*. *Ann. of Math. (2)* 75 (1962), pp. 331–341 (↑ 22).

[BL93] A. Brownstein and R. Lee. *Cohomology of the group of motions of n strings in 3-space*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991).* Vol. 150. Contemp. Math. Amer. Math. Soc., Providence, RI, 1993, pp. 51–61 (↑ 147).

[Bur35] W. Burau. *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg.* Vol. 11. 1. Springer. 1935, pp. 179–186 (↑ 148).

[Bus66] J. van Buskirk. *Braid groups of compact 2-manifolds with elements of finite order*. *Trans. Amer. Math. Soc.* 122 (1966), pp. 81–97 (↑ 85, 104, 109, 131, 134).

[CC21] D. Calegari and L. Chen. *Big mapping class groups and rigidity of the simple circle*. *Ergodic Theory Dynam. Systems* 41.7 (2021), pp. 1961–1987 (↑ 7, 9, 59, 75, 76).

[CC22] D. Calegari and L. Chen. *Normal subgroups of big mapping class groups*. *Trans. Amer. Math. Soc. Ser. B* 9 (2022), pp. 957–976 (↑ 7, 9, 59, 76, 182).

[CP15] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. *Doc. Math.* 20 (2015), pp. 753–805 (↑ 5, 23).

[Cer61] J. Cerf. *Topologie de certains espaces de plongements*. *Bull. Soc. Math. France* 89 (1961), pp. 227–380 (↑ 45, 151).

[Chi72] D. R. J. Chillingworth. *Winding numbers on surfaces. II*. *Math. Ann.* 199 (1972), pp. 131–153 (↑ 166).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504 (↑ 5).

[Coh+91] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram. *The topology of rational functions and divisors of surfaces*. *Acta Math.* 166.3-4 (1991), pp. 163–221 (↑ 4, 17, 19).

[Coh+93] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram. *The homotopy type of rational functions*. *Math. Z.* 213.1 (1993), pp. 37–47 (↑ 19).

[CLM76] F. R. Cohen, T. J. Lada and J. P. May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976, pp. vii+490 (↑ 4, 17, 19).

[CDG13] G. Collinet, A. Djament and J. T. Griffin. *Stabilité homologique pour les groupes d'automorphismes des produits libres*. *Int. Math. Res. Not. IMRN* 19 (2013), pp. 4451–4476 (↑ 51).

[Col89] D. J. Collins. *Cohomological dimension and symmetric automorphisms of a free group*. *Comment. Math. Helv.* 64.1 (1989), pp. 44–61 (↑ 147).

[Dah62] D. M. Dahm. *A generalization of braid theory*. Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, 1962 (↑ 147).

[Dam17] C. Damiani. *A journey through loop braid groups*. *Expo. Math.* 35.3 (2017), pp. 252–285 (↑ 147, 158).

[DMR23] C. Damiani, P. Martin and E. C. Rowell. *Generalisations of Hecke algebras from loop braid groups*. *Pacific J. Math.* 323.1 (2023), pp. 31–65 (↑ 151).

[Dar21] J. Darné. *On the Andreadakis Problem for Subgroups of IA_n*. *Int. Math. Res. Not. IMRN* (Oct. 2021), pp. 14720–14742 (↑ 137).

[DPS22a] J. Darné, M. Palmer and A. Soulié. *Supplementary files containing GAP code used to support the conjectures in Remarks 4.4.75 and 4.4.82 and Conjecture 4.4.95*. Available at the URLs arxiv.org/src/2201.03542v3/anc or mdp.ac/papers/lcs. 2022 (↑ 127, 130, 137).

[DPS22b] J. Darné, M. Palmer and A. Soulié. *When the lower central series stops: a comprehensive study for braid groups and their relatives*. To appear in the Memoirs of the American Mathematical Society; ArXiv:2201.03542. 2022 (↑ iv, vi, 10, 83).

[DM22] M. De Renzi and J. Martel. *Homological Construction of Quantum Representations of Mapping Class Groups*. ArXiv:2212.10940. 2022 (↑ 161).

[Di 06] A. Di Concilio. *Topologizing homeomorphism groups of rim-compact spaces*. *Topology Appl.* 153.11 (2006), pp. 1867–1885 (↑ 80).

