Setup. We have $Z \supseteq A \xrightarrow{f \simeq g} Y$, with A a strong neighbourhood deformation retract in Z, and we would like to conclude that:

$$Z \cup_f Y \quad \text{is homotopy equivalent to} \quad Z \cup_g Y.$$

I will assume that A is closed in Z and that Z is normal, since the proof that I was able to write out in detail below needs these assumptions. Maybe there are weaker assumptions that work too.*

Proof. Let $h\colon A \times [0,1] \to Y$ be a homotopy between f and g. Define $\overline{W} = (Z \times [0,1]) \sqcup Y$ and $W = (Z \times [0,1]) \cup_h Y$. Let

$$q\colon \overline{W} \longrightarrow W$$

be the quotient map. Define subspaces of $\overline{W}$ as follows:

$$\overline{W}_0 = \big((Z \times \{0\}) \cup (A \times [0,1])\big) \sqcup Y$$
$$\overline{W}_1 = \big((Z \times \{1\}) \cup (A \times [0,1])\big) \sqcup Y$$

and note that $Z \cup_f Y \cong q(\overline{W}_0) \subseteq W$ and $Z \cup_g Y \cong q(\overline{W}_1) \subseteq W$. (To see these homeomorphisms, it helps to draw a picture of W.)

Claim: The fact that A is a strong neighbourhood deformation retract in Z means that there exists a strong deformation retraction of $\overline{W}$ onto $\overline{W}_0 \subset \overline{W}$. Similarly, there is a strong deformation retraction of $\overline{W}$ onto $\overline{W}_1 \subset \overline{W}$.

I'll prove this claim below. For now, note that applying q to these strong deformation retractions induces deformation retractions of W onto $q(\overline{W}_0)$ and onto $q(\overline{W}_1)$ respectively. (This is well-defined because the two strong deformation retractions each leave Y and $A \times [0,1]$ fixed at all times, so they are compatible with the equivalence relation corresponding to the partition of $\overline{W}$ into fibres of q.) Hence we have:

$$Z \cup_f Y \cong q(\overline{W}_0) \simeq W \simeq q(\overline{W}_1) \cong Z \cup_g Y,$$

which completes the proof, modulo the claim.

Proof of the claim. We need a strong deformation retraction of $Z \times [0,1]$ onto the subspace $(Z \times \{0\}) \cup (A \times [0,1])$. The idea of this deformation retraction is to compress $Z \times [0,1]$ into the subspace $Z \times [0, 1-t]$ at each time t — except that we only do this "far away" from the subspace $A \times [0,1]$ that should be fixed. Far away means outside of $U \times [0,1]$, where $U \supseteq A$ is an open neighbourhood of A in Z that strongly deformation retracts onto it. Inside $U \times [0,1]$ we have to interpolate between compressing in the $[0,1]$-direction and at the same time doing the deformation retraction of $U \times \{t\}$ onto $A \times \{t\}$ within each slice.

To make this explicit, first choose a specific strong deformation retraction $r\colon U \times [0,1] \to U$ of U onto A. So $r(-,0)$ is the identity, $r(a,t) = a$ for all $a \in A$ and $r(U,1) = A$. Now, recall that we assumed that A is closed in Z and that Z is a normal space. Urysohn's lemma implies that there exists a continuous function $\lambda\colon Z \to [0,1]$ such that $A \subseteq \lambda^{-1}(0)$ and $Z \smallsetminus U \subseteq \lambda^{-1}(1)$. Finally, choose a continuous function $\alpha = (\alpha_1, \alpha_2)\colon [0,1]^2 \times [0,1] \longrightarrow [0,1]^2$ such that

- $\alpha(s, 0, u) = (s, 0)$
- $\alpha(s, t, u) = (s(1-u), t)$ for $t > \frac{1}{2}$
- the image of $\alpha(-,-,1)$ is $([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$

Then we can define a strong deformation retraction

$$k\colon Z \times [0,1] \times [0,1] \longrightarrow Z \times [0,1]$$

of $Z \times [0,1]$ onto $(Z \times \{0\}) \cup (A \times [0,1])$ by

$$k(z,s,u) = \begin{cases} (z, s(1-u)) & z \in Z \smallsetminus U \\ (r(z, 1 - \alpha_2(s, \lambda(z), u)), \alpha_1(s, \lambda(z), u)) & z \in U. \end{cases}$$

*Note: *The definition I am using of* ***strong neighbourhood deformation retract*** *is that A has an open neighbourhood U in Z such that U strongly deformation retracts onto A. I think this is not quite equivalent to the standard definition (since the deformation retraction is defined only on U), which probably explains the need for the extra assumptions that A is closed in Z and that Z is normal in the proof above.*