This is a brief note (which arose during a seminar that I ran on the h-cobordism theorem following the book of Milnor) to explain one technical point in chapter 5. The context is the following. The goal of chapter 5 is to prove the "First Cancellation Theorem", which says that consecutive critical points of a Morse function of indices λ and $\lambda + 1$ respectively may be cancelled, as long as the "right-hand sphere" (in a given level set of the Morse function between the critical points) of the left-hand critical point has exactly one point of intersection with the "left-hand sphere" of the right-hand critical point. (Chapter 6 then uses the Whitney trick to weaken this hypothesis to requiring just that the algebraic intersection number is ± 1.) The first half of chapter 5 proves this theorem assuming something that Milnor calls Assertion 6, which claims that the (unique) trajectory of a certain vector field on the cobordism, which goes between the two critical points, admits a certain special local model. The second half of chapter 5 proves that it does always admit such a local model, by first constructing the local model near each of the critical points, and then trying to glue these two parts of the local model together by following trajectories of the vector field. If they happen to be compatible in this way, then we're done. Otherwise, we need to modify the vector field so that the two local models are compatible when following its trajectories. By an earlier lemma (Lemma 4.7 in the book), this will be possible as long as the diffeomorphism h between two particular level sets of the Morse function, induced by following the trajectories of the vector field, may be isotoped to have certain properties. The technical point arises in seeing how to apply Theorem 5.6 (which tells us how to construct certain isotopies of self-embeddings of Euclidean space) to show that such an isotopy of h exists. This occurs on page 58 of Milnor's book, and the aim of this brief note is just to clarify in more detail how one applies Theorem 5.6 to find an appropriate isotopy of h, to complete the proof of Assertion 6.

We will freely use the notation of chapter 5 of Milnor's book — any unexplained notation may be found there.

Let's start from the top of page 58. At this point we know that it suffices to find an isotopy of h to a new diffeomorphism $\bar{h}$ that coincides with h_0 near p_1 and such that $\bar{h}(S_R(b_1))$ intersects $S'_L(b_2)$ transversely (as submanifolds of $f^{-1}(b_2)$) at p_2 and nowhere else. Equivalently, it suffices to find an isotopy of $h_0^{-1}h$ to a new self-diffeomorphism $\ell\colon f^{-1}(b_1) \to f^{-1}(b_1)$ that coincides with the identity near p_1 and such that $\ell(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ intersect transversely (as submanifolds of $f^{-1}(b_1)$) at p_1 and nowhere else.

Note: from now on we will only ever consider transversality for submanifolds of $f^{-1}(b_1)$, so the word "transverse" will always mean "transverse as submanifolds of $f^{-1}(b_1)$".

Claim. After modifying g_1 if necessary, we may assume that $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ intersect transversely at p_1. Moreover, $h_0^{-1}h$ is orientation-preserving near p_1 and the intersection number of $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ (which is well-defined since they intersect transversely) is the same as the intersection number of $S_R(b_1)$ and $S'_L(b_1)$.

Notes:

- We do not assert that $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ intersect *only* at p_1, just that their intersection at p_1 is transverse.
- It is clear that they intersect at p_1, since $S_R(b_1)$ and $S'_L(b_1)$ intersect at p_1, and $h_0^{-1}h$ fixes this point. The non-trivial claim is that this intersection is transverse.
- Once we know that $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ intersect transversely at p_1, it follows that p_1 is an isolated point of $S_R(b_1) \cap h_0^{-1}h(S'_L(b_1))$, for dimension reasons.

We will get back to this claim later. First we will use it to show that Theorem 5.6 may be applied to prove Assertion 6, as Milnor claims.

Whenever two submanifolds $M, N \subset V$ intersect transversely in an isolated point x (which can only happen if $m + n = v$, where $\dim(M) = m$, $\dim(N) = n$ and $\dim(V) = v$), we can find a coordinate chart $V \supseteq \mathcal{U} \to \mathbb{R}^v$ of V such that x corresponds to 0, $\mathcal{U} \cap M$ corresponds to $\mathbb{R}^m \times \{(0, \ldots, 0)\}$ and $\mathcal{U} \cap N$ corresponds to $\{(0, \ldots, 0)\} \times \mathbb{R}^n$.