[Di 13] A. Di Concilio. *Topologizing homeomorphism groups*. *J. Funct. Spaces Appl.* (2013), Art. ID 894108, 14 (↑ 80).

[Dol62] A. Dold. *Decomposition theorems for $S(n)$-complexes*. *Ann. of Math. (2)* 75 (1962), pp. 8–16 (↑ 50, 58).

[Dom22] G. Domat. *Big pure mapping class groups are never perfect*. *Math. Res. Lett.* 29.3 (2022). Appendix with Ryan Dickmann, pp. 691–726 (↑ 59–62, 65–69, 71, 182).

[DP20] G. Domat and P. Plummer. *First cohomology of pure mapping class groups of big genus one and zero surfaces*. *New York J. Math.* 26 (2020), pp. 322–333 (↑ 59).

[Don84] S. K. Donaldson. *Nahm's equations and the classification of monopoles*. *Comm. Math. Phys.* 96.3 (1984), pp. 387–407 (↑ 4, 17, 18).

[DK92] G. Duchamp and D. Krob. *The lower central series of the free partially commutative group*. *Semigroup Forum* 45.3 (1992), pp. 385–394 (↑ 93).

[Ear78] C. J. Earle. *Families of Riemann surfaces and Jacobi varieties*. *Ann. of Math. (2)* 107.2 (1978), pp. 255–286 (↑ 165).

[EVW16] J. S. Ellenberg, A. Venkatesh and C. Westerland. *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*. *Ann. of Math. (2)* 183.3 (2016), pp. 729–786 (↑ 3).

[FB62] E. Fadell and J. van Buskirk. *The braid groups of E^2 and S^2*. *Duke Math. J.* 29 (1962), pp. 243–257 (↑ 104, 109).

[FH01] E. Fadell and S. Husseini. *Geometry and topology of configuration spaces*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001, pp. xvi+313 (↑ 19).

[FN62] E. Fadell and L. Neuwirth. *Configuration spaces*. *Math. Scand.* 10 (1962), pp. 111–118 (↑ 103–105).

[FR85] M. Falk and R. Randell. *The lower central series of a fiber-type arrangement*. *Invent. Math.* 82.1 (1985), pp. 77–88 (↑ 83, 85, 96).

[FR88] M. Falk and R. Randell. *Pure braid groups and products of free groups*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Amer. Math. Soc., Providence, RI, 1988, pp. 217–228 (↑ 83, 85).

[FRR97] R. Fenn, R. Rimányi and C. Rourke. *The braid-permutation group*. *Topology* 36.1 (1997), pp. 123–135 (↑ 147, 148).

[FW99] G. K. Francis and J. R. Weeks. *Conway's ZIP proof*. *Amer. Math. Monthly* 106.5 (1999), pp. 393–399 (↑ 101).

[FKS18] K. Fritzsch, C. Kottke and M. Singer. *Monopoles and the Sen Conjecture: Part I.* ArXiv:1811.00601. 2018 (↑ 22).

[Fuc70] L. Fuchs. *Infinite abelian groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. Academic Press, New York-London, 1970, pp. xi+290 (↑ 78).

[Fuc73] L. Fuchs. *Infinite abelian groups. Vol. II.* Pure and Applied Mathematics. Vol. 36-II. Academic Press, New York-London, 1973, pp. ix+363 (↑ 77, 78).

[Gal04] S. Galatius. *Mod p homology of the stable mapping class group*. *Topology* 43.5 (2004), pp. 1105–1132 (↑ 61).

[GAP] GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1* (↑ 127, 130, 137).

[God07] V. Godin. *Higher string topology operations.* ArXiv:0711.4859. 2007 (↑ 45).

[Gol73] C. H. Goldberg. *An exact sequence of braid groups*. *Math. Scand.* 33 (1973), pp. 69–82 (↑ 101, 102).

[Gol81] D. L. Goldsmith. *The theory of motion groups*. *Michigan Math. J.* 28.1 (1981), pp. 3–17 (↑ 147).

[GG04a] D. L. Gonçalves and J. Guaschi. *On the structure of surface pure braid groups*. *J. Pure Appl. Algebra* 186.2 (2004). Corrected reprint of: "On the structure of surface pure braid groups" [J. Pure Appl. Algebra **182** (2003), no. 1, 33–64; MR1977999], pp. 187–218 (↑ 105).

[GG04b] D. L. Gonçalves and J. Guaschi. *The braid groups of the projective plane*. *Algebr. Geom. Topol.* 4 (2004), pp. 757–780 (↑ 85).

[GG04c] D. L. Gonçalves and J. Guaschi. *The roots of the full twist for surface braid groups*. *Math. Proc. Cambridge Philos. Soc.* 137.2 (2004), pp. 307–320 (↑ 125).

[GG07] D. L. Gonçalves and J. Guaschi. *The braid groups of the projective plane and the Fadell-Neuwirth short exact sequence*. *Geom. Dedicata* 130 (2007), pp. 93–107 (↑ 85).

[GG09a] D. L. Gonçalves and J. Guaschi. *The lower central and derived series of the braid groups of the finitely-punctured sphere*. *J. Knot Theory Ramifications* 18.5 (2009), pp. 651–704 (↑ 85).

[GG09b] D. L. Gonçalves and J. Guaschi. *The lower central and derived series of the braid groups of the sphere*. *Trans. Amer. Math. Soc.* 361.7 (2009), pp. 3375–3399 (↑ 85).

[GG10] D. L. Gonçalves and J. Guaschi. *Braid groups of non-orientable surfaces and the Fadell-Neuwirth short exact sequence*. *J. Pure Appl. Algebra* 214.5 (2010), pp. 667–677 (↑ 105).

[GG11] D. L. Gonçalves and J. Guaschi. *The lower central and derived series of the braid groups of the projective plane*. *J. Algebra* 331 (2011), pp. 96–129 (↑ 85).

[GG12] D. L. Gonçalves and J. Guaschi. *Surface braid groups and coverings*. *J. Lond. Math. Soc. (2)* 85.3 (2012), pp. 855–868 (↑ 182).

[Gon01] J. González-Meneses. *New presentations of surface braid groups*. *J. Knot Theory Ramifications* 10.3 (2001), pp. 431–451 (↑ 162).

[GP04] J. González-Meneses and L. Paris. *Vassiliev invariants for braids on surfaces*. *Trans. Amer. Math. Soc.* 356.1 (2004), pp. 219–243 (↑ 103).

[GL69] E. A. Gorin and V. Y. Lin. *Algebraic equations with continuous coefficients, and certain questions of the algebraic theory of braids. Mat. Sb. (N.S.)* 78 (120) (1969), pp. 579–610 (↑ 85, 91).

[GM20] J. Guaschi and C. de Miranda e Pereiro. *Lower central and derived series of semi-direct products, and applications to surface braid groups*. *J. Pure Appl. Algebra* 224.7 (2020), pp. 106309, 39 (↑ 85).

[Har85] J. L. Harer. *Stability of the homology of the mapping class groups of orientable surfaces*. *Ann. of Math. (2)* 121.2 (1985), pp. 215–249 (↑ 2).

[HL02] R. Häring-Oldenburg and S. Lambropoulou. *Knot theory in handlebodies*. *Knot theory in handlebodies.* Vol. 11. 6. Knots 2000 Korea, Vol. 3 (Yongpyong). World Scientific (Singapore), 2002, pp. 921–943 (↑ 106).

[Hat83] A. Hatcher. *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$. *Ann. of Math. (2)* 117.3 (1983), pp. 553–607 (↑ 152).

[HW10] A. Hatcher and N. Wahl. *Stabilization for mapping class groups of 3-manifolds*. *Duke Math. J.* 155.2 (2010), pp. 205–269 (↑ 37).

[HEO05] D. F. Holt, B. Eick and E. A. O'Brien. *Handbook of computational group theory*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005, pp. xvi+514 (↑ 144).

[HP23] G. Horel and M. Palmer. *Motivic homological stability of configuration spaces*. *Bull. Lond. Math. Soc.* 55.2 (2023), pp. 892–913 (↑ 6).

[Ibr21] A. Ibrahim. *On matrix representations for generalized braid groups.* ArXiv:2109.03583. 2021 (↑ 155).

[Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (↑ 51).