Using the Claim above, we may therefore choose a coordinate chart $f^{-1}(b_1) \supseteq \mathcal{U} \to \mathbb{R}^n$ of $f^{-1}(b_1)$ such that $\mathcal{U} \cap S_R(b_1)$ corresponds to $\mathbb{R}^a = \mathbb{R}^a \times \{(0, \ldots, 0)\}$ and $\mathcal{U} \cap h_0^{-1}h(S'_L(b_1))$ corresponds to $\mathbb{R}^b = \{(0, \ldots, 0)\} \times \mathbb{R}^b$.

It might not be true that $h_0^{-1}h(\mathcal{U}) \subseteq \mathcal{U}$, so we can't say that $h_0^{-1}h$ restricts to a smooth embedding $\mathcal{U} \to \mathcal{U}$. However, if we write $\mathcal{U}_\epsilon$ for the subset of $\mathcal{U}$ corresponding via the coordinate chart to the open ball of radius ϵ in $\mathbb{R}^n$, we can choose $\epsilon > 0$ sufficiently small that $h_0^{-1}h(\mathcal{U}_\epsilon) \subseteq \mathcal{U}$. Now, via the coordinate chart, $h_0^{-1}h$ restricts to a smooth embedding $k\colon B_\epsilon(0) \to \mathbb{R}^n$, where $B_\epsilon(0)$ is the open ball in $\mathbb{R}^n$ around 0 of radius ϵ. Now we can either apply Theorem 5.6 directly to this embedding — which is valid since Theorem 5.6 is really about the behaviour of embeddings in a small neighbourhood of zero — or, if we want to be more strict and really start with a smooth embedding $\mathbb{R}^n \to \mathbb{R}^n$, then we can choose a diffeomorphism $\mathbb{R}^n \cong B_\epsilon(0)$ that expands radially using a choice of diffeomorphism $[0,\infty) \cong [0,\epsilon)$.

Denote the intersection number of $S_R(b_1)$ and $S'_L(b_1)$ by $\iota \in \{\pm 1\}$. Then the intersection number of $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ is also ι, by the Claim above, and moreover the intersection number of $h_0^{-1}h(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ is also ι. Also, since $S_R(b_1)$ and $S'_L(b_1)$ intersect only at p_1, it is also true that $h_0^{-1}h(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ intersect only at p_1.

Via the coordinate chart $\mathcal{U} \to \mathbb{R}^n$ this translates into saying that $k\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orientation-preserving smooth embedding taking 0 to itself, such that $k(\mathbb{R}^a)$ meets $\mathbb{R}^b$ only at 0. This intersection is transverse, and has intersection number ι. The intersection number of $\mathbb{R}^a$ and $\mathbb{R}^b$ is also ι, so we may fix orientations such that both are $+1$.

Theorem 5.6 therefore applies, and tells us that we may isotope k to a new embedding k_1 with an isotopy constant outside of a small neighbourhood of 0 and also fixing 0, so that k_1 is the identity near 0 and $k_1(\mathbb{R}^a)$ and $\mathbb{R}^b$ intersect only at 0.

Translating back along the coordinate chart $\mathcal{U} \to \mathbb{R}^n$ and extending the isotopy to be constant outside of $\mathcal{U}_\epsilon$, we obtain an isotopy of $h_0^{-1}h$ to a new diffeomorphism ℓ such that ℓ is the identity near p_1 and $\ell(\mathcal{U}_\epsilon \cap S_R(b_1))$ and $\mathcal{U} \cap h_0^{-1}h(S'_L(b_1))$ intersect only at p_1.

Now we claim that $\ell(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ intersect only at p_1, as we wanted. Suppose for a contradiction that $y \neq p_1$ is another point in their intersection and let $x = \ell^{-1}(y)$. Since $h_0^{-1}h(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ intersect only at p_1, and $h_0^{-1}h = \ell$ outside of $\mathcal{U}_\epsilon$, it follows that $x \in \mathcal{U}_\epsilon$ and $y \in \mathcal{U}$, and therefore

$$y \in \ell(\mathcal{U}_\epsilon \cap S_R(b_1)) \cap \mathcal{U} \cap h_0^{-1}h(S'_L(b_1)),$$

which contradicts the previous paragraph. So $\ell(S_R(b_1))$ and $h_0^{-1}h(S'_L(b_1))$ intersect only at p_1, as required. Finally, note that this intersection is transverse, since, in the coordinate chart $\mathcal{U} \to \mathbb{R}^n$, it corresponds to the intersection of $\mathbb{R}^a$ and $\mathbb{R}^b$ at 0 in a sufficiently small neighbourhood. As stated in the third paragraph, this is what was needed to finish the proof of Assertion 6.