[JMM06] C. Jensen, J. McCammond and J. Meier. *The integral cohomology of the group of loops*. *Geom. Topol.* 10 (2006), pp. 759–784 (↑ 147).

[Joh79] D. Johnson. *Homeomorphisms of a surface which act trivially on homology*. *Proc. Amer. Math. Soc.* 75.1 (1979), pp. 119–125 (↑ 167).

[Joh80] D. Johnson. *An abelian quotient of the mapping class group* $\mathcal{I}_g$. *Math. Ann.* 249.3 (1980), pp. 225–242 (↑ 166).

[Joh85] D. Johnson. *The structure of the Torelli group. III. The abelianization of* $\mathfrak{T}$. *Topology* 24.2 (1985), pp. 127–144 (↑ 170).

[Jon17] V. Jones. *Some unitary representations of Thompson's groups* F *and* T. *J. Comb. Algebra* 1.1 (2017), pp. 1–44 (↑ 183).

[Jon19] V. Jones. *On the construction of knots and links from Thompson's groups*. *Knots, low-dimensional topology and applications*. Vol. 284. Springer Proc. Math. Stat. Springer, Cham, 2019, pp. 43–66 (↑ 183).

[Kád+17] Z. Kádár, P. Martin, E. Rowell and Z. Wang. *Local representations of the loop braid group*. *Glasg. Math. J.* 59.2 (2017), pp. 359–378 (↑ 151).

[KT08] C. Kassel and V. Turaev. *Braid groups*. Vol. 247. Graduate Texts in Mathematics. With the graphical assistance of Olivier Dodane. Springer, New York, 2008, pp. xii+340 (↑ 149).

[Ker23] B. von Kerékjártó. *Vorlesungen über Topologie. I.: Flächentopologie.* Vol. 8. Grundlehren Math. Wiss. Springer, Cham, 1923 (↑ 7, 62).

[Knu17] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. *Algebr. Geom. Topol.* 17.5 (2017), pp. 3137–3187 (↑ 5).

[Koh85] T. Kohno. *Série de Poincaré-Koszul associée aux groupes de tresses pures*. *Invent. Math.* 82.1 (1985), pp. 57–75 (↑ 85).

[Kor02] M. Korkmaz. *Low-dimensional homology groups of mapping class groups: a survey*. *Turkish J. Math.* 26.1 (2002), pp. 101–114 (↑ 75, 83).

[KS22] C. Kottke and M. Singer. *Partial compactification of monopoles and metric asymptotics*. *Mem. Amer. Math. Soc.* 280.1383 (2022), pp. vii+110 (↑ 4, 17–22).

[Kra02] D. Krammer. *Braid groups are linear*. *Ann. of Math. (2)* 155.1 (2002), pp. 131–156 (↑ 9, 10, 182).

[Kra19] M. Krannich. *Homological stability of topological moduli spaces*. *Geom. Topol.* 23.5 (2019), pp. 2397–2474 (↑ 18, 25).

[Kre85] M. Kreck. *An extension of results of Browder, Novikov and Wall about surgery on compact manifolds*. Preprint, Universität Mainz. 1985 (↑ 33).

[Kre16] M. Kreck. *Thoughts about a good classification of manifolds*. *Proceedings of the Gökova Geometry-Topology Conference 2015*. Gökova Geometry/Topology Conference (GGT), Gökova, 2016, pp. 187–201 (↑ 33).

[Kun09] Y. Kuno. *A combinatorial formula for Earle's twisted 1-cocycle on the mapping class group* $\mathcal{M}_{g,*}$. *Math. Proc. Cambridge Philos. Soc.* 146.1 (2009), pp. 109–118 (↑ 165, 170).

[Kup20] A. Kupers. *Homological stability for unlinked circles in a 3-manifold*. *Q. J. Math.* 71.1 (2020), pp. 335–358 (↑ 35).

[KM16] A. Kupers and J. Miller. *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*. *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468 (↑ 6).

[KM18] A. Kupers and J. Miller. E_n*-cell attachments and a local-to-global principle for homological stability*. *Math. Ann.* 370.1-2 (2018), pp. 209–269 (↑ 23).