Remark. A key point is that, to apply Theorem 5.6, we should take a coordinate chart $\mathcal{U}$ so that $\mathbb{R}^a$ corresponds to $\mathcal{U} \cap S_R(b_1)$ and $\mathbb{R}^b$ corresponds to $\mathcal{U} \cap h_0^{-1}h(S'_L(b_1))$ — not to $\mathcal{U} \cap S'_L(b_1)$.

Note on the Claim. This claim is very similar to something that Milnor claims on page 58, although he applies the diffeomorphism $h_0^{-1}h$ to the right-hand sphere instead of to the left-hand sphere. He doesn't explicitly make the assumption about $h_0^{-1}h(S_R(b_1))$ and $S'_L(b_1)$ intersecting transversely, but he talks about their intersection number, which is only defined if they intersect transversely, so he is implicitly assuming this.

Here is a rough sketch of how I think one can "modify g_1 if necessary" to ensure that $S_R(b_1)$ and $h_0^{-1}h(S'_L(b_1))$ intersect transversely (this part is less rigorous than above).

The diffeomorphism $h_0^{-1}h$ is the composition $g_1(h')^{-1}g_2^{-1}h$ of four diffeomorphisms. Define $Q = (h')^{-1}g_2^{-1}h(S'_L(b_1))$, so we are interested in the intersection of $g_1(Q)$ and $S_R(b_1)$. Note that Q is contained in the subspace $L_1(b_1) = g_1^{-1}(f^{-1}(b_1))$ of L_1 (see the diagram on page 56). We first modify the embedding $g_1|_Q\colon Q \to f^{-1}(b_1)$ by an isotopy fixing $g_1^{-1}(p_1)$ so that it intersects $S_R(b_1)$ transversely. Then we extend this by the isotopy extension theorem to an isotopy of $g_1|_{L_1(b_1)}$. (See also Lemma 5.3.) This gives us a new embedding $\bar{g}_1\colon L_1(b_1) \to f^{-1}(b_1)$ such that $\bar{g}_1(Q)$ intersects $S_R(b_1)$ transversely. Then we extend $\bar{g}_1$ to an embedding $\hat{g}_1\colon L_1 \to W$ by following trajectories of η and ξ: given a point $x \in L_1$ (let's say $x \in g_1^{-1}(f^{-1}(c))$), follow the trajectory of η until you hit $L_1(b_1)$, then apply $\bar{g}_1$ and then follow the trajectory of ξ backwards until you are in the level $f^{-1}(c)$. It is not necessarily the case that the trajectory of η through every point in L_1 will actually hit $L_1(b_1)$, but we can just replace L_1 by the smaller neighbourhood of 0 consisting of all points of L_1 that do have this property. Then we may replace g_1 with $\hat{g}_1$. Because of how we defined it, property (a) at the top of page 56 still holds for $\hat{g}_1$, so this is a valid replacement. By construction, $\hat{g}_1(Q)$ intersects $S_R(b_1)$ transversely, so if we write $\hat{h}_0 = g_2 h' \hat{g}_1^{-1}$, we deduce that $\hat{h}_0^{-1} h(S'_L(b_1))$ intersects $S_R(b_1)$ transversely.

To ensure also that $\hat{h}_0^{-1} h$ is orientation-preserving near p_1 and that $\hat{h}_0^{-1} h(S'_L(b_1))$ and $S_R(b_1)$ have the correct intersection number, we need one extra step: just after we construct $\bar{g}_1 \colon L_1(b_1) \to f^{-1}(b_1)$ in the previous paragraph, and before extending it by following trajectories, we modify it by precomposition with an appropriate self-diffeomorphism of $L_1(b_1)$ to ensure these two properties. To do this, we make sure that we have chosen the closed neighbourhood L_1 of 0 in $\mathbb{R}^n$ so that it is diffeomorphic to a closed disc and that $L_1(b_1)$ is a closed $(n-1)$-dimensional disc in its boundary (as it looks like in the picture on page 56). Using this identification of $L_1(b_1)$ with D^{n-1}, where we ensure that $g_1^{-1}(p_1) = \bar{g}_1^{-1}(p_1)$ corresponds to $0 \in D^{n-1}$, we can modify $\bar{g}_1$ by precomposing it with appropriate reflections of D^{n-1} to ensure the orientation-preserving and intersection number properties.

NB: *I'm very happy to hear about any corrections if you spot an error in this note.* **Email:** `m@mdp.ac`