[Lab77] J. P. Labute. *The lower central series of the group* $< x, y : x^p = 1 >$. *Proc. Amer. Math. Soc.* 66.2 (1977), pp. 197–201 (↑ 141, 142).

[Law90] R. J. Lawrence. *Homological representations of the Hecke algebra*. *Comm. Math. Phys.* 135.1 (1990), pp. 141–191 (↑ 151).

[Laz54] M. Lazard. *Sur les groupes nilpotents et les anneaux de Lie*. *Ann. Sci. Ecole Norm. Sup. (3)* 71 (1954), pp. 101–190 (↑ 90).

[Lim63] E. L. Lima. *On the local triviality of the restriction map for embeddings*. *Comment. Math. Helv.* 38 (1963), pp. 163–164 (↑ 151).

[LV80] G. Lion and M. Vergne. *The Weil representation, Maslov index and theta series*. Vol. 6. Progress in Mathematics. Birkhäuser, Boston, Mass., 1980, pp. vi+337 (↑ 173, 176, 177).

[LP93] D. D. Long and M. Paton. *The Burau representation is not faithful for* $n \geqslant 6$. *Topology* 32.2 (1993), pp. 439–447 (↑ 159).

[MW07] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford's conjecture*. *Ann. of Math. (2)* 165.3 (2007), pp. 843–941 (↑ 2, 61, 182).

[Mag35] W. Magnus. *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*. *Math. Ann.* 111.1 (1935), pp. 259–280 (↑ 113).

[MKS04] W. Magnus, A. Karrass and D. Solitar. *Combinatorial group theory*. second. Presentations of groups in terms of generators and relations. Dover Publications, Inc., Mineola, NY, 2004, pp. xii+444 (↑ 83, 92, 113).

[MT23] J. Malestein and J. Tao. *Self-similar surfaces: involutions and perfection*. *Michigan Math. J.* (2023). DOI: 10.1307/mmj/20216114 (↑ 59, 62, 71, 182).

[Man97] S. Manfredini. *Some subgroups of Artin's braid group*. *Topology Appl.* 78.1-2 (1997). Special issue on braid groups and related topics (Jerusalem, 1995), pp. 123–142 (↑ 94).

[Man] Manifold Atlas Project Authors. *Parametric connected sum*. Page of the Manifold Atlas Project, available at map.mpim-bonn.mpg.de/Parametric_connected_sum (↑ 33).

[MR22] K. Mann and K. Rafi. *Two results on the end spaces of infinite type surfaces*. ArXiv: 2201.07690. 2022 (↑ 64).

[MR23] K. Mann and K. Rafi. *Large-scale geometry of big mapping class groups*. *Geom. Topol.* 27.6 (2023), pp. 2237–2296 (↑ 64).

[MT14] R. Manthorpe and U. Tillmann. *Tubular configurations: equivariant scanning and splitting*. *J. Lond. Math. Soc. (2)* 90.3 (2014), pp. 940–962 (↑ 29).

[Mar19] D. Margalit. *Problems, questions, and conjectures about mapping class groups*. *Breadth in contemporary topology*. Vol. 102. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2019, pp. 157–186 (↑ 182).

[MS20] S. Mazurkiewicz and W. Sierpiński. *Contribution à la topologie des ensembles dénombrables*. *Fundam. Math.* 1 (1920), pp. 17–27 (↑ 60).

[McC86] J. McCool. *On basis-conjugating automorphisms of free groups*. *Canad. J. Math.* 38.6 (1986), pp. 1525–1529 (↑ 147).

[MS76] D. McDuff and G. Segal. *Homology fibrations and the "group-completion" theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284 (↑ 3).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ 2, 3, 23, 29, 31, 181).

[ML88] R. J. Milgram and P. Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424 (↑ 5).

[MP15a] J. Miller and M. Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. *Q. J. Math.* 66.1 (2015), pp. 265–284 (↑ 3).

[MP15b] J. Miller and M. Palmer. *Scanning for oriented configuration spaces*. *Homology, Homotopy and Applications* 17.1 (2015), pp. 35–66 (↑ 2).

[Moo91] J. A. Moody. *The Burau representation of the braid group B_n is unfaithful for large n*. *Bull. Amer. Math. Soc. (N.S.)* 25.2 (1991), pp. 379–384 (↑ 159).

[Mor89a] S. Morita. *Families of Jacobian manifolds and characteristic classes of surface bundles. I*. *Ann. Inst. Fourier (Grenoble)* 39.3 (1989), pp. 777–810 (↑ 165, 170, 173).

[Mor89b] S. Morita. *Families of Jacobian manifolds and characteristic classes of surface bundles. II*. *Math. Proc. Cambridge Philos. Soc.* 105.1 (1989), pp. 79–101 (↑ 173, 177).

[Mor93] S. Morita. *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*. *Invent. Math.* 111.1 (1993), pp. 197–224 (↑ 166).

[Mum67] D. Mumford. *Abelian quotients of the Teichmüller modular group*. *J. Analyse Math.* 18 (1967), pp. 227–244 (↑ 75).

[Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328 (↑ 2).

[Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups*. ArXiv:1505.04294. 2015 (↑ 6).

[Nic96] W. Nickel. *Computing nilpotent quotients of finitely presented groups*. *Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994)*. Vol. 25. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1996, pp. 175–191 (↑ 127, 130, 137).

[Nun61] R. J. Nunke. *Slender groups*. *Bull. Amer. Math. Soc.* 67 (1961), pp. 274–275 (↑ 77).

[Pal60] R. S. Palais. *Local triviality of the restriction map for embeddings*. *Comment. Math. Helv.* 34 (1960), pp. 305–312 (↑ 44, 45, 47, 55, 151).

[Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711 (↑ 2).

[Pal17] M. Palmer. *A comparison of twisted coefficient systems.* ArXiv:1712.06310. 2017 (↑ 57).

[Pal18a] M. Palmer. *Stability for moduli spaces of manifolds with conical singularities.* ArXiv: 1807.07558. 2018 (↑ iii, v, 6, 7, 31).

[Pal18b] M. Palmer. *Twisted homological stability for configuration spaces*. *Homology, Homotopy and Applications* 20.2 (2018), pp. 145–178 (↑ 6, 18, 23–25, 31, 32, 35, 37, 53–57).

[Pal21] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. *Algebr. Geom. Topol* 21 (2021), pp. 1371–1444 (↑ iii, v, 6, 32, 34–37, 41–45, 47, 50, 55, 56, 58, 152).

[PS19] M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction.* To appear in the Annales Henri Lebesgue; ArXiv: 1910.13423. 2019 (↑ 11, 85, 151).

[PS22a] M. Palmer and A. Soulié. *The Burau representations of loop braid groups*. *C. R. Math. Acad. Sci. Paris* 360 (2022), pp. 781–797 (↑ iv, vi, 12, 147, 191).

[PS22b] M. Palmer and A. Soulié. *The pro-nilpotent Lawrence-Krammer-Bigelow representation.* ArXiv: 2211.01855. 2022 (↑ 11).

[PS22c] M. Palmer and A. Soulié. *Verifying the extended loop braid relations for the reduced Burau representation of* $\mathbf{LB}'_4$. Supplementary materials for the article [PS22a]; Available at the URLs mdp.ac/papers/loop-Burau or arxiv.org/src/2109.11468v2/anc. 2022 (↑ 158).

[PS23] M. Palmer and A. Soulié. *Polynomiality of surface braid and mapping class group representations.* ArXiv: 2302.08827. 2023 (↑ 12).

[PT21] M. Palmer and U. Tillmann. *Configuration-mapping spaces and homology stability*. *Res Math Sci* 8, 38 (2021) (↑ 3, 18, 24).

[PT22] M. Palmer and U. Tillmann. *Point-pushing actions for manifolds with boundary*. *Groups Geom. Dyn.* 16.4 (2022), pp. 1179–1224 (↑ 4, 24).

[PT23] M. Palmer and U. Tillmann. *Homology stability for asymptotic monopole moduli spaces*. *Proc. R. Soc. A* 479, 20230300 (2023) (↑ iii, v, 5, 17).

[PW22a] M. Palmer and X. Wu. *Big mapping class groups with uncountable integral homology.* To appear in Documenta Mathematica; ArXiv:2212.11942. 2022 (↑ iii, v, 8, 59).

[PW22b] M. Palmer and X. Wu. *On the homology of big mapping class groups.* ArXiv:2211.07470. 2022 (↑ iii, v, 8, 59, 61, 71, 76).

[PR00] L. Paris and D. Rolfsen. *Geometric subgroups of mapping class groups*. *J. Reine Angew. Math.* 521 (2000), pp. 47–83 (↑ 123).

[PV18] P. Patel and N. G. Vlamis. *Algebraic and topological properties of big mapping class groups*. *Algebr. Geom. Topol.* 18.7 (2018), pp. 4109–4142 (↑ 61, 69, 70).

[Per13] N. Perlmutter. *Cobordism categories of manifolds with Baas-Sullivan singularities, Part 2.* ArXiv:1306.4045. 2013 (↑ 36, 52).

[Per15] N. Perlmutter. *Cobordism category of manifolds with Baas-Sullivan singularities*. *Münster J. Math.* 8.1 (2015), pp. 119–167 (↑ 36, 52).

[Pow78] J. Powell. *Two theorems on the mapping class group of a surface*. *Proc. Amer. Math. Soc.* 68.3 (1978), pp. 347–350 (↑ 67).

[Qui68] D. G. Quillen. *On the associated graded ring of a group ring*. *J. Algebra* 10 (1968), pp. 411–418 (↑ 83).

[Rad25] T. Radó. *Über den Begriff der Riemannschen Fläche. Acta Litt. Sci. Szeged* 2 (1925), pp. 101–121 (↑ 100).

[Ran13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326 (↑ 2, 5, 23, 31).

[Ric63] I. Richards. *On the classification of noncompact surfaces*. *Trans. Amer. Math. Soc.* 106 (1963), pp. 259–269 (↑ 7, 62, 63, 100).

[Sco70] G. P. Scott. *Braid groups and the group of homeomorphisms of a surface*. *Proc. Cambridge Philos. Soc.* 68 (1970), pp. 605–617 (↑ 162).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ 2, 31, 181).

[Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (↑ 2, 4, 17, 19, 23, 31).

[SS96] G. Segal and A. Selby. *The cohomology of the space of magnetic monopoles*. *Comm. Math. Phys.* 177.3 (1996), pp. 775–787 (↑ 19).

[Sko07] A. Skopenkov. *A new invariant and parametric connected sum of embeddings*. *Fundamenta Mathematicae* 197 (2007), pp. 253–269 (↑ 33).

[Sul67] D. Sullivan. *On the Hauptvermutung for manifolds*. *Bull. Amer. Math. Soc.* 73 (1967), pp. 598–600 (↑ 51, 52).

[Til16] U. Tillmann. *Homology stability for symmetric diffeomorphism and mapping class groups*. *Math. Proc. Cambridge Philos. Soc.* 160.1 (2016), pp. 121–139 (↑ iii, v, 6, 31, 35–37, 39).

[Tra92] R. Trapp. *A linear representation of the mapping class group $\mathcal{M}$ and the theory of winding numbers*. *Topology Appl.* 43.1 (1992), pp. 47–64 (↑ 165, 169).

[Ver01] V. V. Vershinin. *On homology of virtual braids and Burau representation*. Vol. 10. 5. Knots in Hellas '98, Vol. 3 (Delphi). 2001, pp. 795–812 (↑ 150, 155, 156).

[Vla21] N. G. Vlamis. *Three perfect mapping class groups*. *New York J. Math.* 27 (2021), pp. 468–474 (↑ 59).

[Wah13] N. Wahl. *Homological stability for mapping class groups of surfaces*. *Handbook of moduli. Vol. III*. Vol. 26. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2013, pp. 547–583 (↑ 172).

[Wal66] C. T. C. Wall. *Classification problems in differential topology. IV. Thickenings*. *Topology* 5 (1966), pp. 73–94 (↑ 33).

[Wie69] J. Wiegold. $\mathrm{Ext}(Q, Z)$ *is the additive group of real numbers*. *Bull. Austral. Math. Soc.* 1 (1969), pp. 341–343 (↑ 77).

[Zee57] E. C. Zeeman. *A proof of the comparison theorem for spectral sequences*. *Proc. Cambridge Philos. Soc.* 53 (1957), pp. 57–62 (↑ 51).